THE BOCKSTEIN HOMOMORPHISM FOR LOCAL COHOMOLOGY

ANURAG K. SINGH AND ULI WALTHER

1. Introduction

Let R be a polynomial ring over the ring of integers $\mathbb{Z}$, and fix an ideal $\mathfrak{a}$ of R. The connecting homomorphism induced by the local cohomology functor $H^\bullet_{\mathfrak{a}}(-)$ when applied to the short exact sequence

$$0 \longrightarrow R/pR \xrightarrow{p} R/p^2R \longrightarrow R/pR \longrightarrow 0$$

yields the *Bockstein homomorphism* of local cohomology

$$\beta_p^k \colon H^k_{\mathfrak{a}}(R/pR) \longrightarrow H^{k+1}_{\mathfrak{a}}(R/pR) \qquad \text{for } k \geqslant 0.$$

In this note we prove that for all but finitely many prime integers p, these Bockstein homomorphisms are zero, Theorem 3.1.

Our study here is motivated by Lyubeznik's conjecture [Ly2, Remark 3.7], stating that for regular rings R each local cohomology module $H^i_{\mathfrak{a}}(R)$ has finitely many associated prime ideals. This conjecture has been verified for regular rings of positive characteristic by Huneke and Sharp [HS], and for regular local rings of characteristic zero as well as unramified regular local rings of mixed characteristic by Lyubeznik [Ly2, Ly3]. It remains unresolved for polynomial rings over $\mathbb{Z}$, where it implies that almost all Bockstein homomorphisms are zero; Theorem 3.1 provides strong supporting evidence for Lyubeznik's conjecture.

Huneke [Hu, Problem 4] asked whether local cohomology modules of Noetherian rings have finitely many associated prime ideals. The answer to this is negative; in [Si1] the first author constructed an example where, for R a hypersurface, $H^3_{\mathfrak{a}}(R)$ has p-torsion elements for each prime integer p, and hence has infinitely many associated primes, see Example 4.3. In the language of the present article, [Si1] gives an example with infinitely many nonzero Bockstein homomorphisms. The issue of p-torsion appears to be central in studying Lyubeznik's conjecture for finitely generated $\mathbb{Z}$-algebras,

Date: March 5, 2008.
A.K.S. was supported by NSF grants DMS 0600819 and DMS 0608691.
U.W. was supported by NSF grant DMS 0555319 and by NSA grant H98230-06-1-0012.

and the Bockstein homomorphism is a first step towards understanding this issue.

In Section 2 we discuss the definition and basic properties of Bockstein homomorphisms. In Section 3 we prove our main result, Theorem 3.1. In Section 4 we discuss several examples related to Bockstein homomorphisms β_p^k with $k \leqslant 2$. In the last section we establish a close relationship between Bockstein homomorphisms on Stanley–Reisner rings and the Bockstein maps in the singular cohomology of the underlying topological spaces and use it to show the existence of nonzero Bockstein homomorphisms in $\mathbb{Z}[x_0, \dots, x_n]$.

2. Bockstein homomorphisms

Definition 2.1. Let R be a commutative Noetherian ring, and M an R-module. Let p be an element of R which is a non-zerodivisor on M. Let $F^\bullet$ be a covariant δ-functor on the category of R-modules. The exact sequence

$$0 \longrightarrow M \xrightarrow{\ p\ } M \longrightarrow M/pM \longrightarrow 0$$

then induces an exact sequence

$$F^k(M/pM) \xrightarrow{\delta_p^k} F^{k+1}(M) \xrightarrow{\ p\ } F^{k+1}(M) \xrightarrow{\pi_p^{k+1}} F^{k+1}(M/pM)\,.$$

The composition

$$\pi_p^{k+1} \circ \delta_p^k \colon F^k(M/pM) \longrightarrow F^{k+1}(M/pM)$$

is the *Bockstein homomorphism* $\beta_p^k = \beta_p^k(F(M))$.

An elementary check shows that β_p^k is the connecting homomorphism that results from applying $F^\bullet$ to the short exact sequence

$$0 \longrightarrow M/pM \xrightarrow{\ p\ } M/p^2M \longrightarrow M/pM \longrightarrow 0.$$

Let $\mathfrak{a}$ be an ideal of R, generated by elements $f_1, \dots, f_t$. The covariant δ-functors of interest to us are local cohomology $H^\bullet_{\mathfrak{a}}(-)$, Koszul cohomology $H^\bullet(f_1, \dots, f_t; -)$, and $\operatorname{Ext}^\bullet_R(R/\mathfrak{a}, -)$ discussed briefly below.

Setting $\boldsymbol{f}^e = f_1^e, \dots, f_t^e$, the Koszul cohomology modules $H^\bullet(\boldsymbol{f}^e; M)$ are the cohomology modules of the Koszul complex $K^\bullet(\boldsymbol{f}^e; M)$. For each $e \geqslant 1$, one has a map of complexes

$$K^\bullet(\boldsymbol{f}^e; M) \longrightarrow K^\bullet(\boldsymbol{f}^{e+1}; M)\,,$$

and thus a filtered directed system $\{K^\bullet(\boldsymbol{f}^e; M)\}_{e \geqslant 1}$. The direct limit of this system can be identified with the Čech complex $\check{C}^\bullet(\boldsymbol{f}; M)$ displayed below:

$$0 \longrightarrow R \longrightarrow \bigoplus_i R_{f_i} \longrightarrow \bigoplus_{i<j} R_{f_i f_j} \longrightarrow \cdots \longrightarrow R_{f_1 \cdots f_t} \longrightarrow 0\,.$$

The local cohomology modules $H^\bullet_{\mathfrak{a}}(M)$ may be computed as the cohomology modules of $\check{C}^\bullet(\boldsymbol{f}; M)$, or equivalently as the direct limit of the Koszul cohomology modules $H^\bullet(\boldsymbol{f}^e; M)$: the filter condition yields an isomorphism of functors

$$H^k_{\mathfrak{a}}(-) \;\cong\; \varinjlim_e H^k(\boldsymbol{f}^e; -)\,,$$

and each element of $H^k_{\mathfrak{a}}(M)$ lifts to an element of $H^k(\boldsymbol{f}^e; M)$ for $e \gg 0$.

Remark 2.2. Let $F^\bullet$ and $G^\bullet$ be covariant δ-functors on the category of R-modules, and let $\tau\colon F^\bullet \longrightarrow G^\bullet$ be a natural transformation. Since the Bockstein homomorphism is defined as the composition of a connecting homomorphism and reduction mod p, both of which are functorial, the Bockstein homomorphism is functorial as well; one has a commutative diagram

$$\begin{array}{ccc} F^k(M/pM) & \longrightarrow & F^{k+1}(M/pM) \\ {\scriptstyle\tau}\big\downarrow & & \big\downarrow{\scriptstyle\tau} \\ G^k(M/pM) & \longrightarrow & G^{k+1}(M/pM)\,, \end{array}$$

where the horizontal maps are the respective Bockstein homomorphisms. The natural transformations of interest to us are

$$H^\bullet(\boldsymbol{f}^e; -) \longrightarrow H^\bullet(\boldsymbol{f}^{e+1}; -)$$

and

$$H^\bullet(\boldsymbol{f}^e; -) \longrightarrow H^\bullet_{\mathfrak{a}}(-)\,,$$

where $\mathfrak{a} = (f_1, \dots, f_k)$.

Let M be an R-module, let p be an element of R which is a non-zerodivisor on M, and let $\boldsymbol{f} = f_1, \dots, f_t$ and $\boldsymbol{g} = g_1, \dots, g_t$ be elements of R such that $f_i \equiv g_i \bmod p$ for each i. One then has isomorphisms

$$H^\bullet(\boldsymbol{f}; M/pM) \cong H^\bullet(\boldsymbol{g}; M/pM)\,.$$

The Bockstein homomorphisms on $H^\bullet(\boldsymbol{f}; M/pM)$ and $H^\bullet(\boldsymbol{g}; M/pM)$ may not respect these isomorphisms; see Example 2.3. A key point in the proof of Theorem 3.1 is Lemma 2.4, which states that upon passing to the direct limits $\varinjlim_e H^\bullet(\boldsymbol{f}^e; M/pM)$ and $\varinjlim_e H^\bullet(\boldsymbol{g}^e; M/pM)$, the Bockstein homomorphisms commute with the isomorphisms

$$\varinjlim_e H^\bullet(\boldsymbol{f}^e; M/pM) \cong \varinjlim_e H^\bullet(\boldsymbol{g}^e; M/pM)\,.$$

Example 2.3. Let p be a non-zerodivisor on R. Let x be an element of R. The Bockstein homomorphism on Koszul cohomology $H^\bullet(x; R/pR)$ is

$$\begin{array}{rcl} (0 :_{R/pR} x) = H^0(x; R/pR) & \longrightarrow & H^1(x; R/pR) = R/(p,x)R \\ r \bmod (p) & \longmapsto & rx/p \bmod (p,x)\,. \end{array}$$

Let y be an element of R with $x \equiv y \bmod p$. Comparing the Bockstein homomorphisms β, β' on $H^\bullet(x; R/pR)$ and $H^\bullet(y; R/pR)$ respectively, one sees that the diagram

$$\begin{array}{ccc}
\big(0 :_{R/pR} x\big) & \xrightarrow{\ \beta\ } & R/(p,x)R \\
\| & & \| \\
\big(0 :_{R/pR} y\big) & \xrightarrow{\ \beta'\ } & R/(p,y)R
\end{array}$$

does not commute if rx/p and ry/p are different modulo the ideal $(p, y)R$. For an explicit example of this failure take $R = \mathbb{Z}[w, x, z]/(wx - pz)$ and $y = x + p$; then $\beta(\overline{w}) = \overline{z}$, whereas $\beta'(\overline{w}) = \overline{z} + \overline{w}$.

On the positive side, the following diagram does commute, hinting at the proof of the subsequent lemma.

$$\begin{array}{ccccc}
H^0(x; R/pR) & \longrightarrow & H^1(x; R/pR) & \longrightarrow & H^1(x^2; R/pR) \\
\| & & & & \| \\
H^0(y; R/pR) & \longrightarrow & H^1(y; R/pR) & \longrightarrow & H^1(y^2; R/pR)\,.
\end{array}$$

Lemma 2.4. *Let M be an R-module, and let p be an element of R which is M-regular. Suppose $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of R with $\operatorname{rad}(\mathfrak{a}+pR) = \operatorname{rad}(\mathfrak{b}+pR)$. Then there exists a commutative diagram*

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & H^k_{\mathfrak{a}}(M/pM) & \longrightarrow & H^{k+1}_{\mathfrak{a}}(M/pM) & \longrightarrow & \cdots \\
& & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H^k_{\mathfrak{b}}(M/pM) & \longrightarrow & H^{k+1}_{\mathfrak{b}}(M/pM) & \longrightarrow & \cdots
\end{array}$$

where the horizontal maps are the respective Bockstein homomorphisms, and the vertical maps are isomorphisms.

Proof. It suffices to consider the case $\mathfrak{a} = \mathfrak{b} + yR$, where $y \in \operatorname{rad}(\mathfrak{b} + pR)$. For each R-module N, one has an exact sequence

$$\longrightarrow H^{k-1}_{\mathfrak{b}}(N)_y \longrightarrow H^k_{\mathfrak{a}}(N) \longrightarrow H^k_{\mathfrak{b}}(N) \longrightarrow H^k_{\mathfrak{b}}(N)_y \longrightarrow$$

which is functorial in N; see for example [LC, Exercise 14.4]. Using this for

$$0 \longrightarrow M \xrightarrow{\ p\ } M \longrightarrow M/pM \longrightarrow 0\,,$$

one obtains the commutative diagram below, with exact rows and columns.

$$\begin{tikzcd}
H^{k-1}_{\mathfrak{b}}(M/pM)_y \arrow[r] \arrow[d] & H^k_{\mathfrak{b}}(M)_y \arrow[r, "p"] \arrow[d] & H^k_{\mathfrak{b}}(M)_y \arrow[r] \arrow[d] & H^k_{\mathfrak{b}}(M/pM)_y \arrow[d] \\
H^k_{\mathfrak{a}}(M/pM) \arrow[r] \arrow[d, "\theta^k"'] & H^{k+1}_{\mathfrak{a}}(M) \arrow[r, "p"] \arrow[d] & H^{k+1}_{\mathfrak{a}}(M) \arrow[r] \arrow[d] & H^{k+1}_{\mathfrak{a}}(M/pM) \arrow[d, "\theta^{k+1}"] \\
H^k_{\mathfrak{b}}(M/pM) \arrow[r] \arrow[d] & H^{k+1}_{\mathfrak{b}}(M) \arrow[r, "p"] \arrow[d] & H^{k+1}_{\mathfrak{b}}(M) \arrow[r] \arrow[d] & H^{k+1}_{\mathfrak{b}}(M/pM) \arrow[d] \\
H^k_{\mathfrak{b}}(M/pM)_y \arrow[r] & H^{k+1}_{\mathfrak{b}}(M)_y \arrow[r, "p"] & H^{k+1}_{\mathfrak{b}}(M)_y \arrow[r] & H^{k+1}_{\mathfrak{b}}(M/pM)_y
\end{tikzcd}$$

Since $H^\bullet_{\mathfrak{b}}(M/pM)$ is y-torsion, it follows that $H^\bullet_{\mathfrak{b}}(M/pM)_y = 0$. Hence the maps $\theta^\bullet$ are isomorphisms, and the desired result follows. $\square$

3. Main Theorem

We prove our main result:

Theorem 3.1. *Let $R = \mathbb{Z}[x_1, \dots, x_n]$ be a polynomial ring over the integers, and let $\mathfrak{a} = (f_1, \dots, f_t)$ be an ideal of R.*

For each prime integer p which is not a zerodivisor on $H^{k+1}(\boldsymbol{f}; R)$, the Bockstein homomorphism $H^k_{\mathfrak{a}}(R/pR) \longrightarrow H^{k+1}_{\mathfrak{a}}(R/pR)$ is the zero map.

In particular, the Bockstein homomorphisms on $H^\bullet_{\mathfrak{a}}(R/pR)$ are zero for all but finitely many prime integers p.

Proof. The R-modules $H^k(\boldsymbol{f}; R)$ are finitely generated for each k, so

$$\bigcup_k \operatorname{Ass} H^k(\boldsymbol{f}; R)$$

is a finite set of prime ideals. These finitely many prime ideals contain finitely many prime integers, so the last assertion follows from the previous.

Fix a prime number p which is a non-zerodivisor on $H^{k+1}(\boldsymbol{f}; R)$. Let ψ be the endomorphism of R with $\psi(x_i) = x_i^p$ for each $1 \leqslant i \leqslant n$. For each positive integer e, set ${}^{\psi^e}R$ to be the R-module with the same underlying Abelian group as R, and the R-module structure defined via $\psi^e \colon R \longrightarrow R$. Thus, for $r \in R$ and $r' \in {}^{\psi^e}R$, one has

$$r \cdot r' = \psi^e(r) r' .$$

Note that each ${}^{\psi^e}R$ is a flat R-module. Hence, applying $(-) \otimes_R {}^{\psi^e}R$ to the injective homomorphism

$$H^{k+1}(\boldsymbol{f}; R) \xrightarrow{\ p\ } H^{k+1}(\boldsymbol{f}; R) ,$$

we obtain an injective homomorphism

$$H^{k+1}(\psi^e(\boldsymbol{f});R) \xrightarrow{\ p\ } H^{k+1}(\psi^e(\boldsymbol{f});R)\,,$$

where $\psi^e(\boldsymbol{f}) = \psi^e(f_1),\dots,\psi^e(f_t)$. Thus, the connecting homomorphism

$$H^k(\psi^e(\boldsymbol{f});R/pR) \longrightarrow H^{k+1}(\psi^e(\boldsymbol{f});R)$$

is zero, and so is the Bockstein homomorphism

$$H^k(\psi^e(\boldsymbol{f});R/pR) \longrightarrow H^{k+1}(\psi^e(\boldsymbol{f});R/pR)\,. \tag{3.1.1}$$

The families of ideals

$$\big\{\big(\psi^e(\boldsymbol{f})\big)R/pR\big\}_{e\geqslant 1} \qquad \text{and} \qquad \big\{\mathfrak{a}^e R/pR\big\}_{e\geqslant 1}$$

are cofinal, so

$$H^k_{\mathfrak{a}}(R/pR) = \varinjlim_e H^k(\psi^e(\boldsymbol{f});R/pR)\,.$$

Let η be an element of $H^k_{\mathfrak{a}}(R/pR)$. Since the direct limit above is filterd, there exists an integer e and an element $\widetilde{\eta} \in H^k(\psi^e(\boldsymbol{f});R/pR)$ such that $\widetilde{\eta} \longmapsto \eta$. By Remark 2.2 and Lemma 2.4 one has a natural transformation of Bockstein homomorphisms

$$\begin{array}{ccc} H^k(\psi^e(\boldsymbol{f});R/pR) & \longrightarrow & H^{k+1}(\psi^e(\boldsymbol{f});R/pR) \\ \big\downarrow & & \big\downarrow \\ H^k_{\mathfrak{a}}(R/pR) & \longrightarrow & H^{k+1}_{\mathfrak{a}}(R/pR) \end{array}$$

where the map in the upper row is zero by (3.1.1). It follows that η maps to zero in $H^{k+1}_{\mathfrak{a}}(R/pR)$. □

4. Examples

Let p be a prime integer that is a non-zerodivisor in the ring R. Each local cohomology module $H^k_{\mathfrak{a}}(R/pR)$ then has a Frobenius action

$$F\colon H^k_{\mathfrak{a}}(R/pR) \longrightarrow H^k_{\mathfrak{a}}(R/pR)\,.$$

We shall see that $\beta^0_p \circ F = 0 = \beta^1_p \circ F$ for any ring R, whereas $\beta^2_p \circ F$ need not be zero.

In Example 4.3 we exhibit a hypersurface R over $\mathbb{Z}$ with ideal $\mathfrak{a}$ such that $\beta^2_p\colon H^2_{\mathfrak{a}}(R/pR) \longrightarrow H^3_{\mathfrak{a}}(R/pR)$ is nonzero for each prime integer p.

Example 4.1. Let $\mathfrak{a} = (f_1,\dots,f_t) \subseteq R$ and let $[r] \in H^0_{\mathfrak{a}}(R/pR)$. There exists an integer n and $a_i \in R$ such that $rf_i^n = pa_i$ for each i. Then

$$\beta^0_p([r]) = \big[\big(a_1/f_1^n,\dots,a_t/f_t^n\big)\big] \in H^1_{\mathfrak{a}}(R/pR)\,.$$

Note that $\beta^0_p \circ F = 0$ since

$$\beta^0_p\big(F([r])\big) = \big[\big(p^{p-1}a_1^p/f_1^{np},\dots,p^{p-1}a_t^p/f_t^{np}\big)\big] = 0\,.$$

For an example where β_p^0 is nonzero, take $R = \mathbb{Z}[x,r]/(rx-p)$. Then

$$[r] \in H^0_{(x)}(R/pR) \xrightarrow{\beta_p^0} [1/x] \in H^1_{(x)}(R/pR)\,.$$

Example 4.2. Let $\mathfrak{a} = (f_1,\dots,f_t) \subset R$, and let $\eta = [(r_1/f_1^n,\dots,r_t/f_t^n)]$ be an element of $H^1_{\mathfrak{a}}(R/pR)$. For each $i<j$, there exists elements a_{ij} in R and an integer N such that

$$\frac{r_i}{f_i^n} - \frac{r_j}{f_j^n} = \frac{pa_{ij}}{(f_if_j)^N} \qquad \text{in } R_{f_if_j}\,.$$

The Bockstein homomorphism β_p^1 is the map with

$$\beta_p^1(\eta) = \big[\big(\dots,\ a_{ij}/(f_if_j)^N,\ \dots\big)\big]\,.$$

Note that $\beta_p^1\big(F(\eta)\big) = 0$ since

$$(r_i/f_i^n)^p - (r_j/f_j^n)^p \ \in\ p^2R_{f_if_j}\,.$$

Example 4.3. We give an example where β_p^2 is nonzero for each prime integer p; this is adapted from [Si1, Section 4], see also [Si2]. Consider the hypersurface

$$R = \mathbb{Z}[u,v,w,x,y,z]/(ux+vy+wz)$$

and ideal $\mathfrak{a} = (x,y,z)R$. Let p be an arbitrary prime integer. Then the element $(u/yz, -v/xz, w/xy) \in R_{yz} \oplus R_{xz} \oplus R_{xy}$ gives a cohomology class

$$\eta = [(u/yz, -v/xz, w/xy)] \ \in\ H^2_{\mathfrak{a}}(R/pR)\,.$$

It is easily seen that $\beta_p^2(\eta) = 0$; we verify below that $\beta_p^2\big(F(\eta)\big)$ is nonzero. Indeed, if

$$\beta^2\big(F(\eta)\big) = \left[\frac{(ux)^p + (vy)^p + (wz)^p}{p(xyz)^p}\right]$$

is zero, then there exists $k \in \mathbb{N}$ and elements $c_i \in R/pR$ such that

$$\frac{(ux)^p + (vy)^p + (wz)^p}{p}(xyz)^k = c_1x^{p+k} + c_2y^{p+k} + c_3z^{p+k} \tag{4.3.1}$$

in R/pR. Assign weights as follows:

$$\begin{aligned} &x\colon (1,0,0,0), \qquad && u\colon (-1,0,0,1),\\ &y\colon (0,1,0,0), && v\colon (0,-1,0,1),\\ &z\colon (0,0,1,0), && w\colon (0,0,-1,1). \end{aligned}$$

There is no loss of generality in taking the elements c_i to be homogeneous, in which case $\deg c_1 = (-p,k,k,p)$, so c_1 must be a scalar multiple of $u^py^kz^k$. Similarly, c_2 is a scalar multiple of $v^pz^kx^k$ and c_3 of $w^px^ky^k$. Hence

$$\frac{(ux)^p + (vy)^p + (wz)^p}{p}(xyz)^k \ \in\ (xyz)^k\big(u^px^p,\ v^py^p,\ w^pz^p\big)R/pR\,.$$

Canceling $(xyz)^k$ and specializing $u \longmapsto 1, v \longmapsto 1, w \longmapsto 1$, we have

$$\frac{x^p + y^p + (-x-y)^p}{p} \in \big(x^p, y^p\big)\mathbb{Z}/p\mathbb{Z}[x,y]\,,$$

which is easily seen to be false.

For local or graded rings R, the first examples of local cohomology modules $H^i_{\mathfrak{a}}(R)$ with infinitely many associated primes were produced by Katzman [Ka]. The rings in Katzman's examples are hypersurfaces which, however, are not integral domains. Subsequently, Singh and Swanson [SS] constructed families of graded hypersurfaces R over arbitrary fields, for which a local cohomology module $H^i_{\mathfrak{a}}(R)$ has infinitely many associated primes; these hypersurfaces are unique factorization domains which have rational singularities in the characteristic zero case, and are F-regular in the case of positive characteristic.

Example 4.4. Let $E \subset \mathbb{P}^2_{\mathbb{Q}}$ be a smooth elliptic curve. Set $R = \mathbb{Z}[x_0, \dots, x_5]$ and let $\mathfrak{a} \subset R$ be an ideal such that $(R/\mathfrak{a}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the homogeneous coordinate ring of $E \times \mathbb{P}^1_{\mathbb{Q}}$ for the Segre embedding $E \times \mathbb{P}^1_{\mathbb{Q}} \subset \mathbb{P}^5_{\mathbb{Q}}$. For all but finitely many prime integers p, the reduction mod p of E is a smooth elliptic curve E_p, and $R/(\mathfrak{a}+pR)$ is a homogeneous coordinate ring for $E_p \times \mathbb{P}^1_{\mathbb{Z}/p}$; we restrict our attention to such primes.

The elliptic curve E_p is said to be *ordinary* if the Frobenius map

$$F\colon H^1(E_p, \mathcal{O}_{E_p}) \longrightarrow H^1(E_p, \mathcal{O}_{E_p})$$

is injective, and is *supersingular* otherwise. By well-known results on elliptic curves [De, El], there exist infinitely many primes p for which E_p is ordinary, and infinitely many for which E_p is supersingular. The local cohomology module $H^4_{\mathfrak{a}}(R/pR)$ is zero if E_p is supersingular, and nonzero if E_p is ordinary, see [HS, page 75], [Ly4, page 219], [SW1, Corollary 2.2], or [LC, Section 22.1]. It follows that the multiplication by p map

$$H^4_{\mathfrak{a}}(R) \xrightarrow{\ p\cdot\ } H^4_{\mathfrak{a}}(R)$$

is surjective for infinitely many primes p, and not surjective for infinitely many p. Lyubeznik's conjecture implies that this map is injective for almost all primes p, equivalently that the connecting homomorphism

$$H^3_{\mathfrak{a}}(R/pR) \xrightarrow{\ \delta\ } H^4_{\mathfrak{a}}(R)$$

is zero for almost all p. We do not know if this is true. However, Theorem 3.1 implies that the composition

$$\beta^3_p\colon H^3_{\mathfrak{a}}(R/pR) \xrightarrow{\ \delta\ } H^4_{\mathfrak{a}}(R) \longrightarrow H^4_{\mathfrak{a}}(R/pR)$$

is zero for almost all p. It is known that the ideal $\mathfrak{a}R/pR$ is generated up to radical by four elements, [SW1, Theorem 1.1],♣ so the only possibly nonzero

uli: guessing...

Bockstein homomorphism is the map β_p^3 above.

5. Stanley-Reisner rings

It appears that the more pleasing the base ring R, the more likely is the vanishing of local cohomology Bocksteins. For example, while nonzero Bocksteins do exist in fairly simple hypersurface rings and in ramified regular rings by the examples in the previous section, it is not immediately clear whether there are any nonzero Bocksteins in rings of polynomials over $\mathbb{Z}$.

Bockstein maps originated in algebraic topology where they were used, for example, to investigate the topology of lens spaces. A close connection between algebraic topology and commutative algebra is given by Stanley-Reisner ideals and we now focus on this class if ideals.

In this section, p is a prime number as in the previous section, but $\mathfrak{a} \subseteq R$ is now assumed to be a Stanley-Reisner ideal in $R = \mathbb{Z}[x_0, \dots, x_n]$. We shall establish a link between topological Bockstein maps and those on the local cohomology modules $H^\bullet_{\mathfrak{a}}(R)$. In particular, we exhibit ideals in a polynomial ring over $\mathbb{Z}$ with a nonzero Bockstein on local cohomology.

Let us recall some notation.

Definition 5.1. Let Δ_n be the n-simplex, viewed as a simplicial complex (i.e., with its collection of faces), and fix a subcomplex Δ. Then the associated *Stanley-Reisner ideal* is the ideal

$$\mathfrak{a}_\Delta = (\boldsymbol{x}^\sigma \mid \sigma \notin \Delta) \subseteq R$$

generated by the monomials $\boldsymbol{x}^\sigma = \prod_{i=0}^n x_i^{\sigma_i}$ for which σ is not a face of Δ. In particular, if $\{i\} \notin \Delta$ then x_i is in $\mathfrak{a}_\Delta$. The quotient $S_\Delta = R/\mathfrak{a}_\Delta$ is the *Stanley-Reisner ring* of Δ.

A natural source of Stanley-Reisner ideals are triangulations of interesting topological spaces. For example, consider the following simplicial complex corresponding to the minimal triangulation of the real projective plane, induced by the familiar degree-2-map on the boundary of a hemisphere shown below.

The associated Stanley-Reisner ideal is

$$\begin{aligned} \mathfrak{a}_{\mathbb{RP}^2} \;=\; & (x_1x_2x_5,\, x_1x_2x_0,\, x_1x_3x_4,\, x_1x_3x_0,\, x_1x_4x_5, \\ & x_2x_3x_4,\, x_2x_3x_5,\, x_2x_4x_0,\, x_3x_5x_0,\, x_4x_5x_0) \\ & \subseteq \mathbb{Z}[x_0, \dots, x_5]. \end{aligned} \tag{5.1.1}$$

In general, a theorem of Hochster relates the singular (or simplicial, they are equivalent) $\mathbb{Z}/p\mathbb{Z}$-cohomology of the topological space underlying Δ to graded components of the local cohomology modules $H^\bullet_{\mathfrak{a}_\Delta}(R/pR)$. To state the relation succinctly we need two concepts of simplicial topology:

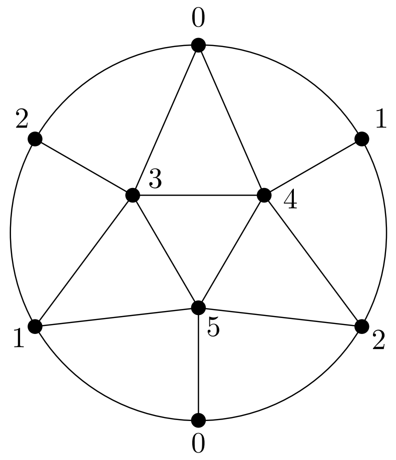

FIGURE 1. The minimal triangulation of the real projective plane

Definition 5.2. Let Δ be a simplicial complex and suppose τ is a subset of its vertex set (but not necessarily a simplex). The *link*, $\mathrm{link}_\Delta(\tau)$, is the set

$$\mathrm{link}_\Delta(\tau) = \{\tau' \mid \tau \cap \tau' = \emptyset, \quad \tau \cup \tau' \in \Delta\},$$

while its *star*, $\mathrm{star}(\tau)$, is simply

$$\mathrm{star}_\Delta(\tau) = \{\tau' \mid \tau \cup \tau' \in \Delta\}.$$

If $\boldsymbol{a} \in \mathbb{Z}^{n+1}$ is a multi-degree then write $\boldsymbol{a}_+$ and $\boldsymbol{a}_-$ for the sets $\{i \mid a_i > 0\}$ and $\{i \mid a_i < 0\}$ respectively. Both can be viewed as subsimplices of Δ_n. Hochster's theorem, [BH, Section 5.3] states the following.

$$\big(H^i_{\mathfrak{m}}(S_\Delta/pS_\Delta)\big)_{\boldsymbol{a}} = \widetilde{H}^{i-|\boldsymbol{a}_-|-1}(\mathrm{link}_{\mathrm{star}_\Delta(\boldsymbol{a}_+)}(\boldsymbol{a}_-); \mathbb{Z}/p\mathbb{Z}).$$

In order to relate these quantities to $H^\bullet_{\mathfrak{a}_\Delta}(R)$ we need

Theorem 5.3 (Bocksteins commute with local duality)**.** *Let* $(S, \mathfrak{m}, \mathbb{K})$ *be a local Gorenstein ring of dimension at least one. Let* $p \in S$ *be a regular element on both* ω_S *and on the finitely generated* S*-module* M*.*

The Bockstein morphism on $\mathrm{Ext}_S^{\dim(S)-i}(M, \omega_S/p\omega_S)$ *induced by*

$$0 \longrightarrow \omega_S \xrightarrow{p} \omega_S \longrightarrow \omega_{S/pS} = \omega_S/p\omega_S \longrightarrow 0$$

is Matlis dual to the Bockstein morphism on $H^i_{\mathfrak{m}}(M/pM)$ *induced by*

$$0 \longrightarrow M \xrightarrow{p} M \longrightarrow M/pM \longrightarrow 0.$$

Proof. Let E be the S-injective hull of $\mathbb{K} = S/\mathfrak{m}$ and denote by $(-)^\vee$ the Matlis duality functor $\mathrm{Hom}_S(-, E_S(\mathbb{K}))$. Let F be a free resolution of M and recall that, by local duality, the complex $F \otimes_S E \cong \mathrm{Hom}_R(F, S)^\vee$ computes $H^\bullet_{\mathfrak{m}}(M)$ as the left derived functors of $M \otimes E = H^d_{\mathfrak{m}}(M)$.

The long exact sequence of local cohomology

$$H^i_{\mathfrak{m}}(M) \xrightarrow{p} H^i_{\mathfrak{m}}(M) \xrightarrow{\pi} H^i_{\mathfrak{m}}(M/pM) \xrightarrow{\delta} H^{i+1}_{\mathfrak{m}}(M)$$

is the (slightly shifted) cohomology exact sequence to the short exact sequence of complexes

$$F \otimes (E_p \longrightarrow E \xrightarrow{p} E) \tag{5.3.1}$$

where $E_p = \operatorname{ann}_E(p) = \operatorname{Hom}_S(S/pS, E) = E_{S/pS}(\mathbb{K})$. Here, the identification of the cohomology groups of $F \otimes E_p$ with $H^\bullet_{\mathfrak{m}}(M/pM)$ uses the fact that p is M-regular. (Namely, if p is M-regular, the exact sequences obtained from $F \otimes (E_p \longrightarrow E \xrightarrow{p} E)$ and $(F \xrightarrow{p} F \longrightarrow F/pF) \otimes E$ coincide while in general there is an error term due to $\operatorname{Tor}_S(\operatorname{ann}_M(p), E) = H_{\mathfrak{m}}(\operatorname{ann}_M(p))$.)

On the other hand, as p is ω_S-regular,

$$\operatorname{Ext}^{d-i}_S(M, \omega_S) \xleftarrow{p} \operatorname{Ext}^{d-i}_S(M, \omega_S) \xleftarrow{\delta} \operatorname{Ext}^{d-i-1}_S(M, \omega_S/p\omega_S) \xleftarrow{\pi} \operatorname{Ext}^{d-i-1}_S(M, E)$$

is the long exact sequence of cohomology to the short exact sequence of complexes

$$\operatorname{Hom}_S(F, (\omega_S \xrightarrow{p} \omega_S \longrightarrow \omega_S/p\omega_S))\,. \tag{5.3.2}$$

It suffices to show that (5.3.1) and (5.3.2) are dual under $(-)^\vee = \operatorname{Hom}_S(-, E)$. Let $F^* = \operatorname{Hom}_S(F, S)$; then

$$\begin{aligned}
&\operatorname{Hom}_S(\operatorname{Hom}_S(F, (\omega_S \xrightarrow{p} \omega_S \longrightarrow \omega_S/p\omega_S)), E)\\
&\quad= \operatorname{Hom}_S(F^* \otimes (\omega_S \xrightarrow{p} \omega_S \longrightarrow \omega_S/p\omega_S), E)\\
&\quad= \operatorname{Hom}_S(F^*, \operatorname{Hom}_S((\omega_S \xrightarrow{p} \omega_S \longrightarrow \omega_S/p\omega_S), E))\\
&\quad= F \otimes \operatorname{Hom}_S((\omega_S \xrightarrow{p} \omega_S \longrightarrow \omega_S/p\omega_S), E)\\
&\quad= F \otimes (E_p \longrightarrow E \xrightarrow{p} E)
\end{aligned}$$

since S is Gorenstein. ☐

Theorem 5.4. *Let $\Delta \subseteq \Delta_n$ be a simplicial complex on $n+1$ vertices, and let $S_\Delta = R/\mathfrak{a}_\Delta$ be its integral Stanley-Reisner ring. If p is a prime number then the following are equivalent:*

1. *For some multi-degree $\boldsymbol{a}$, the topological Bockstein morphism on the singular reduced $(i-|\boldsymbol{a}_-|-1)$-th $\mathbb{Z}/p\mathbb{Z}$-cohomology of $\operatorname{link}_{\operatorname{star}_\Delta(\boldsymbol{a}_+)}(\boldsymbol{a}_-)$ is nonzero.*
2. *The Bockstein morphism $\beta^{n+1-i}_p(H_{\mathfrak{a}_\Delta}(R))$ on the local cohomology group $H^{n+1-i}_{\mathfrak{a}_\Delta}(R/pR)$ is nonzero.*

Note that Condition (1) is satisfied if $\widetilde{H}^{i-1}(\Delta; \mathbb{Z}/p\mathbb{Z})$ admits a nonzero Bockstein map itself, since one then may take $\boldsymbol{a} = 0$, so that $\boldsymbol{a}_- = \boldsymbol{a}_+ = \emptyset$.

Proof. Recall that the ring S of characteristic $p > 0$ is F-pure if and only if for each S-module M the natural map

$$\begin{aligned} \mathrm{Frob}_p \otimes 1\colon S \otimes_S M &\longrightarrow S \otimes_S M, \\ s \otimes m &\longmapsto s^p \otimes m \end{aligned}$$

is an injective map. A particular class of F-pure rings is given by the quotients of R/pR by square-free monomial ideals (compare [Ly1, Mu, SW2]).

uli: shouldn't some japanese be quoted here?

♣Since $\mathfrak{a}_\Delta$ is such an ideal, the maps in the limit

$$H^i_{\mathfrak{a}_\Delta}(R/pR) = \varinjlim_j \mathrm{Ext}^i_{R/pR}(\mathrm{Frob}^e_p(S_\Delta/pS_\Delta), R/pR)$$

are all injective.

We now show that Condition (2) is equivalent to

Condition (3): The Bockstein morphism $\beta^{n+1-i}_p(\mathrm{Ext}_R(S_\Delta, -))$ is nonzero on R/pR.

Namely, $H^{n+1-i}_{\mathfrak{a}_\Delta}(R/pR) = \varinjlim_e \mathrm{Frob}^e_p(\mathrm{Ext}^{n+1-i}_{R/pR}(S_\Delta/pS_\Delta, R/pR))$ and this limit is actually a union by purity. Moreover, regularity of p on S_Δ implies that $\mathrm{Ext}^{n+1-i}_{R/pR}(S_\Delta/pS_\Delta, R/pR)$ and $\mathrm{Ext}^{n+1-i}_R(S_\Delta, R/pR)$ are naturally isomorphic as R-modules. Thus, (2) holds if and only if at least one of the Bockstein morphisms

$$\mathrm{Frob}^e_p(\mathrm{Ext}^{n+1-i}_R(S_\Delta, R/pR)) \longrightarrow \mathrm{Frob}^e_p(\mathrm{Ext}^{n+2-i}_R(S_\Delta, R/pR))$$

is nonzero. Now, functoriality of Bockstein morphisms and injectivity of the limit maps implies that this is the case if and only if (3) holds.

uli: i think m had not been defined before

Let $\mathfrak{m}$ denote the ideal generated by $x_0, \ldots, x_n$.♣ We claim that Condition (3) is equivalent to

Condition (4): The Bockstein morphism $\beta^i_p(H_\mathfrak{m}(-))$ is nonzero on S_Δ/pS_Δ.

Namely, let $\mathfrak{n} = \mathfrak{m} + Rp$; $\mathfrak{n}$ is a maximal ideal of R. By Proposition 5.3, Condition (3) is equivalent to the nonvanishing of the Bockstein $\beta^i_p(H_\mathfrak{n}(-))$ on S_Δ/pS_Δ. There is a natural transformation of functors $H_\mathfrak{n}(-) \longrightarrow H_\mathfrak{m}(-)$, and on p-torsion modules $\Gamma_\mathfrak{n}(-)$ and $\Gamma_\mathfrak{m}(-)$ agree. By Remark 2.2, Conditions (3) and (4) are equivalent.

Finally, Condition (4) is equivalent to Condition (1) as follows.

The Bockstein $\beta^i_p(H^\bullet_\mathfrak{m}(S_\Delta)$ is computed from the long exact sequence

$$\cdots \longrightarrow H^i_\mathfrak{m}(S_\Delta/pS_\Delta) \longrightarrow H^i_\mathfrak{m}(S_\Delta) \longrightarrow H^i_\mathfrak{m}(S_\Delta) \longrightarrow H^{i+1}_\mathfrak{m}(S_\Delta/pS_\Delta) \longrightarrow \cdots.$$

Inspection shows that the $\mathbb{Z}^{n+1}$-graded component in degree $\boldsymbol{a}$ of this exact sequence carries precisely the same simplicial structure as the sequence one would naturally use to compute the topological Bockstein morphism $\beta^{i-|\boldsymbol{a}_+|-1}_p(\widetilde{H}^\bullet(\mathrm{link}_{\mathrm{star}_\Delta(\boldsymbol{a}_+)}(\boldsymbol{a}_-); \mathbb{Z}/p\mathbb{Z})$. For example, the proof of [BH, Lemma

5.3.7] spells this out in detail; one does not need the base ring to be a field. Hence the image of $\beta^i_p(H^\bullet_{\mathfrak{m}}(S_\Delta))$ is nontrivial if and only if Condition (2) holds.

□

Example 5.5. We return to the real projective plane. As is well-known, the topological Bockstein $\beta^1_2(H_{sing}(\mathbb{RP}^2))$ is nonzero. Namely, $H^2(\mathbb{RP}^2;\mathbb{Z})$ is generated by the 2-cocycle $c_2 = (1,2,3)^*$ (or any other 2-simplex) where the star indicates that we consider cochains. Orienting all 2-simplices of the hemisphere positively, equivalence modulo coboundary allows the move the chosen simplex around. Within the hemisphere this preserves orientation but moving across the boundary reverses it. It follows that $2c_2$ is zero in $H^2(\mathbb{RP}^2;\mathbb{Z})$ whence c_2 generates an element in $H^1(\mathbb{RP}^2;\mathbb{Z}/2\mathbb{Z})$. An easy check shows that this is the class of $c_1 = (1,2)^* - (2,5)^* + (5,6)^* - 6,3)^* + (3,1)^*$. The 2-Bockstein on the class of c_1 is nonzero since it is the class of c_2 taken modulo 2.

The Stanley-Reisner ideal $\mathfrak{a}_\Delta$ is generated by the cubics shown in (5.1.1), and its minimal free resolution is of the form

$$R \longrightarrow R^7 \longrightarrow R^{15} \longrightarrow R^{10} \longrightarrow R$$

where the top degree map is the transpose of the the peculiar matrix

$$R^7 \xrightarrow{(x_1,x_2,x_3,x_4,x_5,x_6,2)} R.$$

In particular, S_Δ/pS_Δ is Cohen-Macaulay if and only if $p > 2$.

Example 5.6. It is interesting to contemplate the higher Bockstein homomorphisms $\beta^k_{p^i}(H_{\mathfrak{a}}(R))$. Of course, these can only be nonzero if $\operatorname{Ext}^k_R(R/\mathfrak{a}, R)$ has an associated prime containing p. However, it can happen that all Bocksteins $\beta^i_p(H_{\mathfrak{a}_\Delta}(R))$ with $p \in \operatorname{Spec}(\mathbb{Z})$ are zero while some higher Bocksteins $\beta^i_{p^e}(H_{\mathfrak{a}_\Delta}(R))$ are not. For the most elementary example consider the Stanley-Reisner ideal associated to the space X obtained from a hemisphere via a fourfold covering of the boundary. The simplicial complex is depicted below.

The corresponding Stanley-Reisner ideal is composed of the six quadrics

$$x_4x_7,\ x_5x_7,\ x_5x_8,\ x_5x_0,\ x_6x_0,\ x_7x_0$$

and the 28 cubics

$$\begin{gathered}x_1x_2x_3,\ x_1x_2x_6,\ x_1x_2x_0,\ x_1x_3x_4,\ x_1x_3x_7,\ x_1x_4x_5, x_1x_4x_6,\\ x_1x_4x_8,\ x_1x_5x_6,\ x_1x_6x_8,\ x_1x_7x_8,\ x_1x_8x_0,\ x_2x_3x_5,\ x_2x_3x_8,\\ x_2x_4x_5,\ x_2x_4x_6,\ x_2x_4x_8,\ x_2x_4x_0,\ x_2x_6x_7, x_2x_6x_8, x_2x_7x_8,\\ x_3x_4x_6, x_3x_4x_8, x_3x_4x_0, x_3x_5x_6,\ x_3x_6x_7,\ x_3x_6x_8,\ x_3x_8x_0\ .\end{gathered}$$

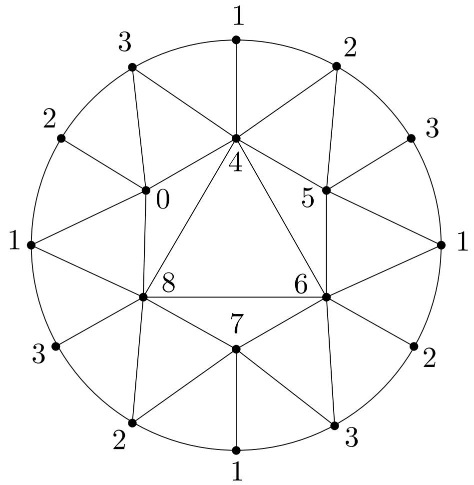

FIGURE 2. A quotient of the real projective plane

One can show in parallel to the previous example that the class $c_2 = (1, 2, 3)^*$ generates $H^2(X; \mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z}$. It follows that there is $c_1 \in H^1(X; \mathbb{Z}/4\mathbb{Z})$ mapping to c_2 under the connecting homomorphism, and the 4-Bockstein of c_1 is nonzero. On the other hand, no element of $\mathbb{Z}/4\mathbb{Z}$ that is killed by 2 survives reduction modulo 2, and thus the 2-Bockstein on $H^1(X; \mathbb{Z}/2\mathbb{Z})$ must be zero. Correspondingly, the Bockstein on $H^7_{\mathfrak{a}_\Delta}(R/pR)$ is nontrivial if $p = 4$. This time, the resolution of $\mathfrak{a}_\Delta$ ends in the transpose of

$$R^9 \xrightarrow{(-2x_1, 3x_1, x_1, -x_3, 2x_2, -3x_2, 3x_3, -x_2, 2x_3, 3x_5, x_7, -x_0, -x_4, -x_6, -x_8, 4)} R.$$

Remark 5.7. One easily checks that for any collection of prime numbers $p_1, \dots, p_k$ the quotient of the hemisphere obtained from a $p_1 \cdots p_k$-fold covering has a Stanley-Reisner ideal with nonvanishing p-Bocksteins precisely if $p = p_i$ for some i.

REFERENCES

[BH] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, revised edition, Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, Cambridge, 1998.

[De] M. Deuring, *Die Typen der Multiplikatorenringe elliptischer Funktionenkörper*, Abh. Math. Sem. Hansischen Univ. **14** (1941), 197–272.

[El] N. D. Elkies, *The existence of infinitely many supersingular primes for every elliptic curve over* $\mathbb{Q}$, Invent. Math. **89** (1987), 561–567.

[HS] R. Hartshorne and R. Speiser, *Local cohomological dimension in characteristic* p, Ann. of Math. (2) **105** (1977), 45–79.

[Hu] C. Huneke, *Problems on local cohomology*, in: Free resolutions in commutative algebra and algebraic geometry (Sundance, Utah, 1990), 93–108, Res. Notes Math. **2**, Jones and Bartlett, Boston, MA, 1992.

[HS] C. Huneke and R. Sharp, *Bass numbers of local cohomology modules*, Trans. Amer. Math. Soc. **339** (1993), 765–779.

[LC] S. B. Iyengar, G. J. Leuschke, A. Leykin, Anton; C. Miller, E. Miller, A. K. Singh, and U. Walther, *Twenty-four hours of local cohomology*, Grad. Stud. Math. **87**, American Mathematical Society, Providence, RI, 2007.

[Ka] M. Katzman, *An example of an infinite set of associated primes of a local cohomology module*, J. Algebra **252** (2002), 161–166.

[Ly1] G. Lyubeznik, *On the local cohomology modules $H^i_{\mathfrak{a}}(R)$ for ideals $\mathfrak{a}$ generated by monomials in an R-sequence*, in: Complete Intersections, Lecture Notes in Math. **1092**, Springer, 1984, pp. 214–220.

[Ly2] G. Lyubeznik, *Finiteness properties of local cohomology modules (an application of D-modules to commutative algebra)*, Invent. Math. **113** (1993), 41–55.

[Ly3] G. Lyubeznik, *Finiteness properties of local cohomology modules for regular local rings of mixed characteristic: the unramified case*, Comm. Alg. **28** (2000), 5867–5882.

[Ly4] G. Lyubeznik, *On the vanishing of local cohomology in characteristic $p > 0$*, Compos. Math. **142** (2006), 207–221.

[Mu] M. Mustaţă, *Local cohomology at monomial ideals*, J. Symbolic Comput. **29** (2000), 709–720.

[Si1] A. K. Singh, *p-torsion elements in local cohomology modules*, Math. Res. Lett. **7** (2000), 165–176.

[Si2] A. K. Singh, *p-torsion elements in local cohomology modules II*, in: Local cohomology and its applications (Guanajuato, 1999), 155–167, Lecture Notes in Pure and Appl. Math. **226**, Dekker, New York, 2002.

[SS] A. K. Singh and I. Swanson, *Associated primes of local cohomology modules and of Frobenius powers*, Int. Math. Res. Not. **33** (2004), 1703–1733.

[SW1] A. K. Singh and U. Walther, *On the arithmetic rank of certain Segre products*, Contemp. Math. **390** (2005) 147–155.

[SW2] A. K. Singh and U. Walther, *Local cohomology and pure morphisms*, Illinois J. Math. **51** (2007), 287–298.

Department of Mathematics, University of Utah, 155 South 1400 East, Salt Lake City, UT 84112, USA

E-mail address: singh@math.utah.edu

Department of Mathematics, Purdue University, 150 N. University Street, West Lafayette, IN 47907, USA

E-mail address: walther@math.purdue.edu