MA 523/Fall 2015
Homework 3
(Due Thursday, October 8 in class or before 3pm in MATH 714)

1. Let $u$ be the solution of
$$\begin{cases}
\Delta u = 0 \text{ in } \mathbb{R}^n_+ \\
u = g \text{ on } \partial \mathbb{R}^n_+
\end{cases}$$
given by Poisson’s formula for $\mathbb{R}^n_+$. Assume $g$ is bounded and $g(x) = |x|$ for $x \in \partial \mathbb{R}^n_+$, $|x| \leq 1$. Show $Du$ is not bounded near $x = 0$.

2. Let $U^+$ denote the open half-ball $B(0,1) \cap \mathbb{R}^n_+$. Assume $u \in C(\overline{U^+})$ is harmonic in $U^+$, with $u = 0$ on $\partial U^+ \cap \{x_n = 0\}$. Set
$$v(x) = \begin{cases}
u(x) & \text{if } x_n \geq 0 \\
-u(x_1, \ldots, x_{n-1}, -x_n) & \text{if } x_n < 0
\end{cases}$$
for $x \in U = B(0,1)$. Prove $v$ is harmonic in $B(0,1)$.

3. Show that for $n = 2$ the function
$$v = \frac{1}{8\pi} r^2 \log r, \; r = |x - \xi|$$
is a fundamental solution for the operator $\Delta^2$.

4. Find all solutions with spherical symmetry of the biharmonic equation $\Delta^2 u = 0$ in $n$ dimensions.

5. Let $L = \Delta + c$ in $n = 3$ dimensions, where $c$ is a positive constant.
   (a) Find all solutions of $Lu = 0$ with spherical symmetry.
   (b) Prove that
$$K(x, \xi) = -\frac{\cos(\sqrt{c}r)}{4\pi r}, \; r = |x - \xi|$$
is a fundamental solution for $L$ with pole $\xi$.
   (c) Show that a solution $u$ of $Lu = 0$ in $B(\xi, \rho)$ for $\sin(\sqrt{c}\rho) \neq 0$ has the modified mean value property
$$u(\xi) = \frac{\sqrt{c}\rho}{\sin(\sqrt{c}\rho)} \frac{1}{4\pi \rho^2} \int_{\partial B(\xi, \rho)} u(x) \, d\sigma(x).$$