MA 511, Session 43

Review

1) Let A be a $n \times n$ matrix. The following conditions are all equivalent for A:

i) A is singular

ii) There is $b \in \mathbb{R}^n$ such that $Ax = b$ has no solution

iii) 0 is an eigenvalue of A

iv) $Ax = 0$ has infinitely many solutions $x \in \mathbb{R}^n$

v) $\det A = 0$

vi) $\operatorname{rank} A < n$

vii) $\operatorname{nullity} A > 0$

2) Let $V = \mathcal{P}_3$ be the vector space of polynomials of degree ≤ 3, and let $W = \{f \in V : f(1) = 0\}$. What is the dimension of W?

Solution: If $p \in W$, then $p(t) = (t-1)\,q(t)$ with $q \in \mathcal{P}_2$. Hence, a basis for W is

$$v_1(t) = (t-1), \quad v_2(t) = (t-1)t, \quad v_3(t) = (t-1)t^2,$$

and thus the dimension of W is 3.

3) Let U be a **unitary** matrix. Then,

i) U is unitarily similar to a diagonal matrix

ii) If λ is an eigenvalue of U, then $|\lambda| = 1$

iii) $U^H = U^{-1}$, thus U is invertible

iv) $\|Ux\| = \|x\|$, for any $x \in \mathbb{R}^n$

v) $(Uy, Ux) = (y, x)$, for any $x, y \in \mathbb{R}^n$

vi) $|\det U| = 1$

vii) If U is (real) orthogonal, the only singular value of U is $\sigma_1 = \cdots = \sigma_n = 1$ (a singular value decomposition of U is $U = (UQ)\mathbb{I}Q^T$, where Q is any $n \times n$ orthogonal matrix)

4) Let H be a **Hermitian** matrix. Then,

i) H is unitarily similar to a diagonal matrix

ii) If λ is an eigenvalue of H, then $\lambda \in \mathbb{R}$

iii) All eigenspaces of H are mutually orthogonal

iv) If H is (real) symmetric, its null space is the orthogonal complement of its column space

5) Let $V = \mathcal{M}_{2\times 2}$ be the vector space of (real) 2×2 matrices, and let $W = \left\{ A \in V : A \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0 \right\}$. Prove that W is a subspace of V and find a basis for W and its dimension.

<u>Solution</u>: (i) $0 \in W$ since $0 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0$;

(ii) If $A, B \in W$, then $A + B \in W$ since

$$(A+B)\begin{pmatrix} 1 \\ 2 \end{pmatrix} = A\begin{pmatrix} 1 \\ 2 \end{pmatrix} + B\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0 + 0 = 0;$$

(iii) If $A \in W$ and $c \in \mathbb{R}$, then $cA \in W$ since

$$(cA)\begin{pmatrix} 1 \\ 2 \end{pmatrix} = cA\begin{pmatrix} 1 \\ 2 \end{pmatrix} = c0 = 0.$$

Thus, W is a subspace of V.

Let now $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a generic element of V. If $A \in W$, then $A \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0$. This results in the following system of 2 equations for the 4 unknowns a, b, c, d:

$$\begin{cases} a + 2b = 0 \\ c + 2d = 0. \end{cases}$$

Thus, a generic element of W has the form

$$\begin{pmatrix} a & -\frac{a}{2} \\ c & -\frac{c}{2} \end{pmatrix} = a \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & -\frac{1}{2} \end{pmatrix},$$

whereby $\dim W = 2$ and a basis for W is, for example,

$$\begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & -\frac{1}{2} \end{pmatrix}.$$

6) Suppose that V is the same but W is now the subspace of V consisting of all 2×2 matrices X such that

$$X \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 0.$$

Show that $\dim W = 0$.

<u>Solution</u>: Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then, $X \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 0$ means

$$\begin{cases} a + 3b \qquad\qquad = 0 \\ 2a + 4b \qquad\qquad = 0 \\ \qquad\qquad c + 3d = 0 \\ \qquad\qquad 2c + 4d = 0. \end{cases}$$

We easily find $a = b = c = d = 0$, i.e. $W = \{0\}$.