Harmonic maps from manifolds of L^∞-Riemannian metrics

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Abstract. *For a bounded domain $\Omega \subset \mathbf{R}^n$ endowed with L^∞-metric g, and a C^5-Riemannian manifold $(N,h) \subset \mathbf{R}^k$ without boundary, let $u \in W^{1,2}(\Omega, N)$ be a weakly harmonic map, we prove that (1) $u \in C^\alpha(\Omega, N)$ for $n = 2$, and (2) for $n \geq 3$, if, in additions, $g \in VMO(\Omega)$ and u satisfies the quasi-monotonicity inequality (1.5), then there exists a closed set $\Sigma \subset \Omega$, with $H^{n-2}(\Sigma) = 0$, such that $u \in C^\alpha(\Omega \setminus \Sigma, N)$ for some $\alpha \in (0,1)$.*

§1. Introduction

For $n \geq 2$, let Ω be a bounded domain in $\mathbf{R}^n$. Throughout this paper, let g be a bounded (or L^∞), measurable Riemannian metric on $\mathbf{R}^n$, namely, there exists $\Lambda > 0$ such that $g = \sum_{\alpha,\beta=1}^n g_{\alpha\beta}\, dx_\alpha\, dx_\beta$ satisfies:

$$\Lambda^{-1} I_n \leq (g_{\alpha\beta})(x) \leq \Lambda I_n, \ \forall x \in \mathbf{R}^n. \tag{1.1}$$

Let $(N,h) \subset \mathbf{R}^k$ be a compact, at least C^5-Riemannian manifold without boundary, isometrically embedded into an Euclidean space $\mathbf{R}^k$. For $1 < p < \infty$, define the Sobolev space $W^{1,p}(\Omega, N)$ by

$$W^{1,p}(\Omega, N) := \{u : \Omega \to \mathbf{R}^k \mid \ E_p(u) < +\infty, \ u(x) \in N \text{ for a. e. } x \in \Omega\}$$

where

$$E_p(u) = \int_\Omega (\sum_{i=1}^k |\nabla u^i|_g^2)^{\frac{p}{2}}\, dv_g$$

is the p-th Dirichlet energy of u w.r.t. g,

$$|\nabla u^i|_g^2 = \sum_{\alpha,\beta=1}^n g^{\alpha\beta} \frac{\partial u^i}{\partial x_\alpha} \frac{\partial u^i}{\partial x_\beta}, \ 1 \leq i \leq k,$$

where $(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$, and $dv_g = \sqrt{g}\, dx = \sqrt{\det(g_{\alpha\beta})}\, dx$ is the volume element of (Ω, g).

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Let $d_g(x,y)$ and $d_0(x,y) \equiv |x-y|$ be the distance functions w.r.t. g and g_0 (the Euclidean metric) respectively. Since g is L^∞-Riemannian metric on $\mathbf{R}^n$, it is easy to see that there exists $0 < C_\Lambda < +\infty$ such that

$$C_\Lambda^{-1} d_0(x,y) \le d_g(x,y) \le C_\Lambda d_0(x,y),\ \forall x,\ y \in \mathbf{R}^n. \tag{1.2}$$

In particular, $f \in C^\alpha(\Omega, N)$ w.r.t. g iff $f \in C^\alpha(\Omega, N)$ w.r.t. g_0, and for any open set $U \subset \mathbf{R}^m$ and $1 \le p < +\infty$,

$$C_\Lambda^{-1} \int_U |h|_g^p \, dv_g \le \int_U |h|^p \, dx \le C_\Lambda \int_U |h|_g^p \, dv_g \tag{1.3}$$

holds for any vector field $h \in L^p(U, \mathbf{R}^n)$, here $|h| = (\sum_{i=1}^n h_i^2)^{\frac{1}{2}}$ and dx is the volume element of g_0.

Definition 1. A map $u \in W^{1,2}(\Omega, N)$ is a weakly harmonic map, if it is a critical point of $E_2(\cdot)$.

It is readily seen that any weakly harmonic map $u \in W^{1,2}(\Omega, N)$ satisfies the harmonic map equation:

$$\Delta_g u + A_g(u)(\nabla u, \nabla u) = 0, \text{ in } \mathcal{D}'(\Omega) \tag{1.4}$$

where $\Delta_g = \frac{1}{\sqrt{g}} \sum_{\alpha,\beta=1}^n \frac{\partial}{\partial x_\alpha}(\sqrt{g} g^{\alpha\beta} \frac{\partial}{\partial x_\beta})$ is the Laplace-Beltrami operator of (Ω, g), and $A(y)(\cdot,\cdot) : T_yN \times T_yN \to (T_yN)^\perp,\ y \in N$ is the second fundamental form of $N \subset \mathbf{R}^k$, and

$$A_g(u)(\nabla u, \nabla u) = \sum_{\alpha,\beta=1}^n g^{\alpha\beta} A(u)(\frac{\partial u}{\partial x_\alpha}, \frac{\partial u}{\partial x_\beta}).$$

Regularity of harmonic maps from manifolds with C^∞-Riemannian metrics g has been extensively studied by many people. Schoen-Uhlenbeck [SU], Giaquinta-Guisti [GG] independently proved that any minimizing harmonic map is smooth off a closed set whose Hausdorff dimension is at most $(n-3)$. Hélein [H1,2] proved that any weakly harmonic map from a Riemannian surface is smooth. Evans [E] and Bethuel [B] proved that any stationary harmonic map in dimensions at least three is smooth off a closed set of zero $(n-2)$-dimensional Hausdorff measure.

In this paper, we are mainly interested in seeking *the minimal regularity assumption on Riemannian metrics g such that any weakly harmonic map $u \in W^{1,2}(\Omega, N)$ enjoys (partial) Hölder continuity.*

In this context, our first theorem is

Theorem A. *For $n = 2$ and a L^∞-Riemannian metric g on $\mathbf{R}^n$, let $u \in W^{1,2}(\Omega, N)$ be a weakly harmonic map. Then $u \in C^\alpha(\Omega, N)$ for some $\alpha \in (0, 1)$.*

Remark 1. For $n \geq 2$, if, in addition, $g \in C^{m,\beta}(\Omega)$ for some $m \geq 0$ and $\beta \in (0, 1)$ and $N \in C^{m+5}$, then theorem A and the theory of higher regularity of harmonic maps (cf. Giaquinta [G]) imply that if $u \in C^\alpha(\Omega, N)$, then $u \in C^{m+1,\delta}(\Omega, N)$ for $\delta = \min\{\alpha, \beta\}$.

For $n \geq 3$, Riviére [R] constructed an example of weakly harmonic map from B^3 to S^2 that is singular everywhere. It turns out that the stationarity or suitable energy monotonicity inequality plays a crucial role for the partial regularity of weakly harmonic maps. To this end, we introduce

Definition 2 (quasi-monotonicity inequality). A map $u \in W^{1,2}(\Omega, N)$ enjoys the quasi-monotonicity inequality property, if there exist $K = K(n, g) > 0$ and $r_0 = r_0(n, g) > 0$ such that for any $x \in \Omega$ and $0 < r \leq R < \min\{r_0, \mathrm{dist}(x, \partial\Omega)\}$, we have

$$r^{2-n} \int_{B_r(x)} |\nabla u|^2 \, dx \leq K R^{2-n} \int_{B_R(x)} |\nabla u|^2 \, dx. \tag{1.5}$$

Remark 2. (a) For $n = 2$, (1.5) holds automatically for $u \in W^{1,2}(\Omega, N)$ with $K = 1$.

(b) For $n \geq 3$ and $g \in C^2(\Omega)$, it is well-known that both minimizing harmonic maps and stationary (or C^2)-harmonic maps enjoy the quasi-monotonicity inequality property (cf. [SU], Preiss [P], and Schoen [S]).

(c) In proposition 5.1 and 5.2 below, we verify that for $n \geq 3$, both minimizing harmonic maps w.r.t. Dini continuous g and stationary harmonic maps w.r.t. Lipschitz continuous g enjoy the quasi-monotonicity inequality property.

It is also well-known that certain regularity of the coefficients is necessary for the regularity of second order elliptic systems (cf. [G]). To this end, we recall

Definition 3. (a) For any open set $U \subset \mathbf{R}^n$, a function $f \in \mathrm{BMO}(U)$, if $f \in L^1_{\mathrm{loc}}(U)$ and

$$[f]_{\mathrm{BMO}(U)} := \sup\{\frac{1}{|B_r(x)|} \int_{B_r(x)} |f - f_{x,r}| \mid B_r(x) \subset U\} < \infty$$

where $f_{x,r} = \frac{1}{|B_r(x)|} \int_{B_r(x)} f$.

(b) For any open set $U \subset \mathbf{R}^n$, a function $f \in \mathrm{VMO}(U)$, if $f \in \mathrm{BMO}(U)$ and

$$\lim_{r \to 0} \sup_{x \in U} [f]_{\mathrm{BMO}(U \cap B_r(x))} = 0.$$

Now we are ready to state our second theorem.

Theorem B. *For $n \geq 3$ and $g \in VMO(\Omega)$, suppose that $u \in W^{1,2}(\Omega, N)$ is a weakly harmonic map satisfying the quasi-monotonicity inequality (1.5). Then there exist a closed set $\Sigma \subset \Omega$, with $H^{n-2}(\Sigma) = 0$, and $\alpha \in (0,1)$ such that $u \in C^{\alpha}(\Omega \setminus \Sigma, N)$. Here H^{n-2} denotes the $(n-2)$-dimensional Hausdorff measure w.r.t. g_0.*

We would like to mention that Shi [Sy] proved the partial regularity theorem, similar to theorem B, for minimizing harmonic maps from manifolds with L^∞-Riemannian metrics. However, the argument in [Sy] relies heavily on the minimality property. Our method is of PDE nature and partly motivated by the techniques developed by [H1,2], [B], [E].

The paper is written as follows. In §2, for any C^5-Riemannian manifold N, we outline the Coulomb gauge frame construction by Hélein [H] on $u^*TN|_\Omega$ with respect to g. In §3, we utilize the $W_0^{1,p}$-solvablity theorem on $\nabla \cdot (A\nabla u) = \nabla \cdot F$ by Meyers [M] ($n = 2$) and Di Fazio [D] ($n \geq 3$) for bounded measurable elliptic matrix A to obtain the Div-Curl decomposition theorem on (Ω, g). In §4, we establish the decay Lemma on the $M^{p,n-p}$ norm of u, $\|u\|_{M^{p,n-p}(\cdot)}$, under the smallness condition of $\|\nabla u\|_{M^{2,n-2}(\cdot)}$. In §5, we provide two examples in which the quasi-monotonicity inequality (1.5) holds. In §6, we make some final remarks.

§2. **Construction of Coulomb gauge frame**

In this section, we sketch the Coulomb gauge frame construction on u^*TN by Hélein [H1,2] to (Ω, g) for any C^5-Riemannian manifold N and L^∞-Riemannian metric g on $\mathbf{R}^n$.

Let $l = \dim(N)$. For any ball $B \subset \Omega$, $\{e_i\}_{i=1}^l \subset W^{1,2}(B, \mathbf{R}^k)$ is called to be a frame of u^*TN on B, if $\{e_i(x)\}_{i=1}^l$ forms an orthonormal base of $T_{u(x)}N$ for a.e. $x \in B$.

For a vector field $V = (V_1, \cdots, V_n) : \Omega \to \mathbf{R}^n$, define the divergence of V w.r.t. g by

$$\mathrm{div}_g(V) = \sum_{\alpha,\beta=1}^{n} \frac{\partial}{\partial x_\alpha}(\sqrt{g} g^{\alpha\beta} V_\beta).$$

First we have

Lemma 2.1. *Assume that there exist a C^5-Riemannian manifold $\hat{N} \subset \mathbf{R}^k$ and a totally geodesic, isometric embedding $i : N \to \hat{N}$. If $u \in W^{1,2}(\Omega, N)$ solves (1.4), then $\hat{u} = i \circ u \in W^{1,2}(\Omega, \hat{N})$ also solves (1.4).*

Proof. Straightforward calculations (cf. Jost [J]) imply that

$$\Delta_g \hat{u} = \nabla i(u)(\Delta_g u) + \sum_{\alpha,\beta=1}^{n} g^{\alpha\beta} (\nabla^2 i)(u)(\frac{\partial u}{\partial x_\alpha}, \frac{\partial u}{\partial x_\beta})$$

$$= \nabla i(u)(A_g(u)(\nabla u, \nabla u))$$
$$= \hat{A}_g(\hat{u})(\nabla \hat{u}, \nabla \hat{u})$$

where $\hat{A}$ denotes the second fundamental form of $\hat{N}$ in $\mathbf{R}^k$. ∎

With help of Lemma 2.1 and the enlargement construction by Hélein [H1,2], we may assume that N is parallelizable so that we have

Proposition 2.2. *Assume that $N \in C^5$ is parallelizable and g is L^∞-Riemannian metric on $\mathbf{R}^n$. Let $\Omega \subset \mathbf{R}^n$ be a bounded domian and $B \subset \Omega$ be a ball. If $u \in W^{1,2}(B, N)$, then there exists a Coulomb gauge frame $\{e_i\}_{i=1}^l \subset W^{1,2}(B, \mathbf{R}^k)$ of u^*TN on B, i.e.*

$$div_g(\langle \nabla e_i, e_j \rangle) = 0 \quad in\ B,\ 1 \le i, j \le l \tag{2.1}$$

$$\sum_{\alpha,\beta=1}^{n} g^{\alpha\beta} \langle \frac{\partial e_i}{\partial x_\beta}, e_j \rangle x_\beta = 0 \quad on\ \partial B,\ 1 \le i, j \le l, \tag{2.2}$$

and

$$\sum_{i=1}^{l} \int_B |\nabla e_i|^2 \, dx \le C \int_B |\nabla u|^2 \, dx. \tag{2.3}$$

Proof. As N is parallelizable, there exists a smooth orthonormal frame $\{\hat{e}_i(y)\}_{i=1}^l$ of TN. For $1 \le i \le l$, define $\bar{e}_i(x) = \hat{e}_i(u(x))$ for a.e. $x \in B$. Then $\{\bar{e}_i\}_{i=1}^l$ forms a frame of u^*TN on B. Denote $\mathrm{SO}(l)$ as the special orthonormal group of order l, consider the minimization problem:

$$\inf\{\sum_{i,j=1}^{l} \int_B |\nabla(R_{ij}\bar{e}_j)|_g^2 \, dv_g : R = (R_{ij}) \in W^{1,2}(B, \mathrm{SO}(l))\}. \tag{2.4}$$

By the direct method, there is $R^0 \in W^{1,2}(B, SO(l))$ such that $e_\alpha(x) = \sum_{\beta=1}^{l} R^0_{\alpha\beta}(x)\bar{e}_\beta(x)$, $1 \le \alpha \le l$, satisfies

$$\sum_{\alpha=1}^{l} \int_B |\nabla e_\alpha|_g^2 \, dv_g \le \sum_{\alpha,\beta=1}^{l} \int_B |\nabla(R_{\alpha\beta}\bar{e}_\beta)|_g^2 \, dv_g, \quad \forall R \in W^{1,2}(B, \mathrm{SO}(l)). \tag{2.5}$$

In particular, we have

$$\sum_{\alpha=1}^{l} \int_B |\nabla e_\alpha|_g^2 \, dv_g \le \sum_{\alpha,\beta=1}^{l} \int_B |\nabla(\delta_{\alpha\beta}\bar{e}_\beta)|_g^2 \, dv_g \le C \int_B |\nabla u|_g^2 \, dv_g. \tag{2.6}$$

This, combined with (1.3), implies (2.3). Moreover, the first variation similar to [H1,2] implies that $\langle \nabla e_i, e_j \rangle$, $1 \le i, j \le l$, satisfies the Euler-Lagrange equation (2.1) and the Neumann condition (2.2). Hence the proof is complete. ■

§3. **Div-curl decomposition**

In this section, we prove that if the metric g is either L^∞ for $n = 2$ or in $VMO(\Omega)$ for $n \ge 3$, then the div-curl decomposition holds, namely, *any $F \in L^p(\Omega, \mathbf{R}^n)$ can be decomposed into the sum of ∇G, with $G \in W_0^{1,p}(\Omega)$, and a div_g-free $H \in L^p(\Omega, \mathbf{R}^n)$, for p sufficiently close to $\frac{n}{n-1}$*. The key ingredients are $W_0^{1,p}$-solvability results by Meyers [M] for $n = 2$, and Di Fazio [D] for $n \ge 3$.

More precisely, we have

Theorem 3.1. *Let g be L^∞-Riemannian metric on $\mathbf{R}^n$ and $B \subset \Omega \subset \mathbf{R}^n$ be a ball. If, in addition, $g \in VMO(\Omega)$ for $n \ge 3$, then there exists $\delta_0 = \delta(n, g) > 0$ such that for $p \in (\frac{n}{n-1} - \delta_0, \frac{n}{n-1} + \delta_0)$ and any $F \in L^p(B, \mathbf{R}^n)$ there exist $G \in W_0^{1,p}(B)$ and $H \in L^p(B, \mathbf{R}^n)$, with $div_g(H) = 0$ in Ω, such that*

$$F = \nabla G + H \quad in \ \ B, \tag{3.1}$$

and

$$\|\nabla G\|_{L^p(B)} + \|H\|_{L^p(B)} \le C(p, g) \|F\|_{L^p(B)} \tag{3.2}$$

where $L^p(B)$ is L^p-space w.r.t. g_0.

The proof of Theorem 3.1 relies on the following $W_0^{1,p}$-solvability result.

Proposition 3.2 [M]. *For $n \ge 2$ and any ball $B \subset \Omega$, assume that $A = (a_{ij}) \in L^\infty(B, \mathbf{R}^{n\times n})$ is symmetric and uniformly elliptic, then there exists $\delta_0 = \delta_0(n) > 0$ such that, for any $p \in (2 - \delta_0, 2 + \delta_0)$ and $F \in L^p(B, \mathbf{R}^n)$, there exists a unique solution $u \in W_0^{1,p}(B)$ to the Dirichlet problem:*

$$\begin{aligned} \sum_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u}{\partial x_j}) &= \sum_{i=1}^n \frac{\partial F_i}{\partial x_i}, && in\ B, \\ u &= 0, && on\ \partial B. \end{aligned} \tag{3.3}$$

Moreover,

$$\|\nabla u\|_{L^p(B)} \le C(p, A) \|F\|_{L^p(B)}. \tag{3.4}$$

Proposition 3.3 [D]. *For $n \geq 3$ and ball $B \subset \Omega$, assume that $A = (a_{ij}) \in L^\infty \cap$ VMO$(B, \mathbf{R}^{n\times n})$ is symmetric and uniformly elliptic, then for any $p \in (1, +\infty)$ and $F \in L^p(B, \mathbf{R}^n)$, there exists a unique solution $u \in W_0^{1,p}(B)$ to (3.3) satisfying (3.4).*

Proof of Theorem 3.1. Consider the Dirichlet problem:

$$\operatorname{div}_g(\nabla G) = \operatorname{div}_g(F), \quad \text{in } B \tag{3.5}$$
$$G = 0, \qquad \text{on } \partial B.$$

Observe that (3.5) is equivalent to

$$\sum_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial G}{\partial x_j}) = \sum_{i=1}^n \frac{\partial \hat{F}_i}{\partial x_i}, \text{ in } B \tag{3.6}$$
$$G = 0, \qquad \text{on } \partial B$$

where $a_{ij} = \sqrt{g}g^{ij}$ and $\hat{F}_i = \sum_{j=1}^n \sqrt{g}g^{ij}F_j$. Since g satisfies (1.1), it is easy to see that $(a_{ij}) \in L^\infty(B, \mathbf{R}^{n\times n})$ is symmetric and uniformly elliptic. Moreover, we have $\|\hat{F}\|_{L^p(B)} \leq \|F\|_{L^p(B)}$. For $n = 2$, Proposition 3.2 implies that there exists $\delta_0 > 0$ such that (3.5) is uniquely solvable in $W_0^{1,p}(B)$ for any $p \in (2 - \delta_0, 2 + \delta_0)$. For $n \geq 3$, since $g \in$ VMO(B) implies $(a_{ij}) \in$ VMO(B), Proposition 3.3 implies (3.5) is uniquely solvable in $W_0^{1,p}(B)$ for any $1 < p < \infty$. Set $H = F - \nabla G$, (3.5) implies $\operatorname{div}_g(H) = 0$ in B. Moreover, for any $p \in (\frac{n}{n-1} - \delta_0, \frac{n}{n-1} + \delta_0)$, (3.4) yields

$$\|H\|_{L^p(B)} \leq \|F\|_{L^p(B)} + \|\nabla G\|_{L^p(B)} \leq C\|F\|_{L^p(B)}. \tag{3.7}$$

The completes the proof of Theorem 3.1. ■

§4. **Decay Estimate in Morrey Spaces**

In this section, we prove both theorem A and B. The crucial step is to establish that under the smallness condition of $\|\nabla u\|_{M^{2,n-2}(B)}$, $\|u\|_{M^{p,n-p}(B_r)}$ decays as r^α for some $\alpha \in (0,1)$. The ideas are suitable modifications of techniques developed by Hélein [H1,2], Evans [E], and Bethuel [B]. In order to achieve it, we need two new ingredients: (1) the div-curl decomposition Proposition 3.1, and (2) a new approach to estimate the L^p norm of div_g-free vector fields.

First we define Morrey spaces.

Definition 4.1. For $1 \leq p \leq n$ and any open set $U \subset \mathbf{R}^n$, the Morrey space $M^{p,n-p}(U)$ is defined by

$$M^{p,n-p}(U) = \{f \in L^p(U) \mid \|f\|^p_{M^{p,n-p}(U)} \equiv \sup_{B_r(x) \subset U} \{r^{p-n} \int_{B_r(x)} |f|^p \, dx\} < +\infty\}.$$

Now we have

Lemma 4.1 (ϵ_0-decay estimate). *For any bounded domain $\Omega \subset \mathbf{R}^n$ and L^∞-Riemannian metric g on $\mathbf{R}^n$. If, in addition, $g \in VMO(\Omega)$ for $n \geq 3$, then there exist $\delta_n > 0$, $\epsilon_0 = \epsilon_0(g, N) > 0$, and $\theta_0 = \theta_0(g, N) \in (0, \frac{1}{2})$ such that if $u \in W^{1,2}(\Omega, N)$ is a weakly harmonic map satisfying the quasi-monotonicity inequality (1.5), and for $B_r(x) \subset \Omega$,*

$$r^{2-n} \int_{B_r(x)} |\nabla u|^2_g \, dv_g \leq \epsilon_0^2 \tag{4.1}$$

then, for any $p \in (\frac{n}{n-1} - \delta_n, \frac{n}{n-1})$,

$$\|\nabla u\|_{M^{p,n-p}(B_{\theta_0 r}(x))} \leq \frac{1}{2} \|\nabla u\|_{M^{p,n-p}(B_r(x))}. \tag{4.2}$$

Proof of Lemma 4.1.

By Lemma 2.1, assume that N is parallelizable. For $x \in \Omega$ and $r > 0$, let $g_{x,r}(y) = g(x + ry)$ and $u_{x,r}(y) = u(x + ry)$ for $y \in B$. Observe that $g_{x,r}$ is L^∞-Riemannian metric on B and $u_{x,r} \in W^{1,2}(B, N)$ is a weakly harmonic map w.r.t. $g_{x,r}$, satisfies the quasi-monotonicity inequality (1.5), and

$$\int_B |\nabla u|^2_{g_{x,r}} \, dv_{g_{x,r}} = r^{2-n} \int_{B_r(x)} |\nabla u|^2_g \, dv_g \leq \epsilon_0^2. \tag{4.3}$$

Hence, without loss of generality, assume $x = 0$ and $r = 1$. It follows from (1.5) that there exists $K > 0$ such that

$$\|\nabla u\|_{M^{2,n-2}(B_{\frac{1}{2}})} \leq K \|\nabla u\|_{L^2(B)} \leq K \epsilon_0^2. \tag{4.4}$$

For any $\theta \in (0, \frac{1}{2})$, let $B_{2\theta} \subset B_{\frac{1}{2}}$ be an arbitrary ball of radius 2θ and $\eta \in C_0^\infty(B)$ be such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ on B_θ, $\eta = 0$ outside $B_{2\theta}$, and $|\nabla \eta| \leq 2\theta^{-1}$. Denote the average of u over $B_{2\theta}$ by $u_{2\theta} = \frac{1}{|B_{2\theta}|} \int_{B_{2\theta}} u \, dv_g$, and $|B_{2\theta}|$ is the volume of $B_{2\theta}$ w.r.t. g.

Let $\{e_\alpha\}_{\alpha=1}^l \in W^{1,2}(B_{2\theta}, \mathbf{R}^k)$ be the Coulomb gauge frame of u^*TN on $B_{2\theta}$ given by Proposition 2.2.

Let

$$\langle p,q\rangle=\sum_{i=1}^n p_iq_i,\ \langle p,g\rangle_g=\sum_{i,j=1}^n g^{ij}p_iq_j,\ p=(p_1,\cdots,p_n),\ q=(q_1,\cdots,q_n)\in\mathbf{R}^n$$

denote the inner products w.r.t. g_0 and g on $\mathbf{R}^n$ respectively.

By Theorem 3.1, there exists $\delta_n>0$ such that for any $p\in(\frac{n}{n-1}-\delta_n,\frac{n}{n-1})$, there are $\phi_\alpha\in W_0^{1,p}(B_{2\theta})$ and $\psi_\alpha\in L^p(B_{2\theta})$ such that

$$\langle\nabla((u-u_{2\theta})\eta),e_\alpha\rangle=\nabla\phi_\alpha+\psi_\alpha,\quad \operatorname{div}_g(\psi_\alpha)=0,\ \text{in } B_{2\theta}, \tag{4.5}$$

and

$$\|\nabla\phi_\alpha\|_{L^p(B_{2\theta})}+\|\psi_\alpha\|_{L^p(B_{2\theta})}\le C\|\nabla((u-u_{2\theta})\eta)\|_{L^p(B_{2\theta})}\le C\|\nabla u\|_{L^p(B_{2\theta})} \tag{4.6}$$

where we have used the Poincaré inequality in the last inequality of (4.6).

Using the Coulomb gauge frame $\{e_\alpha\}_{\alpha=1}^l$, (1.4) can be written as:

$$\operatorname{div}_g(\langle\nabla u,e_\alpha\rangle)=\sum_{\beta=1}^l\sum_{i,j=1}^n g^{ij}\langle\frac{\partial u}{\partial x_i},\langle\frac{\partial e_\alpha}{\partial x_j},e_\beta\rangle\rangle e_\beta\quad\text{in } B_{2\theta}. \tag{4.7}$$

We estimate ϕ_α,ψ_α as follows. Let $\phi_\alpha^{(1)}\in W^{1,2}(B_\theta)$ be the weak solution of

$$\sum_{i,,j=1}^n\frac{\partial}{\partial x_i}(a_{ij}\frac{\partial\phi_\alpha^{(1)}}{\partial x_j})=0,\qquad \text{in } B_\theta \tag{4.8}$$

$$\phi_\alpha^{(1)}=\phi_\alpha,\quad \text{on } \partial B_\theta. \tag{4.9}$$

where $a_{ij}=\sqrt{g}g^{ij}$, $1\le i,j\le n$. Let $\phi_\alpha^{(2)}=\phi_\alpha-\phi_\alpha^{(1)}$, then $\phi_\alpha^{(2)}$ satisfies

$$\sum_{i,j=1}^n\frac{\partial}{\partial x_i}(a_{ij}\frac{\partial\phi_\alpha^{(2)}}{\partial x_j})=\sum_{\beta=1}^l\sum_{i,j=1}^n g^{ij}\langle\frac{\partial u}{\partial x_i},\langle\frac{\partial e_\alpha}{\partial x_j},e_\beta\rangle\rangle e_\beta,\ \text{in } B_\theta, \tag{4.10}$$

$$\phi_\alpha^{(2)}=0,\qquad \text{on } \partial B_\theta. \tag{4.11}$$

Step I(a). Estimation of $\nabla\phi_\alpha^{(1)}$.

It is well-known (cf. [GT]) that there exists $\delta\in(0,1)$ such that $\phi_\alpha^{(1)}\in C^\delta(B_\theta)$, and for any $0<r\le\frac{\theta}{2}$ and $p>1$,

$$[\phi_\alpha^{(1)}]^p_{C^\delta(B_r)}\le C\theta^{p-n}\int_{B_\theta}|\nabla\phi_\alpha^{(1)}|^p\,dx,\ 0<r\le\frac{\theta}{2}.$$

On the other hand, since $\phi_\alpha^{(2)} \in W_0^{1,2}(B_\theta)$ satisfies

$$\sum_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial \phi_\alpha^{(2)}}{\partial x_j}) = \sum_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial \phi_\alpha}{\partial x_j}), \quad \text{in } B_\theta,$$

Theorem 3.1 implies that there exists $\delta_n > 0$ such that, for $p \in (\frac{n}{n-1} - \delta_n, \frac{n}{n-1})$,

$$\|\nabla \phi_\alpha^{(2)}\|_{L^p(B_\theta)} \le C\|\nabla \phi_\alpha\|_{L^p(B_\theta)} \le C\|\nabla u\|_{L^p(B_{2\theta})}.$$

In particular, we have

$$\|\nabla \phi_\alpha^{(1)}\|_{L^p(B_\theta)} \le \|\nabla \phi_\alpha\|_{L^p(B_\theta)} + \|\nabla \phi_\alpha^{(2)}\|_{L^p(B_\theta)} \le C\|\nabla u\|_{L^p(B_{2\theta})},$$

and, for $0 < r \le \frac{\theta}{2}$ and $p \in (\frac{n}{n-1} - \delta_n, \frac{n}{n-1})$,

$$[\phi_\alpha^{(1)}]^p_{C^\delta(B_r)} \le C\theta^{p-n}\int_{B_{2\theta}} |\nabla u|^p \, dx.$$

This, combined with the Cacciopolli inequality, implies that for any $\tau \in (0, \frac{1}{4})$ and $p \in (\frac{n}{n-1} - \delta_n, \frac{n}{n-1})$, we have

$$\begin{aligned}(\tau\theta)^{p-n}\int_{B_{\tau\theta}} |\nabla \phi_\alpha^{(1)}|^p \, dx &\le C[\phi_\alpha^{(1)}]^p_{C^\delta(B_{2\tau\theta})} \\ &\le C\tau^{p\delta}\theta^{p-n}\int_{B_{2\theta}} |\nabla u|^p \, dx \\ &\le C\tau^{p\delta}\|\nabla u\|_{M^{p,n-p}(B_1)}.\end{aligned} \tag{4.12}$$

Step I (b). Estimation of $\nabla \phi_\alpha^{(2)}$.

First, we claim

There exists $\delta_n > 0$ such that for any $p \in (\frac{n}{n-1} - \delta_n, \frac{n}{n-1})$, if $f \in W_0^{1,p}(B_\theta)$ then

$$\|\nabla f\|_{L^p(B_\theta)} \le C \sup\{\int_{B_\theta} \langle \nabla f, \nabla v\rangle_g \, dv_g : v \in W_0^{1,p'}(B_\theta), \|\nabla v\|_{L^{p'}(B_\theta)} = 1\} \tag{4.13}$$

where $p' = \frac{p}{p-1}$.

To see (4.13), observe that by L^p-duality, there exists $v \in L^{p'}(B_\theta)$, with $\|v\|_{L^{p'}(B_\theta)} = 1$, such that

$$\|\nabla f\|_{L^p(B_\theta)} \le C\int_{B_\theta} \langle \nabla f, v\rangle_g \, dv_g. \tag{4.14}$$

On the other hand, by Theorem 3.1, there exists $\delta_n > 0$ such that if $p \in (\frac{n}{n-1} - \delta_n, \frac{n}{n-1})$, then there exist $v_1 \in W_0^{1,p'}(B_\theta)$ and $v_2 \in L^{p'}(B_\theta, \mathbf{R}^n)$, with $\mathrm{div}_g(v_2) = 0$ in B_θ, such that

$$v = \nabla v_1 + v_2 \ \text{ in } B_\theta, \ \ \|\nabla v_1\|_{L^{p'}(B_\theta)} + \|v_2\|_{L^{p'}(B_\theta)} \leq C\|v\|_{L^{p'}(B_\theta)}. \tag{4.15}$$

This and (4.14) imply

$$\begin{aligned}\|\nabla f\|_{L^p(B_\theta)} &\leq C(\int_{B_\theta} \langle \nabla f, \nabla v_1\rangle_g \, dv_g + \int_{B_\theta} \langle \nabla f, v_2\rangle_g \, dv_g) \\ &= C\int_{B_\theta} \langle \nabla f, \nabla v_1\rangle_g \, dv_g,\end{aligned}$$

where we have used $\mathrm{div}_g(v_2) = 0$ in the last step. Hence (4.13) holds.

Applying (4.13) to eqn. (4.7), we have that for $p \in (\frac{n}{n-1} - \delta_n, \frac{n}{n-1})$, there exists $v \in W_0^{1,p'}(B_\theta)$ such that

$$\begin{aligned}\|\nabla \phi_\alpha^{(2)}\|_{L^p(B_\theta)} &\leq C\int_{B_\theta} \langle \nabla \phi_\alpha^{(2)}, \nabla v\rangle_g \, dv_g \\ &= -C\sum_{\beta=1}^{l}\sum_{i,j=1}^{n}\int_{B_\theta} \sqrt{g} g^{ij} \langle \frac{\partial u}{\partial x_i}, \langle \frac{\partial e_\alpha}{\partial x_j}, e_\beta\rangle\rangle (e_\beta v)\, dx.\end{aligned} \tag{4.16}$$

To estimate the right hand side, we need the Hardy-BMO duality theorem (cf. [FS]) and the tri-linear estimate (cf. [CLMS], [E]).

Proposition 4.2 ([E]). *Suppose that $f \in W^{1,2}(\mathbf{R}^n)$, $h \in L^2(\mathbf{R}^n, \mathbf{R}^n)$ with $div(h) = \sum_{i=1}^n \frac{\partial h_i}{\partial x_i} = 0$, and $v \in BMO(\mathbf{R}^n)$. Then we have*

$$|\int_{\mathbf{R}^n} \langle \nabla f, h\rangle v \, dx| \leq C\|\nabla f\|_{L^2(\mathbf{R}^n)}\|h\|_{L^2(\mathbf{R}^n)}\|v\|_{BMO(\mathbf{R}^n)}. \tag{4.17}$$

Let $\hat{u} : \mathbf{R}^n \to \mathbf{R}^k$ be an extension of u such that

$$\|\nabla \hat{u}\|_{L^2(\mathbf{R}^n)} \leq C\|\nabla u\|_{L^2(B_{2\theta})}, \ [\hat{u}]_{\mathrm{BMO}(\mathbf{R}^n)} \leq C[u]_{\mathrm{BMO}(B_{2\theta})}. \tag{4.18}$$

Let $w_\alpha^i = \sum_{\beta=1}^l \sum_{j=1}^n \sqrt{g} g^{ij} \langle \frac{\partial e_\alpha}{\partial x_j}, e_\beta\rangle$, $1 \leq i \leq n$, and $w_\alpha = (w_\alpha^1, \cdots, w_\alpha^n)$. Then, by (2.1), we have

$$\mathrm{div}(w_\alpha) = \sum_{i=1}^n \frac{\partial w_\alpha^i}{\partial x_i} = \sqrt{g}\sum_{\beta=1}^l \mathrm{div}_g(\langle \nabla e_\alpha, e_\beta\rangle) = 0 \ \text{ on } B_{2\theta}.$$

This, combined with (2.2), implies that there exists an extension $\hat{w}_\alpha \in L^2(\mathbf{R}^n, \mathbf{R}^n)$ of w_α such that

$$\text{div}(\hat{w}_\alpha) = 0 \text{ in } \mathbf{R}^n,\ \|\hat{w}_\alpha\|_{L^2(\mathbf{R}^n)} \le C\|w_\alpha\|_{L^2(B_{2\theta})} \le C\|\nabla u\|_{L^2(B_{2\theta})}. \tag{4.19}$$

Putting (4.17)-(4.19) into (4.16), we have

$$\begin{aligned}
\|\nabla\phi_\alpha^{(2)}\|_{L^p(B_\theta)} &\le -C\int_{\mathbf{R}^n} \langle \nabla u, \hat{\omega}_\alpha\rangle (ve_\alpha)\,dx \\
&= C\int_{\mathbf{R}^n} \langle \hat{u}, \hat{w}_\alpha\rangle \nabla(ve_\alpha)\,dx \\
&\le C[\hat{u}]_{\text{BMO}(\mathbf{R}^n)}\|\hat{w}_\alpha\|_{L^2(\mathbf{R}^n)}\|\nabla(ve_\alpha)\|_{L^2(\mathbf{R}^n)} \\
&\le C\|\nabla u\|_{L^2(B_{2\theta})}[u]_{\text{BMO}(B_{2\theta})}\|\nabla(ve_\alpha)\|_{L^2(B_\theta)}. \qquad (4.20)
\end{aligned}$$

To estimate $\|\nabla(ve_\alpha)\|_{L^2(B_\theta)}$, note that for $p \in (1, \frac{n}{n-1})$, $p' = \frac{p}{p-1} > n$ and hence the Sobolev embedding theorem implies $v \in W_0^{1,p'}(B_\theta) \subset C_0^{1-\frac{n}{p'}}(B_\theta)$, and

$$\|v\|_{L^\infty(B_\theta)} \le C\theta^{1-\frac{n}{p'}} = C\theta^{1-n+\frac{n}{p}}. \tag{4.21}$$

Moreover, by Hölder inequality, we have

$$\|\nabla v\|_{L^2(B_\theta)} \le C\theta^{\frac{n}{2}-\frac{n}{p'}}\|\nabla v\|_{L^{p'}(B_\theta)} \le C\theta^{\frac{n}{p}-\frac{n}{2}}. \tag{4.22}$$

Therefore we have

$$\begin{aligned}
\|\nabla(ve_\alpha)\|_{L^2(B_\theta)} &\le C(\|\nabla v\|_{L^2(B_\theta)} + \|v\|_{L^\infty(B_\theta)}\|\nabla e_\alpha\|_{L^2(B_\theta)}) \qquad (4.23)\\
&\le C\theta^{\frac{n}{p}-\frac{n}{2}}[1 + \theta^{1-\frac{n}{2}}\|\nabla u\|_{L^2(B_{2\theta})}] \\
&\le C\theta^{\frac{n}{p}-\frac{n}{2}}(1 + \|\nabla u\|_{M^{2,n-2}(B_1)}) \\
&\le C\theta^{\frac{n}{p}-\frac{n}{2}}(1 + \epsilon_0) \le C\theta^{\frac{n}{p}-\frac{n}{2}}.
\end{aligned}$$

Putting (4.23) into (4.20), and combining with (4.12), we have, for any $\tau \in (0, \frac{1}{4})$,

$$\{(\tau\theta)^{p-n}\int_{B_{\tau\theta}} |\nabla\phi_\alpha|^p\,dx\}^{\frac{1}{p}} \le C[\tau^\delta + \tau^{1-\frac{n}{p}}\epsilon_0]\|\nabla u\|_{M^{p,n-p}(B_1)} \tag{4.24}$$

where we have used the Poincaré inequality:

$$[u]_{\text{BMO}(B_{2\theta})} \le C\|\nabla u\|_{M^{p,n-p}(B_{2\theta})} \le C\|\nabla u\|_{M^{p,n-p}(B_1)}. \tag{4.25}$$

Step II. Estimation of ψ_α.

It follows from (4.5) and Proposition 4.2 that we have

$$
\begin{aligned}
\int_{B_\theta} |\psi_\alpha|_g^2\, dv_g &= \sum_{i,j=1}^n \int_{B_\theta} a_{ij}\psi_\alpha^i \psi_\alpha^j\, dx \\
&= \sum_{i,j=1}^n \int_{B_\theta} a_{ij}\psi_\alpha^i \langle \frac{\partial((u-u_{2\theta})\eta)}{\partial x_j}, e_\alpha\rangle\, dx \\
&\quad - \sum_{i,j=1}^n \int_{B_\theta} a_{ij}\psi_\alpha^i \frac{\partial \phi_\alpha}{\partial x_j}\, dx \\
&= - \sum_{i,j=1}^n \int_{B_\theta} a_{ij}\psi_\alpha^i \langle (u-u_{2\theta})\eta, \frac{\partial e_\alpha}{\partial x_j}\rangle\, dx \\
&\le C\|\psi_\alpha\|_{L^2(B_\theta)}\|\nabla e_\alpha\|_{L^2(B_\theta)}[(u-u_{2\theta})\eta]_{\mathrm{BMO}(B_\theta)} \\
&\le C\|\psi_\alpha\|_{L^2(B_\theta)}\|\nabla u\|_{L^2(B_\theta)}\|\nabla u\|_{M^{p,n-p}(B_1)}
\end{aligned}
\tag{4.26}
$$

where we have used the fact $\mathrm{div}_g(\psi_\alpha)=0$, i.e.

$$\sum_{i,j=1}^n \int_{B_\theta} a_{ij}\psi_\alpha^i \frac{\partial \eta}{\partial x_j}\, dx = 0,\ \forall \eta \in W_0^{1,2}(B_\theta),$$

and

$$[(u-u_{2\theta})\eta]_{\mathrm{BMO}(B_\theta)} \le C[u]_{\mathrm{BMO}(B_{2\theta})} \le C\|\nabla u\|_{M^{p,n-p}(B_1)}. \tag{4.27}$$

By Hölder inequality, (4.26) yields

$$\{\theta^{p-n}\int_{B_\theta} |\psi_\alpha|^p\, dx)^{\frac{1}{p}} \le C\epsilon_0\|\nabla u\|_{M^{p,n-p}(B_1)}. \tag{4.28}$$

It follows from (4.5), (4.24), and (4.28) that for any $\tau \in (0, \frac{1}{4})$, any ball $B_{2\theta} \subset B_{\frac{1}{2}}$,

$$\{(\tau\theta)^{p-n}\int_{B_{\tau\theta}} |\nabla u|^p\, dx\}^{\frac{1}{p}} \le C(\tau^\delta + \tau^{1-\frac{n}{p}}\epsilon_0)\|\nabla u\|_{M^{p,n-p}(B_1)}. \tag{4.29}$$

Taking superum over all balls $B_{2\theta} \subset B_{\frac{1}{2}}$, we have

$$\|\nabla u\|_{M^{p,n-p}(B_{\frac{\tau}{2}})} \le C(\tau^\delta + \tau^{1-\frac{n}{p}}\epsilon_0)\|\nabla u\|_{M^{p,n-p}(B_1)}. \tag{4.30}$$

Therefore, by choosing $\tau = \tau_1 = 4C^{\frac{-1}{\delta}}$ and $\epsilon_0 = \frac{1}{4C}\tau_0^{\frac{n}{p}-1}$ sufficiently small, we have, for $\tau_0 = \frac{\tau_1}{2} > 0$,

$$\|\nabla u\|_{M^{p,n-p}(B_{\tau_0})} \le \frac{1}{2}\|\nabla u\|_{M^{p,n-p}(B_1)}. \tag{4.31}$$

This completes the proof of Lemma 4.1. ■

Proof of Theorem A. For $n = 2$, the absolute continuity of $\int |\nabla u|^2$ implies that there exists $r_0 > 0$ such that

$$\int_{B_r(x)} |\nabla u|^2\, dx \le \epsilon_0^2, \quad \forall r \le r_0, \quad x \in \Omega. \tag{4.31}$$

Hence, applying Lemma 4.1 repeatedly, we have that for some $p \in (1, 2)$ and $\tau_0 \in (0, \frac{1}{2})$,

$$(\tau_0^m r_0)^{p-2} \int_{B_{\tau_0^m r_0}(x)} |\nabla u|^p \le 2^{-pm} \epsilon_0^p, \quad \forall m \ge 1, \quad \forall x \in \Omega. \tag{4.32}$$

This implies that there exists $\alpha_0 \in (0, 1)$ such that

$$r^{p-2} \int_{B_r(x)} |\nabla u|^p \le C(\epsilon_0, p) r^{\alpha}, \quad \forall r \in (0, r_0), \quad x \in \Omega. \tag{4.33}$$

Hence, by Morrey's Lemma (cf. [G]), we conclude $u \in C^{\alpha}(\Omega, N)$. This completes the proof of Theorem A. ■

Proof of Theorem B. Define

$$\Sigma = \{x \in \Omega : \lim_{r \downarrow 0} r^{2-n} \int_{B_r(x)} |\nabla u|^2 \ge \epsilon_0^2\}.$$

It is well-known (cf. [SU]) that $H^{n-2}(\Sigma) = 0$. Moreover, by Lemma 4.1, $\Sigma \subset \Omega$ is a closed set. For any $x_0 \in \Omega \setminus \Sigma$, there exists $r_0 > 0$ such that $B_{2r_0}(x_0) \cap \Sigma = \emptyset$, and

$$r^{2-n} \int_{B_r(x)} |\nabla u|^2 \le \epsilon_0^2, \quad \forall x \in B_{r_0}(x_0), \quad r \le r_0.$$

Therefore, by Lemma 4.1, we have that for some $p \in (1, \frac{n}{n-1})$ and $\tau_0 \in (0, 1)$,

$$(\tau_0^m r_0)^{p-n} \int_{B_{\tau_0^m r_0}(x)} |\nabla u|^p \le 2^{-pm} \epsilon_0^p, \quad \forall m \ge 1, \quad \forall x \in B_{r_0}(x_0). \tag{4.34}$$

This implies that there is $\alpha \in (0, 1)$ such that

$$r^{p-n} \int_{B_r(x)} |\nabla u|^p \le C(\epsilon_0, p) r^{p\alpha}, \quad \forall x \in B_{r_0}(x_0), \quad \forall r \in (0, r_0). \tag{4.35}$$

Hence, by Morrey's Lemma, we conclude $u \in C^{\alpha}(B_{r_0}(x_0), N)$ and $u \in C^{\alpha}(\Omega \setminus \Sigma, N)$. ■

§5. **Quasi-monotonicity inequality**

In this section, we derive the quasi-monotonicity inequality (1.5) for two classes of harmonic maps in dimensions $n \geq 3$: (1) minimizing harmonic maps w.r.t. Dini-continuous metrics g, and (2) stationary harmonic maps w.r.t. Lipschitz continuous metrics g.

Definition 5.1. A map $u \in W^{1,2}(\Omega, N)$ is a minimizing harmonic map, if

$$\int_\Omega |\nabla u|_g^2 \, dv_g \leq \int_\Omega |\nabla v|_g^2 \, dv_g, \ \forall v \in W^{1,2}(\Omega, N) \text{ with } v|_{\partial\Omega} = u|_{\partial\Omega}. \tag{5.1}$$

Recall that $f : \Omega \to \mathbf{R}^{n\times n}$ is Dini-continuous, if there exist $r_0 > 0$ and a monotonically non-decreasing $\omega : [0, r_0] \to \mathbf{R}_+$, with $\omega(0) = 0$ and $\int_0^{r_0} \frac{\omega(t)}{t} \, dt < \infty$, such that

$$|f(x) - f(y)| \leq \omega(|x - y|), \ \forall x, y \in \Omega, \ |x - y| \leq r_0. \tag{5.2}$$

Proposition 5.1. *For $n \geq 3$, suppose that g is a Dini-continuous metric on Ω and $u \in W^{1,2}(\Omega, N)$ is a minimizing harmonic map. Then u satisfies the quasi-monotonicity inequality (1.5).*

Proof. It suffices to prove (1.5) for $x = 0 \in \Omega$. Assume $g_0 = g(0)$ is the Euclidean metric on $\mathbf{R}^n$. For $0 < r < \min\{r_0, \mathrm{dist}(0, \partial\Omega)\}$, define

$$\begin{aligned} v(x) &= u(\frac{rx}{|x|}), \ x \in B_r \\ &= u(x), \quad\ \ x \in \Omega \setminus B_r. \end{aligned}$$

Then the minimality of u implies

$$\int_{B_r} |\nabla u|_g^2 \, dv_g \leq \int_{B_r} |\nabla v|_g^2 \, dv_g. \tag{5.3}$$

It follows from the Dini-continuity of g that

$$\max_{x \in B_r} |g(x) - g_0| \leq \omega(r), \ \forall 0 < r \leq \min\{r_0, \mathrm{dist}(0, \partial\Omega)\},$$

where ω is the modular of continuity of g. This and (5.3) imply that there exists $C_0 > 0$ such that

$$(1 - C_0\omega(r)) \int_{B_r} |\nabla u|^2 \, dx \leq \int_{B_r} |\nabla v|^2 \, dx, \ \forall 0 < r \leq \min\{r_0, \mathrm{dist}(0, \partial\Omega)\}. \tag{5.4}$$

Direct calculations imply

$$\int_{B_r} |\nabla v|^2 \, dx = \frac{r}{n-2} \int_{\partial B_r} (|\nabla u|^2 - |\frac{\partial u}{\partial r}|^2) \, dH^{n-1}.$$

Therefore we have, for $0 < r \le \min\{r_0, \mathrm{dist}(0, \partial\Omega)\}$,

$$(5.5) \qquad (n-2)(1 - C_0\omega(r))r^{1-n} \int_{B_r} |\nabla u|^2 \, dx \le r^{2-n} \int_{\partial B_r} |\nabla u|^2 \, dH^{n-1} - r^{2-n} \int_{\partial B_r} |\frac{\partial u}{\partial r}|^2 \, dH^{n-1}.$$

This yields, for $0 < r \le \min\{r_0, \mathrm{dist}(0, \partial\Omega)\}$,

$$(5.6) \qquad \begin{aligned} &\frac{d}{dr}\{e^{\{(n-2)C_0 \int_0^r \frac{\omega(t)}{t} \, dt\}} r^{2-n} \int_{B_r} |\nabla u|^2 \, dx\} \\ &\ge e^{\{(n-2)C_0 \int_0^r \frac{\omega(t)}{t} \, dt\}} r^{2-n} \int_{\partial B_r} |\frac{\partial u}{\partial r}|^2 \, dH^{n-1} \\ &\ge r^{2-n} \int_{\partial B_r} |\frac{\partial u}{\partial r}|^2 \, dH^{n-1}. \end{aligned}$$

Integrating (5.6), we have, for $0 < r \le R \le \min\{r_0, \mathrm{dist}(0, \partial\Omega)\}$,

$$(5.7) \qquad \begin{aligned} &\int_{B_R \setminus B_r} |x|^{2-n} |\frac{\partial u}{\partial r}|^2 \, dx + r^{2-n} \int_{B_r} |\nabla u|^2 \, dx \\ &\le e^{\{(n-2)C_0 \int_0^R \frac{\omega(t)}{t} \, dt\}} R^{2-n} \int_{B_R} |\nabla u|^2 \, dx. \end{aligned}$$

This implies (1.5) holds for $K = e^{\{(n-2)C_0 \int_0^{r_0} \frac{\omega(t)}{t} \, dt\}}$. ∎

Next we consider stationary harmonic maps.

Definition 5.2. A weakly harmonic map $u \in W^{1,2}(\Omega, N)$ is a stationary harmonic map, if it is a critical point of E_2 w.r.t. the domain variations:

$$(5.8) \qquad \frac{d}{dt}|_{t=0} \int_\Omega |\nabla u(x + tX(x))|_g^2 \, dv_g = 0, \ \forall X \in C_0^1(\Omega, \mathbf{R}^n).$$

We have

Proposition 5.2. *For $n \ge 3$, let g be a Lipschitz continuous Riemannian metric on Ω. Then any stationary map $u \in W^{1,2}(\Omega, N)$ satisfies (1.5) for some $K = K(n, g) > 0$.*

Proof. For simplicity, assume $x = 0 \in \Omega$ and $g(0) = g_0$. Define the energy-stress tensor

$$S_{\alpha\beta} = \frac{1}{2}|\nabla u|_g^2 g_{\alpha\beta} - \langle \frac{\partial u}{\partial x_\alpha}, \frac{\partial u}{\partial x_\beta} \rangle, \quad 1 \le \alpha, \beta \le n.$$

Then it is well-known (cf. [H2]) that the stationarity (5.8) implies

$$\sum_{\alpha,\beta=1}^{n}\int_{\Omega}(L_X g^{\alpha\beta})S_{\alpha\beta}\,dv_g=0 \tag{5.9}$$

where

$$L_X g^{\alpha\beta}=\sum_{\gamma=1}^{n}[X_\gamma\frac{\partial g^{\alpha\beta}}{\partial x_\gamma}-\frac{\partial X_\alpha}{\partial x_\gamma}g^{\gamma\beta}-\frac{\partial X_\beta}{\partial x_\gamma}g^{\gamma\alpha}]$$

is the Lie derivative of $(g^{\alpha\beta})$ with respect to X.

For $B_r\subset\Omega$, and $\eta(x)=\eta(|x|)\in C_0^1(B_r)$ with $0\le\eta\le 1$, let $X(x)=x\eta(|x|)$. Then we have

$$\frac{\partial X_\alpha}{\partial x_\gamma}=\delta_{\alpha\gamma}\eta(|x|)+\eta'(|x|)\frac{x_\alpha x_\gamma}{|x|},\quad 1\le\alpha,\gamma\le n,$$

and

$$L_X g^{\alpha\beta}=\eta(|x|)\sum_{\gamma=1}^{n}x_\gamma\frac{\partial g^{\alpha\beta}}{\partial x_\gamma}-2\eta(|x|)g^{\alpha\beta}-2\eta'(|x|)\sum_{\gamma=1}^{n}\frac{x_\beta x_\gamma}{|x|}g^{\alpha\gamma}.$$

Since g is Lipschitz continuous, there exist $r_0>0$ and $C_0>0$ depending on $\mathrm{Lip}(g)$ such that

$$\|\nabla g^{\alpha\beta}\|_{L^\infty(B_r)}\le C_0\mathrm{Lip}(g),\ \forall 0<r\le r_0. \tag{5.10}$$

Let $I\equiv\sum_{\alpha,\beta,\gamma=1}^{n}\int_{B_r}x_\gamma\eta(|x|)\frac{\partial g^{\alpha\beta}}{\partial x_\gamma}S_{\alpha\beta}\,dv_g$. Then we have

$$\begin{aligned}|I|&\le\sum_{\alpha,\beta,\gamma=1}^{n}\int_{B_r}|x_\gamma||\frac{\partial g^{\alpha\beta}}{\partial x_\gamma}||S_{\alpha\beta}|\,dv_g\\&\le r\|\nabla g^{\alpha\beta}\|_{L^\infty(B_r)}\sum_{\alpha,\beta=1}^{n}\int_{B_r}|S_{\alpha\beta}|\,dv_g\le Cr\int_{B_r}|\nabla u|_g^2\,dv_g\end{aligned}$$

for $C=C_0\mathrm{Lip}(g)$.

Set $II\equiv-2\sum_{\alpha,\beta=1}^{n}\int_{B_r}\eta(|x|)g^{\alpha\beta}S_{\alpha\beta}\,dv_g$. Then we have

$$\begin{aligned}II&=-2\sum_{\alpha,\beta=1}^{n}\int_{B_r}\eta(|x|)g^{\alpha\beta}(\frac{1}{2}|\nabla u|_g^2 g_{\alpha\beta}-\langle\frac{\partial u}{\partial x_\alpha},\frac{\partial u}{\partial x_\beta}\rangle)\,dv_g\\&=(2-n)\int_{B_r}\eta(|x|)|\nabla u|_g^2\,dv_g.\end{aligned}$$

For $III \equiv -2\sum_{\alpha,\beta,\gamma=1}^n \int_{B_r} \eta'(|x|)\frac{x_\beta x_\gamma}{|x|} g^{\alpha\gamma} S_{\alpha\beta}\, dv_g$, we have

$$\begin{aligned}
III &= -2 \sum_{\alpha,\beta,\gamma=1}^n \int_{B_r} \eta'(|x|)\frac{x_\beta x_\gamma}{|x|} g^{\alpha\gamma}(\frac{1}{2}|\nabla u|_g^2 g_{\alpha\beta} - \langle \frac{\partial u}{\partial x_\alpha}, \frac{\partial u}{\partial x_\beta}\rangle)\, dv_g \\
&= -\int_{B_r} \eta'(|x|)|x||\nabla u|_g^2\, dv_g \\
&+ 2 \sum_{\alpha,\beta,\gamma=1}^n \int_{B_r} \eta'(|x|)\frac{x_\beta x_\gamma}{|x|} g^{\alpha\gamma}\langle \frac{\partial u}{\partial x_\alpha}, \frac{\partial u}{\partial x_\beta}\rangle\, dv_g \\
&= IV + V.
\end{aligned}$$

Observe that (5.10) implies, for $0 < r \le r_0$,

$$g^{\alpha\gamma}(x) = \delta_{\alpha\gamma} + h_{\alpha\gamma}(x), \quad |h_{\alpha\gamma}|(x) \le C_0 \mathrm{Lip}(g)|x|, \ \forall x \in B_r, \ \forall 1 \le \alpha, \gamma \le n.$$

Hence we have

$$V = 2\int_{B_r} |x|\eta'(|x|)|\frac{\partial u}{\partial r}|^2\, dv_g + 2\sum_{\alpha,\gamma=1}^n \int_{B_r} \eta'(|x|) x_\gamma h_{\alpha\gamma} \langle \frac{\partial u}{\partial x_\alpha}, \frac{\partial u}{\partial r}\rangle\, dv_g. \tag{5.11}$$

As

$$0 = \sum_{\alpha,\beta=1}^n \int_\Omega (L_X g^{\alpha\beta}) S_{\alpha\beta}\, dv_g = I + II + III,$$

we have

$$\begin{aligned}
&(2-n)\int_{B_r} \eta(|x|)|\nabla u|_g^2\, dv_g - \int_{B_r} |x|\eta'(|x|)(|\nabla u|_g^2 - 2|\frac{\partial u}{\partial r}|^2)\, dv_g \\
&\ge -Cr\int_{B_r} |\nabla u|_g^2\, dv_g - 2\sum_{\alpha,\gamma=1}^n \int_{B_r} \eta'(|x|) x_\gamma h_{\alpha\gamma}\langle \frac{\partial u}{\partial x_\alpha}, \frac{\partial u}{\partial r}\rangle\, dv_g.
\end{aligned} \tag{5.12}$$

For small $\epsilon > 0$, let $\eta = \eta_\epsilon(|x|) \in C_0^{0,1}(B_r)$ be such that $\eta_\epsilon(t) = 1$ for $0 \le t \le r - \epsilon$, $\eta_\epsilon(t) = 0$ for $t \ge r$, and $\eta'_\epsilon(t) = -\frac{1}{\epsilon}$ for $r - \epsilon \le t \le r$. Putting η into (5.12) and sending ϵ to zero, we obtain

$$\begin{aligned}
&(2-n)\int_{B_r} |\nabla u|_g^2\, dv_g + r\int_{\partial B_r} |\nabla u|_g^2\, dH_g^{n-1} \\
&\ge 2r\int_{\partial B_r} |\frac{\partial u}{\partial r}|^2\, dH_g^{n-1} - Cr\int_{B_r} |\nabla u|_g^2\, dv_g \\
&+ 2\sum_{\alpha,\gamma=1}^n \int_{\partial B_r} x_\gamma h_{\alpha\gamma}\langle \frac{\partial u}{\partial x_\alpha}, \frac{\partial u}{\partial r}\rangle\, dH_g^{n-1} \\
&\ge 2r\int_{\partial B_r} |\frac{\partial u}{\partial r}|^2\, dH_g^{n-1} - Cr\int_{B_r} |\nabla u|_g^2\, dv_g \\
&- Cr^3\int_{\partial B_r} |\nabla u|_g^2\, dH_g^{n-1}
\end{aligned} \tag{5.13}$$

where dH_g^{n-1} is the $(n-1)$-dimensional Hausdorff measure w.r.t. g, and we have used the Hölder inequality in the last step:

$$
\begin{aligned}
&2\sum_{\alpha,\gamma=1}^{n} |\int_{\partial B_r} x_\gamma h^{\alpha\gamma}\langle\frac{\partial u}{\partial\alpha},\frac{\partial u}{\partial r}\rangle\, dH_g^{n-1}| \\
&\leq r\int_{\partial B_r} |\frac{\partial u}{\partial r}|^2\, dH_g^{n-1} + r(\sum_{\alpha,\gamma=1}^{n}\max_{B_r}|h^{\alpha\gamma}|^2)\int_{\partial B_r}|\nabla u|_g^2\, dH_g^{n-1} \\
&\leq r\int_{\partial B_r} |\frac{\partial u}{\partial r}|^2\, dH_g^{n-1} + Cr^3\int_{\partial B_r}|\nabla u|_g^2\, dH_g^{n-1}.
\end{aligned}
$$

Let $f(r) = \int_{B_r}|\nabla u|_g^2\, dv_g$, we have $f'(r) = \int_{\partial B_r}|\nabla u|_g^2\, dH^{n-1}$ for a.e. $r > 0$. Hence (5.13) yields

$$(2-n+Cr)f(r) + r(1+Cr)f'(r) \geq r\int_{\partial B_r}|\frac{\partial u}{\partial r}|^2\, dH_g^{n-1}.$$

In particular, there exists a small $r_0 > 0$ depending on g such that for $0 < r \leq r_0$,

$$(2-n+O(r))f(r) + rf'(r) \geq \frac{r}{2}\int_{\partial B_r}|\frac{\partial u}{\partial r}|^2\, dH_g^{n-1} \tag{5.14}$$

where $C^{-1}r \leq O(r) \leq Cr$. Therefore we have, $0 < r \leq r_0$,

$$\frac{d}{dr}(e^{O(r)}r^{2-n}f(r)) \geq \frac{1}{2}e^{O(r)}r^{2-n}\int_{\partial B_r}|\frac{\partial u}{\partial r}|^2\, dH_g^{n-1}. \tag{5.15}$$

Integrating (5.15) over $0 < r \leq R \leq r_0$, we have

$$e^{O(R)}R^{2-n}f(R) \geq r^{2-n}f(r) + \frac{1}{2}\int_{B_R\setminus B_r}|x|^{2-n}|\frac{\partial u}{\partial r}|^2\, dv_g. \tag{5.16}$$

This, combined with (1.3), implies (1.5) with $K = e^{O(r_0)}$. ■

Remark 5.1. The monotonicity inequality (5.15) has been derived by Garofalo-Lin [GL] for second order elliptic equations with divergence structure by a different method.

§6. **Final remarks**

This section is devoted to some further discussions on theorem A and B. The first remark asserts that for $n \geq 3$, $g \in \mathrm{VMO}(\Omega)$ can be weaken. The second remark concerns the optimal Hausdorff dimension estimate on minimizing harmonic map from domains with

Dini continuous metrics. The third remark concerns the blow-up analysis of stationary harmonic maps from domians with Lipschitz continuous Riemannian metrics.

Theorem 6.1. *For $n \geq 3$, there exists $\delta_0 > 0$ such that if g is a L^∞-Riemannian metric on Ω with $[g]_{BMO(\Omega)} \leq \delta_0$ and $u \in W^{1,2}(\Omega, N)$ is a weakly harmonic map satisfying the quasi-monotonicity inequality (1.5), then there are $\alpha \in (0,1)$ and closed subset $\Sigma \subset \Omega$, with $H^{n-2}(\Sigma) = 0$, such that $u \in C^\alpha(\Omega \setminus \Sigma, N)$.*

Proof. It follows from the same arguments as in theorem B, except that we need to replace Proposition 3.3 by the following proposition, due to Byun-Wang [SW] (see also Caffarelli-Peral [CP]).

Lemma 6.2. *For $n \geq 3$ and ball $B \subset \Omega$, assume that $A = (a_{ij}) \in L^\infty(B, \mathbf{R}^{n\times n})$ is symmetric, and uniformly elliptic with ellipticity constant $\Lambda > 0$. For any $p \in (1, +\infty)$ and $F \in L^p(B, \mathbf{R}^n)$, there exists $\delta_p > 0$ such that if $[g]_{BMO(B)} \leq \delta_p$, then there exists a unique solution $G \in W_0^{1,p}(B)$ to the Dirichlet problem:*

$$(6.1) \qquad \sum_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial G}{\partial x_j}) = \sum_{i=1}^n \frac{\partial F_i}{\partial x_i}, \quad \text{in } B$$

$$(6.2) \qquad G = 0, \qquad \text{on } \partial B.$$

Moreover,

$$(6.3) \qquad \|\nabla G\|_{L^p(B)} \leq C([A]_{BMO(B)}, n, \Lambda)\|F\|_{L^p(B)}.$$

■

Theorem 6.2. *For $n \geq 3$ and a Dini-continuous Riemannian metric g in $\Omega \subset \mathbf{R}^n$, if $u \in W^{1,2}(\Omega, N)$ is a minimizing harmonic map, then there exist $\alpha \in (0,1)$ and closed subset $\Omega \subset \Omega$, which is discrete for $n = 3$ and has Hausdorff dimension at most $(n-3)$ for $n \geq 4$, such that $u \in C^\alpha(\Omega \setminus \Sigma, N)$.*

Proof. Note that the Dini-continuity of g implies $g \in \mathrm{VMO}(\Omega)$. Since u is a minimizing harmonic map, Proposition 5.1 implies that u satisfies the monotonicity inequality (5.7). Define

$$(6.2) \qquad \Sigma = \{x \in \Omega \mid \Theta(u,x) \equiv \lim_{r\downarrow 0} r^{2-n} \int_{B_r(x)} |\nabla u|^2 \geq \epsilon_0^2\}$$

where ϵ_0 is given by Lemma 4.1. Then, by theorem B, we have that $u \in C^\alpha(\Omega \setminus \Sigma, N)$ for some $\alpha \in (0, 1)$.

To prove the Hausdorff dimension estimate of Σ, define the rescalled map $u_{x_0,r_i}(x) = u(x_0 + r_i x) : B_2 \to N$ for any $x_0 \in \Sigma$ and $r_i \downarrow 0$. It is easy to see that u_{x_0,r_i} is minimizing harmonic map w.r.t. $g_i(x) = g(x_0 + r_i x)$. Since g is Dini-continuous, we know $g_i \to g_0$, the Euclidean metric, uniformly on B_2.

It follows from Luckhaus' extension Lemma (see [Ls]) and the minimality of u that there exists a minimizing harmonic map $\phi \in W^{1,2}(B_2, N)$ w.r.t. g_0 such that after taking possible subsequences, $u_{x_0,r_i}(x) \equiv u(x_0+r_i x) \to \phi$ strongly in $W^{1,2}(B_2, N)$. Moreover, the monotonicity inequality (5.7) yields $\frac{\partial \phi}{\partial r} = 0$ a.e. in B_2 and $\phi(x) = \phi(\frac{x}{|x|})$ for a.e. $x \in B_2$. Now we can apply Federer's dimension reduction argument (cf. [SU]) to conclude that Σ is discrete for $n = 3$, and has Hausdorff dimension at most $(n - 3)$ for $n \geq 4$. ■

Theorem 6.3. *For $n \geq 3$ and a Lipschitz continuous metric g on $\Omega \subset \mathbf{R}^n$. Assume that N doesn't support nonconstant harmonic maps from S^2. If $u \in W^{1,2}(\Omega, N)$ is a stationary harmonic map, then there exist $\alpha \in (0, 1)$ and closed subset $\Sigma \subset \Omega$, which is discrete for $n = 4$, and has Hausdorff dimension at most $(n-4)$ for $n \geq 5$, such that $u \in C^\alpha(\Omega \setminus \Sigma, N)$.*

Proof. Note that the Lipschitz continuity of g implies $g \in \mathrm{VMO}(\Omega)$. It follows from the stationarity and Proposition 5.2 that u satisfies the monotonicity inequality (5.16). Therefore, Theorem B implies $u \in C^\alpha(\Omega \setminus \Sigma, N)$ for some $\alpha \in (0, 1)$, with Σ given by (6.2).

For any $x_0 \in \Sigma$ and $r_i \downarrow 0$, $u_{x_0,r_i} \in W^{1,2}(B_2, N)$ are stationary harmonic maps w.r.t. g_i. It follows from (5.16) that there is a harmonic map $\phi \in W^{1,2}(B_2, N)$ w.r.t.g_0, which is homogeneous of degree zero, such that after passing to subsequences, $u_{x_0,r_i}(x) \equiv u(x_0 + r_i x) \to \phi$ weakly in $W^{1,2}(B_2, N)$. One can check the blow-up analysis by Lin [L] applies to stationary harmonic maps w.r.t. Lipschitz continuous metrics g as long as we have theorem B, (5.16), and N doesn't support harmonic S^2's. In particular, $u_{x_0,r_i} \to \phi$ strongly in $W^{1,2}(B_2, N)$. With this strong convergence, one can show Σ is discrete for $n = 4$, and has Hausdorff dimension at most $(n - 4)$ for $n \geq 5$. ■

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