$\boxed{\text{Solutions to Midterm I}}$:

1 a) $y' + \frac{2}{t} y = \frac{1+\cos t}{t^2}, \quad y(\pi/2) = 1$

$\mu(t) = e^{\int 2/t} = t^2 \rightarrow (t^2 y)' = 1 + \cos t$

$\Rightarrow \quad t^2 y - \left(\frac{\pi}{2}\right)^2 \cdot 1 = \int_{\pi/2}^{t} (1+\cos t) = (t - \pi/2) + \left(\sin t - \sin \frac{\pi}{2}\right)$

$\Rightarrow \quad \boxed{t^2 y = \frac{\pi^2}{4} + t - \pi/2 + \sin t - 1}$

b) $y' + \frac{4t}{1+t^2} y = (1+t^2)^{-3}$

$\Rightarrow \mu(t) = e^{\int \frac{4t}{1+t^2} dt} = e^{2\ln(1+t^2)} = (1+t^2)^2$

$\Rightarrow \left((1+t^2)^2 y\right)' = (1+t^2)^{-1} \Rightarrow (1+t^2)^2 y - (1+1^2)^2 \cdot 2$

$= \int_1^t \frac{1}{1+t^2} dt$

$= \tan^{-1} t \Big|_1^t$

$= \tan^{-1} t - \frac{\pi}{4}$

$\Rightarrow \quad \boxed{(1+t^2)^2 y = 8 - \frac{\pi}{4} + \tan^{-1} t}$

2) a) $\frac{dy}{dx} = \frac{1+3x^2}{3y^2 - 6y} \quad y(0) = 1$

$\Rightarrow \int (3y^2 - 6y)\,dy = \int (1+3x^2)\,dx$

$\Rightarrow \quad y^3 - 3y^2 = x + x^3 + c$ set $x=0$, $y=1$, we

have $1^3 - 3\cdot 1^2 = 0 + 0^3 + c \Rightarrow c = -2$. $\boxed{y^3 - 3y^2 = x + x^3 - 2}$

b) $\begin{cases} \dfrac{dy}{dx} = \dfrac{x^2+3xy+y^2}{x^2} = 1+3\dfrac{y}{x}+\left(\dfrac{y}{x}\right)^2 \\ y(1)=1 \end{cases}$ — homogeneous

$$v = y/x \Rightarrow y = xv \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = 1+3v+v^2$$

$$x\frac{dv}{dx} = 1+2v+v^2 = (1+v)^2$$

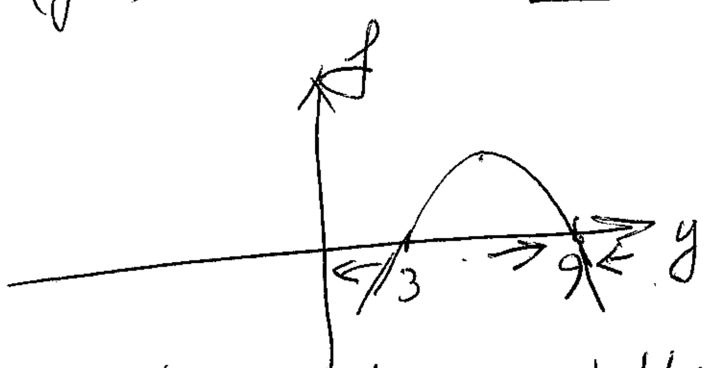
$$\int \frac{dv}{(1+v)^2} = \int \frac{1}{x}dx \qquad \frac{(1+v)^{-1}}{-1} = \ln x + C$$

$$\frac{(1+1)^{-1}}{-1} = \ln 1 + c \Rightarrow c = -\frac{1}{2}$$

$$\Rightarrow -\frac{1}{1+v} = \ln x - \frac{1}{2}$$

$$\Rightarrow \boxed{\frac{1}{1+y/x} = \frac{1}{2} - \ln x}$$

3). $f(y) = (y-3)(9-y) = 0 \Rightarrow \boxed{y_1 = 3 \text{ and } y_2 = 9}$



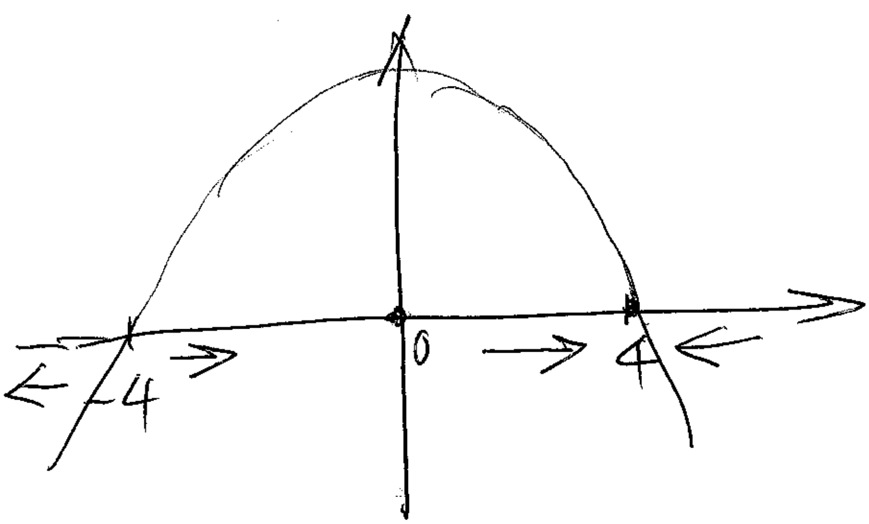
$y_1 = 3$ unstable, $y_2 = 9$ stable

$$\int \frac{dy}{(y-3)(9-y)} = \int dt \Rightarrow \frac{1}{6}\left(\ln|y-3| - \ln|y-9|\right) = t + c$$

$$\Rightarrow \ln\left|\frac{y-3}{y-9}\right| = 6t + c$$

$$\Rightarrow \boxed{\frac{y-3}{y-9} = ce^{6t}}$$

4). $f(y) = y^2(16-y^2) \Rightarrow y_1 \equiv 0,\ y_2 \equiv 4,\ y_3 \equiv -4$



$$\begin{cases} y_1 \equiv 0 & \text{- semistable} \\ y_2 \equiv 4 & \text{- stable} \\ y_3 \equiv -4 & \text{- unstable} \end{cases}$$

5). $M_y = e^x \cos y - 2\sin x$

$N_x = e^x \cos y - 2\sin x$

$\Rightarrow \psi(x,y) = \int (e^x \sin y - 2y\sin x + x^2)\,dx + G(y)$

$= e^x \sin y + 2y\cos x + x^3/3 + G(y)$

$\psi_y = e^x \cos y + 2\cos x + G'(y)$

$\parallel$

$e^x\cos y + 2\cos x + y^2$

$\Rightarrow G'(y) = y^2 \Rightarrow G(y) = y^3/3$

$\Rightarrow \boxed{e^x \sin y + 2\cos x\, y + \frac{x^3}{3} + \frac{y^3}{3} = C}$

6). $M_y = e^{2xy} + 2xy\,e^{2xy}$ $\parallel$ $\Rightarrow \boxed{b=1}$

$N_x = b e^{2xy} + b x(2y) e^{2xy}$

7. $Q(0) = 100\,g$

$$\begin{cases} \dfrac{dQ}{dt} = 0.2 \times 5 - \dfrac{Q}{200 * t} \\ Q(0) = 100 \end{cases}$$

$\Rightarrow$ Left to the readers -:)

8. $t_0 = 0 \qquad t_1 = 0.1 \qquad t_2 = 0.2 \qquad t_3 = 0.3$

$y_0 = 1 \qquad y_1 = 1.1 \qquad y_2 = a$

$\underline{h = 0.1}$

$$y_1 = y_0 + (t_0^2 + y_0^2) \cdot h = 1 + (0^2 + 1^2) \cdot 0.1 = 1.1$$

$$y_2 = y_1 + (t_1^2 + y_1^2) \cdot h = \underbrace{1.1 + (0.1^2 + 1.1^2) \cdot 0.1}_{\overset{?}{=}} = a$$

$$y_3 = y_2 + (t_2^2 + y_2^2) \cdot h = \underbrace{a + (0.2^2 + a^2) \cdot 0.1}_{} = \cdots$$

9. 1) $r^2+5r+3=0 \Rightarrow r=\frac{-5\pm\sqrt{13}}{2}$

$y(t)=c_1 e^{\frac{-5+\sqrt{13}}{2}t}+c_2 e^{\frac{-5-\sqrt{13}}{2}t}$

2) $r^2-2r+5=0 \Rightarrow r=1\pm 2i$

$y_c(t)=e^t(c_1\cos 2t+c_2\sin 2t)$

10. $y''-\frac{2}{t^2}y'+\frac{3+t}{t^2}y$

$\frac{dW}{dt}\cancel{+}\frac{2}{t^2}W=0 \Longrightarrow W(t)=ce^{\int\frac{2}{t^2}dt}=ce^{-2t^{-1}}$

$W(2)=3 \quad W(4)=?$

$\frac{W(4)}{W(2)}=\frac{e^{-2\cdot 4^{-1}}}{e^{-2\cdot 2^{-1}}}=\frac{e^{-\frac{1}{2}}}{e^{-1}}=e^{\frac{1}{2}}$.

$\Rightarrow \boxed{W(4)=3e^{\frac{1}{2}}}$