1. Suppose $u$ is smooth and solves $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0, \infty)$.
   (i) Show $u_\lambda(x, t) = u(\lambda x, \lambda^2 t)$ also solves the heat equation for each $\lambda \in \mathbb{R}$.
   (ii) Use (i) to show $v(x, t) = x \cdot Du(x, t) + 2tu_t(x, t)$ solves the heat equation as well.

2. Assume $n = 1$ and $u(x, t) = v\left(\frac{x^2}{t}\right)$.
   (a) Show $u_t = u_{xx}$ if and only if
      \[4zv''(z) + (2 + z)v'(z) = 0 \quad (z > 0).\]
   (b) Show that the general solution of (1) is
      \[v(z) = c \int_0^z e^{-\frac{s^2}{4}} ds + d.\]
   (c) Differentiate $v\left(\frac{x^2}{t}\right)$ with respect to $x$ and select the constant $c$ properly, so as to obtain the fundamental solution $\Phi$ for $n = 1$.

3. Write down an explicit formula for a solution of
   \[
   \begin{cases}
   u_t - \Delta u + cu = f & \text{in } \mathbb{R}^n \times (0, \infty) \\
   u = g & \text{on } \mathbb{R}^n \times \{t = 0\},
   \end{cases}
   \]
   where $c \in \mathbb{R}$.

4. Given $g : [0, \infty) \to \mathbb{R}$, with $g(0) = 0$, derive the formula
   \[u(x, t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} g(s) ds\]
   for a solution of the initial/boundary-value problem
   \[
   \begin{cases}
   u_t - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0, \infty) \\
   u = 0 & \text{on } \mathbb{R}_+ \times \{t = 0\} \\
   u = g & \text{on } \{x = 0\} \times [0, \infty).
   \end{cases}
   \]
   (Hint: Let $v(x, t) = u(x, t) - g(t)$ and extend $v$ to $\{x < 0\}$ by odd reflection.)

5. We say $v \in C^2_1(\mathbb{R}^n \times (0, T))$ is a subsolution of the heat equation if
   \[v_t - \Delta v \leq 0 \quad \text{in } \mathbb{R}^n \times (0, T).\]
   Show that
   (i) if $u$ is a solution of the heat equation and $\phi : \mathbb{R} \to \mathbb{R}$ is smooth and convex, then $v = \phi(u)$ is a subsolution to the heat equation.
   (ii) if $u$ solves the heat equation, then $v := |Du|^2 + u_t^2$ is a subsolution of the heat equation.
6. Let \( u_1(x,t), \ldots, u_n(s,t) \) be \( n \) solutions of \( u_t = u_{ss} \). Prove that
\[
\begin{align*}
  u(x,t) &= u(x_1, \ldots, x_n, t) = \Pi_{k=1}^{n} u_k(x_k, t)
\end{align*}
\]
solves \( u_t = \Delta u \) in \( \mathbb{R}^n \times (0, +\infty) \).

7. Let \( n = 1 \) and \( \mu \) be a positive constant. Let \( u(x,t) \) be a positive solution of class \( C^2 \) of
\[
\begin{align*}
  u_t &= \mu u_{xx}, \quad \text{in} \; \mathbb{R} \times (0, \infty).
\end{align*}
\]
Show that \( \theta = -\frac{2\mu u_x}{u} \) solves the Burger equation:
\[
\begin{align*}
  \theta_t + \theta \theta_x &= \mu \theta_{xx}.
\end{align*}
\]

8. Define for \( x, y, t \in \mathbb{R}, t \neq 0 \)
\[
K(x, y, t) = (4\pi |t|)^{-\frac{1}{2}} \exp\left(-\frac{(x - y)^2}{4t}\right).
\]
Show that
\[
K(x, 0, s + t) = \int K(x, y, t)K(y, 0, s) dy
\]
holds
(a) when \( s > 0, t > 0 \)
(b) when \( 0 < t < -s \).