MA 511, Session 44

Review

1) Intersections and sums of subspaces.

Let $V = \mathcal{M}_{2\times 2}$, and consider the 2 subspaces of V,

$$W_1 = \left\{ A \in V : A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\},$$

$$W_2 = \left\{ A \in V : A = A^T \right\}.$$

Find $W_1 \cap W_2$ and $W_1 + W_2$.

Solution: We find that

$$W_1 = \text{ span } \left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \right\}$$

and

$$W_2 = \text{ span } \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

This immediately implies

$$W_1 + W_2 = \text{ span } \left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

We can readily check that the first 4 of these matrices are linearly independent:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = a\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} + c\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

gives the 4×4 system

$$\begin{cases} a + c = 0 \\ -a + d = 0 \\ b + d = 0 \\ -b = 0 \end{cases}$$

which implies $b = 0$ and then $d = 0$, which makes $a = 0$ and thus $c = 0$. Thus, $W_1 + W_2 = \mathcal{M}_{2\times 2}$, since there is no other subspace of V of dimension 4. As for their intersection,

$$\begin{aligned} W_1 \cap W_2 &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : b = c, a + b = 0, c + d = 0 \right\} \\ &= \operatorname{span}\left\{ \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \right\} \end{aligned}$$

and $\dim W_1 \cap W_2 = 1$.