MA 511, Session 39

Review

1) The matrix exponential: Let A be a $n \times n$ matrix. Then,

$$e^A = \mathbb{I} + A + \tfrac{A^2}{2} + \cdots = \sum_{n=0}^{\infty} \frac{A^n}{n!} \quad (A^0 = \mathbb{I}).$$

It follows that $e^A e^{-A} = \mathbb{I}$ so that e^A is nonsingular. We also have $e^A e^B = e^{A+B}$ *if A and B commute.* Otherwise this is not true.

The system $\frac{dx}{dt} = Ax$ has e^{At} as fundamental matrix, which means its columns are n linearly independent solutions of the system, and hence a basis for the solution space. All of this is true even for defective matrices.

Suppose A is real symmetric. Then, its eigenvalues are real, $\lambda_1, \ldots, \lambda_n$ (repeated by multiplicity) and A can be diagonalized

$$S^{-1}AS = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ & & \vdots & \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} = \Lambda, \text{ i.e. } \ A = S\Lambda S^{-1}.$$

Then, from its definition e^A is also symmetric and

$$e^A = Se^\Lambda S^{-1} = S \begin{pmatrix} e^{\lambda_1} & 0 & \dots & 0 \\ 0 & e^{\lambda_2} & \dots & 0 \\ & & \vdots & \\ 0 & 0 & \dots & e^{\lambda_n} \end{pmatrix} S^{-1}.$$

Therefore, since similarity transformations preserve eigenvalues, we see that the eigenvalues of e^{At} are $e^{\lambda_1 t}, \dots, e^{\lambda_n t}$ and so e^A must be positive definite.

2) Given $A = \begin{pmatrix} 2 & 2 & -1 \\ -1 & -1 & 1 \\ -1 & -2 & 2 \end{pmatrix}$, find M such that $M^{-1}AM = J$ is a Jordan canonical form for A.

Solution: The characteristic polynomial is $p(\lambda) = -(\lambda - 1)^3$, so that the only eigenvalue of A is $\lambda = 1$ with algebraic multiplicity 3. Next we look for linearly independent eigenvectors.

The system $(A-\mathbb{I})x = 0$ has 2 free variables and we easily find that

$$S_1 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}.$$

It follows that J must consist of 2 Jordan blocks, necessarily one of size 2 and one of size 1. Let now x_1, x_2, x_3 denote the columns of M. Then,

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = A \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}.$$

As always, $Ax_1 = \lambda_1 x_1$ ($\lambda_1 = 1$ here), that is, x_1 is an eigenvector

$$x_1 = c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

The second column gives $Ax_2 = x_1 + \lambda x_2$, which is the same as $(A - \lambda\mathbb{I})x_2 = x_1$, that is x_2 is a generalized eigenvector, solution of

$$\begin{pmatrix} 1 & 2 & -1 \\ -1 & -2 & 1 \\ -1 & -2 & 1 \end{pmatrix} x_2 = \begin{pmatrix} c_1 \\ c_2 \\ c_1 + 2c_2 \end{pmatrix},$$

that is, with $x_2 = (x, y, z)^T$,

$$\begin{cases} x + 2y - z = c_1 \\ -x - 2y + x = c_2 \\ -x - 2y + x = c_1 + 2c_2. \end{cases}$$

Using row operations (Gaussian elimination), we find

$$\begin{cases} x + 2y - z = c_1 \\ 0 = c_1 + c_2 \\ 0 = 2c_1 + 2c_2, \end{cases}$$

which requires $c_1 = -c_2$ for solutions to exist. We chose $c_1 = 1$, $c_2 = -1$ and then $y = z = 0$. Thus we obtain

$$x_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \qquad x_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Finally, for x_3 we need an eigenvector that is linearly independent with x_1, for example $x_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Thus,

$$M = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix},$$

and it is immediate to see that $AM = MJ$ indeed.

3) Let us find now an orthogonal matrix Q such that $Q^T AQ = T$ is upper triangular, for the matrix A from last example.

<u>Solution</u>: First, we take a unit eigenvector as the first column of an orthogonal matrix Q_1, and take another two columns orthonormal with it:

$$Q_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

Then,

$$Q_1^{-1} AQ_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -1 & -1 & 1 \\ \frac{3}{\sqrt{2}} & \frac{4}{\sqrt{2}} & -\frac{3}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -\frac{2}{\sqrt{2}} \\ 0 & \frac{4}{\sqrt{2}} & 3 \end{pmatrix}$$

Next, we let $A_2 = \begin{pmatrix} -1 & -\frac{2}{\sqrt{2}} \\ \frac{4}{\sqrt{2}} & 3 \end{pmatrix}$. Its characteristic polynomial is

$p(\lambda) = (-1-\lambda)(3-\lambda)+4 = \lambda^2 - 2\lambda + 1 = (\lambda-1)^2$. We need an eigenvector of A_2, that is a nontrivial solution of $-2x_1 - \frac{2}{\sqrt{2}}x_2 = 0$. We choose it to be $\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$. Then we define an orthogonal 2×2 matrix $\tilde{Q}_2$ with this vector as first column, for example

$$\tilde{Q}_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix},$$

and then we define the 3×3 orthogonal matrix

$$Q_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & \sqrt{2} \\ 0 & -\sqrt{2} & 1 \end{pmatrix}.$$

Finally, the desired orthogonal matrix is $Q = Q_1 Q_2$:

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \cdot \frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & \sqrt{2} \\ 0 & -\sqrt{2} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{pmatrix},$$

and the upper triangular matrix is

$$
\begin{aligned}
T &= Q^T A Q \\
&= \frac{1}{6}\begin{pmatrix} \sqrt{3} & 0 & \sqrt{3} \\ -\sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & 2 & -1 \end{pmatrix}\begin{pmatrix} 2 & 2 & -1 \\ -1 & -1 & 1 \\ -1 & -2 & 2 \end{pmatrix}\begin{pmatrix} \sqrt{3} & -\sqrt{2} & 1 \\ 0 & \sqrt{2} & 2 \\ \sqrt{3} & \sqrt{2} & -1 \end{pmatrix} \\
&= \frac{1}{6}\begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & -18\sqrt{2} \\ 0 & 0 & 6 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -3\sqrt{2} \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}
$$