Aronsson's equations on Carnot-Carathéodory spaces

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ABSTRACT. Let $(\mathbf{R}^n, d_X)$ be a Carnot-Carathéodory metric space generated by a family of smooth vector fields $\{X_i\}_{i=1}^m$ satisfying Hörmander's finite rank condition, and $\mathcal{H}_X = \{(x, \sum_{i=1}^m a_i X_i(x)) \,|x \in \mathbf{R}^n, (a_i)_{i=1}^m \in \mathbf{R}^m\}$ be the horizontal tangent bundle generated by $\{X_i\}_{i=1}^m$. Assume that $H = H(x,p) \in C^1(\mathcal{H}_X)$ is quasiconvex in p-variable. We prove that any absolute minimizer $u \in W_X^{1,\infty}(\Omega)$ to $F_\infty(v,\Omega) = \text{ess sup}_{x\in\Omega} H(x, Xv(x))$ is a viscosity solution of the Aronsson equation

$$\mathcal{A}^X[u] := X(H(x, Xu(x))) \cdot H_p(x, Xu(x)) = 0, \quad \text{in } \Omega.$$

§1. Introduction

For $1 \le m, n$, let $\{X_i\}_{i=1}^m \subset C^\infty(\mathbf{R}^n, \mathbf{R}^n)$ be a family of smooth vector fields satisfying Hörmander's finite rank condition, i.e., there is an integer $r \ge 1$ such that $\{X_i\}_{i=1}^m$ and their commutators up to order r span $\mathbf{R}^n$ everywhere. For $x \in \mathbf{R}^n$, let

$$\mathcal{H}(x) = \text{span}\{X_1(x), \cdots, X_m(x)\}$$

be the horizontal tangent space at x. Let

$$\mathcal{H}_X = \{(x, \mathcal{H}(x)) \,|x \in \mathbf{R}^n\}$$

be the subbundle of the tangent bundle $T\mathbf{R}^n$ generated by $\{X_i\}_{i=1}^m$, called a horizontal tangent bundle. Endow an inner product on $\mathbf{R}^n$ such that $\{X_i\}_{i=1}^m$ be an orthonormal set. Recall that an absolutely continuous curve $\xi : [0,T] \to \mathbf{R}^n$ is a *horizontal* curve, if there are measurable functions $a_i(t) : [0,T] \to \mathbf{R}$, $1 \le i \le m$, such that

$$(1.1) \qquad \sum_{i=1}^m a_i^2(t) = 1, \ \xi'(t) = \sum_{i=1}^m a_i(t) X_i(\xi(t)) \ \text{ for a. e. } x \in [0,T].$$

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It is readily seen from (1.1) that $t \in [0,T]$ is the arclength parameter of ξ, whose length is T. Since $\{X_i\}_{i=1}^m$ satisfies Hörmander's condition, it is well-known (cf. Nagel-Stein-Wainger [NSW]) that there exists at least one *horizontal* curve joining any pair of points in $\mathbf{R}^n$. Hence we can introduce the Carnot-Carathéodory distance (cf. [NSW]):

(1.2)

$$d_X(x,y) = \inf\{T \geq 0 \mid \exists \text{ a horizontal curve } \xi : [0,T] \to \mathbf{R}^n \text{ with } \xi(0) = x, \xi(T) = y\}$$

for any $x, y \in \mathbf{R}^n$. Moreover, for any compact set $K \subset \mathbf{R}^n$, there exists $C_K > 0$ such that

$$C_K^{-1}\|x-y\| \leq d_X(x,y) \leq C_K\|x-y\|^{\frac{1}{r}}, \ \forall x, y \in K, \tag{1.3}$$

where $\|\cdot\|$ is the Euclidean distance on $\mathbf{R}^n$.

Typical examples of Carnot-Carathéodory metric spaces include (i) the Euclidean space $(\mathbf{R}^n, \|\cdot\|)$ generated by $\{\frac{\partial}{\partial x_i}\}_{i=1}^n$, and (ii) the Heisenberg group $\mathbf{H}^n \equiv \mathbf{C}^n \times \mathbf{R}$, the simplest Carnot group of step two, endowed with the group law:

$$(z,t)\cdot(z',t') = (z_1 + z_1', \cdots z_n + z_n', t + t' + 2\mathrm{Im}(\sum_{i=1}^n z_i \bar{z_i'})), \ \forall (z,t), (z',t') \in \mathbf{C}^n \times \mathbf{R},$$

whose Lie algebra $h = V_1 + V_2$ with $V_1 = \mathrm{span}\{X_i, Y_i\}_{1 \leq i \leq n}$ and $V_2 = \mathrm{span}\{T\}$, where

$$X_i = \frac{\partial}{\partial x_i} = 2y_i\frac{\partial}{\partial t}, \ Y_i = \frac{\partial}{\partial y_i} + 2x_i\frac{\partial}{\partial t}, \ 1 \leq i \leq n, \ T = 4\frac{\partial}{\partial t}.$$

For any bounded domain $\Omega \subset \mathbf{R}^n$ and $u : \Omega \to \mathbf{R}$, denote by $Xu := (X_1u, \cdots, X_mu)$ the horizontal gradient of u. The horizontal Sobolev space, $W_X^{1,\infty}(\Omega)$, is defined by

$$W_X^{1,\infty}(\Omega) := \{u : \Omega \to \mathbf{R} \mid \|u\|_{W_X^{1,\infty}(\Omega)} \equiv \|u\|_{L^\infty(\Omega)} + \|Xu\|_{L^\infty(\Omega)} < +\infty\},$$

and the horizontal Lipschitz space is defined by

$$\mathrm{Lip}_X(\Omega) := \{u : \Omega \to \mathbf{R} \mid \|u\|_{\mathrm{Lip}_X(\Omega)} \equiv \sup_{x,y\in\Omega, x\neq y} \frac{|u(x)-u(y)|}{d_X(x,y)} < +\infty\}.$$

It is known (cf. Garofalo-Nieu [GN], Franchi-Serapioni-Serra [FSS]) that $u \in W_X^{1,\infty}(\Omega)$ iff $u \in \mathrm{Lip}_X(\Omega)$.

Definition 1.1. For a continuous function $H \in C(\mathcal{H}_X)$, define the L^∞-functional

$$F_\infty(v,\Omega) = \mathrm{esssup}_{x\in\Omega} H(x, Xv(x)), \ \forall v \in W_X^{1,\infty}(\Omega).$$

A function $u : \Omega \to \mathbf{R}$ is an *absolute minimizer* of H if, for any $U \subset\subset \Omega$, $u \in W^{1,\infty}_X(U)$ and

$$F_\infty(u, U) \le F_\infty(v, U), \ \forall v \in W^{1,\infty}_X(U), \ v = u \text{ on } \partial U. \tag{1.4}$$

Formal calculations yield that an absolute minimizer $u : \Omega \to \mathbf{R}$ of H satisfies the (subelliptic) Aronsson equation:

$$\mathcal{A}^X[u] := \sum_{i=1}^m X_i(H(x, Xu(x))) \cdot H_{p_i}(x, Xu(x)) = 0, \text{ in } \Omega. \tag{1.5}$$

Let $\mathcal{S}^m$ be the set of symmetric $m \times m$ matrices, equiped with the usual order. Note that the Aronsson operator $\mathcal{A}^X : \Omega \times \mathbf{R}^m \times \mathcal{S}^m \to \mathbf{R}$ given by

$$\mathcal{A}^X(x, p, M) = \sum_{i,j=1}^m H_{p_i}(x, p) H_{p_j}(x, p) M_{ij} + \sum_{i=1}^m X_i H(x, p) H_{p_i}(x, p)$$

is *degenerately elliptic*, i.e. for any $(x, p) \in \Omega \times \mathbf{R}^m$,

$$\mathcal{A}^X(x, p, M) \le \mathcal{A}^X(x, p, N), \ \forall M, N \in \mathcal{S}^m, \text{ with } M \le N. \tag{1.6}$$

Therefore we can adapt the notion of viscosity solutions by Crandall-Lions [CL] (cf. also [CIL]) to define

Definition 1.2. A function $u \in C(\Omega)$ is a viscosity subsolution (or supersolution, resp.) of (1.5), if for any $(x_0, \phi) \in \Omega \times C^2(\Omega)$ such that

$$0 = (\phi - u)(x_0) \le (\text{ or } \ge)(\phi - u)(x), \ \forall x \in \Omega,$$

then $\mathcal{A}^X[\phi](x_0) \ge$ (or $\le$)0. A function $u \in C(\Omega)$ is a viscosity solution of (1.5) if it is both a viscosity subsolution and a viscosity supersolution of (1.5).

Definition 1.3. A function $f : \mathbf{R}^m \to R$ is *quasiconvex* if

$$\{p \in \mathbf{R}^m \mid f(p) \le \lambda\} \text{ is convex, for any } \lambda \in \mathbf{R}, \tag{1.7}$$

or equivalently,

$$f(tp + (1-t)q) \le \max\{f(p), f(q)\}, \text{ for any } p, q \in \mathbf{R}^m \text{ and } t \in [0, 1]. \tag{1.8}$$

A typical quasiconvex function f, which may not be convex, can be constructed by letting $f(p) = g \circ h(p)$, where $g : \mathbf{R} \to \mathbf{R}$ is a monotone function and $h : \mathbf{R}^m \to \mathbf{R}$ is a convex function.

The second author has proved in Wang [W] that *any absolute minimizer* $u : \Omega \to \mathbf{R}$ *of* H *is a viscosity solution to the Aronsson equation (1.5), provided that (i)* $H = H(x,p) \in C^2(\mathcal{H}_X)$ *is quasiconvex in* p*-variable, and (ii)* $H_p(0,0) = 0$ *and* $H(x,\cdot)$ *is homogeneous of degree* $\alpha \geq 1$. See Bieske [B1,2] and Bieske-Capogna [BC] for earlier works on *absolutely minimal horizontally Lipschitz extensions* on Carnot groups.

Since equation (1.5) is defined for $H \in C^1(\mathcal{H}_X)$, it is a very natural question to ask *whether the above result by* [W] *remains true if we weaken* $H \in C^1(\mathcal{H}_X)$.

In this paper we answer this question affirmatively by proving the following theorem.

Theorem 1.4. *For any family of vector fields* $\{X_i\}_{i=1}^m$ *satisfying Hörmander's finite rank condition, if* $H = H(x,p) \in C^1(\mathcal{H}_X)$ *is quasiconvex in* p*-variable for any* $x \in \Omega$*, then any absolute minimizer* $u : \Omega \to \mathbf{R}$ *is a viscosity solution of the Aronsson equation (1.5).*

The study of absolute minimizers was initiated by Aronsson [A1,2,3] in dimension one. Jensen established in his seminal paper [J] the equivalence between infinity harmonic functions and absolute minimizing Lipschitz extensions, and their uniqueness as well. Later, Juutinen [Jp] extended the main theorem of [J] to Riemannian manifold settings. In the Euclidean setting, Barron-Jensen-Wang [BJW] provided a general study on absolute minimizers and established that any absolute minimizer for suitable $H(p,z,x) \in C^2(\mathbf{R}^n \times \mathbf{R} \times \Omega)$ is a viscosity solution of the Aronsson equation:

$$H_p(\nabla u, u, x) \cdot (H(\nabla u, u, x))_x = 0. \tag{1.9}$$

Subsequently, Crandall [C] gave a simpler proof of this result of [BJW] under weaker hypotheses. The techniques employed by [BJW] and [C] rely crucially on $H \in C^2(\mathbf{R}^n \times \mathbf{R} \times \Omega)$, because of the construction of local, C^2 solutions to the Hamilton-Jacobi equation $H(\nabla\psi, \psi, x) = k$. Very recently, Crandall-Wang-Yu [CWY] found a new proof of this theorem even for $H \in C^1(\mathbf{R}^n \times \mathbf{R} \times \Omega)$. The new observation made by [CWY] is to use global, viscosity solutions to the Hamilton-Jacobi equation associated with $H \in C^1(\mathbf{R}^n \times \mathbf{R} \times \Omega)$ as comparison functions to absolute minimizers.

Bieske-Capogna [BC] extended the idea of [C] to derive the subelliptic infinity Laplace equation for an absolute minimizing horizontal Lipschitz extension on Carnot groups. Wang [W] made a new observation based on [C] to derive the Aronsson equation for any absolute minimizer of $H \in C^2(\mathcal{H}_X)$ associated with any family of Hörmander's vector fields. Here we aim to modify and extend the observation made in [CWY] to the Carnot-Carathéodory space $(\mathbf{R}^n, d_X)$. Roughly speaking, if $\phi \in C^2(\Omega)$ is a upper test function for an absolute minimizer $u \in W_X^{1,\infty}(\Omega)$, at x_0, then we show in §3 below that there exists

$x_r \neq x_0$ such that

$$\phi(x_r) - \phi(x_0) \geq \max_{\{p \in \mathcal{H}(x_0), H(x_0,p) \leq H(x_0, X\phi(x_0))\}} \langle p, P_{\mathcal{H}(x_0)}(x_r - x_0) \rangle_{\mathcal{H}(x_0)}. \tag{1.10}$$

This turns out to be sufficient to show that u is a viscosity subsolution of the Aronsson equation (1.5).

We would like to point out that Crandall-Evans-Gariepy [CEG] has shown that an absolute minimizing Lipschitz extension can also be characterized by the comparison principle with cones, which has been subsequently extended by Gariepy-Wang-Yu [GWY] to absolute minimizers to quasiconvex Hamiltonians. This characterization for absolute minimizers in term of comparison principle with cone type functions has also been obtained for some non-Euclidean spaces including Grushin spaces by [B2], Finsler metric spaces by Champion-De Pascale [CD], and metric-measure spaces by Juutinen-Shanmugalingam [JP].

The paper is organized as follows. In §2, we establish some preliminary properties of absolute minimizers. In §3, we give a proof of theorem 1.4.

§2. Some preliminary results

This section is devoted to some basic facts on absolute minimizer and the construction of viscosity solutions to Hamilton-Jacobi equation $H(x, Xv) = k$.

Let d_X be the Carnot-Carathéodory distance given by §1, and define subelliptic balls

$$B_r(x_0) = \{x \in \mathbf{R}^n \mid d_X(x, x_0) < r\}, \ \overline{B}_r(x_0) = \{y \in \mathbf{R}^n \mid d_X(x, x_0) \leq r\}.$$

First we have

Proposition 2.1. *Let $H = H(x,p) \in C(\mathcal{H}_X)$ be quasiconvex in p-variable. Let $U \subset\subset \Omega$ be a bounded open set.*

(a) Suppose $(x_0, \phi) \in U \times C^1(U)$, and $v \in Lip_X(U)$. If ϕ touches v at x_0 from above, i.e.

$$0 = (\phi - v)(x_0) \leq (\phi - v)(x), \ \forall x \in U \tag{2.1}$$

then

$$H(x_0, X\phi(x_0)) \leq \lim_{r \downarrow 0} esssup_{B_r(x_0)} H(x, Xv(x)). \tag{2.2}$$

(b) Let u be an absolute minimizer for H in Ω. Assume that $x_0 \in U$ and $w \in Lip_X(U)$ satisfy

$$(w-u)(x_0) \le 0 \le (w-u)(x),\ \forall x \in \partial U, \tag{2.3}$$

then

$$\lim_{r\downarrow 0} esssup_{B_r(x_0)} H(x, Xu(x)) \le esssup_U H(x, Xw(x)). \tag{2.4}$$

Proof. First observe that, by continuity of H, we have

$$\lim_{r\downarrow 0} \text{esssup}_{B_r(x_0)} H(x, Xv(x)) = \lim_{r\downarrow 0} \text{esssup}_{B_r(x_0)} H(x_0, Xv(x)). \tag{2.5}$$

By replacing ϕ by $\phi(x) + \|x - x_0\|^2$, we may assume that for $r > 0$ small,

$$0 = (\phi - v)(x_0) < (\phi - v)(x)\ \forall x \in \overline{B}_r(x_0) \setminus \{x_0\}. \tag{2.6}$$

For $0 < \epsilon \le \frac{r}{2}$, let $v_\epsilon(x) = \int_{\mathbf{R}^n} \eta_\epsilon(x-y)v(y)\,dy \in C^\infty(B_{\frac{r}{2}}(x_0))$ be a standard modification of v and $x_\epsilon \in \bar{B}_{\frac{r}{2}}(x_0)$ satisfy

$$(\phi - v_\epsilon)(x_\epsilon) = \min_{x \in \overline{B}_{\frac{r}{2}}(x_0)} (\phi - v_\epsilon)(x).$$

By (2.6), we have $\lim_{\epsilon\downarrow 0} x_\epsilon = x_0$. Hence, for small ϵ, we have $X\phi(x_\epsilon) = Xv_\epsilon(x_\epsilon)$, and

$$H(x_\epsilon, X\phi(x_\epsilon)) = H(x_\epsilon, X(v_\epsilon)(x_\epsilon)). \tag{2.7}$$

We claim

$$|X(v_\epsilon)(x_\epsilon) - (Xv)_\epsilon(x_\epsilon)| \le C\|X\|_{C^1(B_r(x_0))}\|u\|_{W_X^{1,\infty}(B_r(x_0))}\omega(r), \tag{2.8}$$

where $\omega(r)$ denotes the modular of continuity of d_X with respect to $\|\cdot\|$.

To see (2.8), let $X_i(x) = \sum_{j=1}^n a_{ij}(x)\frac{\partial}{\partial x_j}$ for $x \in \mathbf{R}^n$ and $1 \le i \le m$, with $(a_{ij}) \in C^\infty(\mathbf{R}^n, \mathbf{R}^{nm})$. Then, for $1 \le i \le m$ and $x \in B_{\frac{r}{2}}(x_0)$, we have

$$\begin{aligned} &(X_i v)_\epsilon(x) - X_i(v_\epsilon)(x) \\ &= \int_{\mathbf{R}^n} \eta_\epsilon(x-y)(\sum_{j=1}^n a_{ij}(y)\frac{\partial}{\partial y_j})\{v(y) - v(x)\}\,dy \end{aligned}$$

$$
\begin{aligned}
&-\int_{\mathbf{R}^n}\sum_{j=1}^n a_{ij}(x)\frac{\partial\eta_\epsilon(x-y)}{\partial x_j}\{v(y)-v(x)\}\,dy\\
=&\sum_{j=1}^n\int_{\mathbf{R}^n}[-\frac{\partial}{\partial y_j}(a_{ij}(y)\eta_\epsilon(x-y))-a_{ij}(x)\frac{\partial\eta_\epsilon(x-y)}{\partial x_j}](v(y)-v(x))\,dy\\
=&\sum_{j=1}^n\int_{\mathbf{R}^n}(a_{ij}(y)-a_{ij}(x))\frac{\partial\eta_\epsilon(x-y)}{\partial x_j}(v(y)-v(x))\,dy\\
&+\sum_{j=1}^n\int_{\mathbf{R}^n}\frac{\partial a_{ij}(y)}{\partial y_j}\eta_\epsilon(x-y)(v(y)-v(x))\,dy.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&|(X_iv)_\epsilon(x)-X_i(v_\epsilon)(x)|\\
&\le C\max_{1\le j\le n}\|\nabla a_{ij}\|_{L^\infty(B_r(x_0))}\int_{\mathbf{R}^n}\{\eta_\epsilon(x-y)|v(y)-v(x)|\\
&\quad+\|y-x\||\nabla\eta_\epsilon(x-y)||v(y)-v(x)|\}\,dy\\
&\le C\|X_i\|_{C^1(B_r(x_0))}\|v\|_{\mathrm{Lip}_X(B_r(x_0))}\max_{\|y-x\|\le r}d_X(y,x)\\
&\le C\|X_i\|_{C^1(B_r(x_0))}\|v\|_{\mathrm{Lip}_X(B_r(x_0))}\omega(r)
\end{aligned}
$$

and hence (2.8) follows. Since $\|(Xv)_\epsilon\|_{L^\infty(B_{\frac{r}{2}}(x_0))}\le\|Xv\|_{L^\infty(B_r(x_0))}$, it follows from (2.8) that for small $r>0$, $|(Xv_\epsilon)(x_\epsilon)|\le\|Xv\|_{L^\infty(B_r(x_0))}+1$. Hence

$$
\begin{aligned}
&H(x_\epsilon,X(v_\epsilon)(x_\epsilon))\\
&\le H(x_\epsilon,(Xv)_\epsilon(x_\epsilon))\\
&+\max_{x\in B_r(x_0)}\max_{\{|p|\le\|Xv\|_{L^\infty(B_r(x_0))}+1\}}\{|H_p(x,p)||X(v_\epsilon)(x_\epsilon)-(Xv)_\epsilon(x_\epsilon)|\}\\
&\le H(x_\epsilon,(Xv)_\epsilon(x_\epsilon))+C\omega(r)\\
&\le H(x_0,(Xv)_\epsilon(x_\epsilon))+\{\max_{x\in B_r(x_0)}\max_{\{|p|\le\|Xv\|_{L^\infty(B_r(x_0))}+1\}}|\nabla_xH(x,p)|\}r+C\omega(r)\\
&\le \mathrm{esssup}_{x\in B_r(x_0)}H(x_0,Xv(x))+C(r+\omega(r))
\end{aligned}\tag{2.9}
$$

where we have used the quasiconvexity of $H(x_0,p)$ in p-variable:

$$
H(x_0,(Xv)_\epsilon(x_\epsilon))\le\mathrm{esssup}_{B_{\frac{r}{2}}(x_0)}H(x_0,Xv_\epsilon(x))\le\mathrm{esssup}_{B_r(x_0)}H(x_0,Xv(x)).
$$

Taking r into zero and noting $\lim_{r\downarrow 0}\omega(r)=0$, (2.9) and (2.7) imply (2.2).

To prove (b). Set, for small $\epsilon>0,\delta>0$,

$$
w_{\epsilon,\delta}(x)=w(x)+\epsilon\|x-x_0\|^2-\delta,\ x\in U.
$$

Then $u(x_0) - w_{\epsilon,\delta}(x_0) \geq \delta > 0$, and, for $x \in \partial U$,

$$u(x) - w_{\epsilon,\delta}(x) \leq u(x) - w(x) - \epsilon \min_{\partial U} \|x - x_0\|^2 + \delta \leq \delta - \epsilon \min_{x \in \partial U} \|x - x_0\|^2 < 0$$

provided that we choose ϵ and δ such that

$$\delta - \epsilon \min_{x \in \partial U} \|x - x_0\|^2 < 0. \tag{2.10}$$

Hence there exists another open connected component V of $\{x \in U \mid u(x) - w_{\epsilon,\delta}(x) > 0\}$ such that $x_0 \in V$ and $V \subset\subset U$. Since $u = w_{\epsilon,\delta}$ on ∂V, the absolute minimality of u implies that

$$\begin{aligned} \text{esssup}_{B_r(x_0)} H(x, Xu(x)) &\leq \text{esssup}_{B_r(x_0)} H(x, Xw_{\epsilon,\delta}(x)) \\ &\leq \text{esssup}_V H(x, Xw_{\epsilon,\delta}(x)) \\ &\leq \text{esssup}_U H(x, Xw_{\epsilon,\delta}(x)). \end{aligned}$$

By sending $r \downarrow 0$ and then $\epsilon, \delta \downarrow 0$, (2.3) then follows. ■

Similar to [CWY], the second observation is that we may assume

$$\lim_{\{p \in \mathcal{H}(x) : \|p\| \to +\infty\}} H(x,p) = +\infty \text{ uniformly for } x \in \overline{\Omega}. \tag{2.11}$$

In fact, as in [CWY] §2, let $u \in W_X^{1,\infty}(\Omega)$ be the absolute minimizer of H under consideration and

$$R = \|Xu\|_{L^\infty(\Omega)} + 1, M = \min\{H(x,p) \mid x \in \overline{\Omega}, p \in \mathcal{H}(x), \text{ with } \|p\| \leq R\}, \tag{2.12}$$

and define

$$\hat{H}(x,p) = \max\{H(x,p), \|p - P_R(p)\| + M\}, \ \forall (x,p) \in \mathcal{H}_X, \tag{2.13}$$

where $P_R : \mathbf{R}^m \to \mathbf{R}^m$ is given by

$$\begin{aligned} P_R(p) &= p, \qquad \|p\| \leq R, \\ &= R\frac{p}{\|p\|}, \ \|p\| \geq R. \end{aligned}$$

It is easy to see that $\hat{H}$ is quasiconvex in p-variable and satisfies (2.11), $H \leq \hat{H}$, and

$$H(x, Xu(x)) = \hat{H}(x, Xu(x)) \text{ for a. e. } x \in \overline{\Omega}.$$

Thus u is also an absolute minimizer for $\hat{H}$. Finally, if $\phi \in C^1(\Omega)$ touches u from above at x_0, then proposition 2.1(a) implies that $|X\phi|(x_0) < R$ and hence $\hat{H}_p(x_0, X\phi(x_0))(= H_p(x_0, X\phi(x_0))$ exists.

Now we indicate how to construct viscosity solutions of the Hamiltonian-Jacobi equation $H(x, X\Phi(x)) = k$. Let $P_{\mathcal{H}(x)} : \mathbf{R}^n \to \mathcal{H}(x)$, $x \in \mathbf{R}^n$, be the orthogonal projection map. For $k \in \mathbf{R}$, $x \in B_r(x_0)$ and $p \in \mathbf{R}^n$, define

$$L(x, p, k) = \max_{\{q \in \mathcal{H}(x) \mid H(x,q) \le k\}} \langle q, P_{\mathcal{H}(x)}(p) \rangle_{\mathcal{H}(x)}. \tag{2.14}$$

Set

$$k_0(r) = \max_{x \in \overline{B}_r(x_0)} \min_{q \in \mathcal{H}(x)} H(x, q). \tag{2.15}$$

Notice that by (2.11), $k_0(r) < +\infty$.

For $L(x, p, k)$, we have

Proposition 2.2. *If $H = H(x, p) \in C(\mathcal{H}_X)$ is quasiconvex in p-variable and satisfies the coercivity condition (2.11). Then, for any $x \in \overline{B}_r(x_0)$, $p \in \mathbf{R}^n$ and $k \ge k_0(r)$, we have*

(1) $x \to L(x, p, k)$ is upper-semicontinuous,

(2) $p \to L(x, p, k)$ is Lipschitz continuous with respect to the Euclidean distance $\|\cdot\|$, and its Lipschitz constant depends only on k,

(3) $p \to L(x, p, k)$ is convex, positively 1-homogeneous, and $L(x, p, k) = 0$ for any $p \perp \mathcal{H}(x)$,

(4) If $M > 0$, then there is $k_M > 0$ such that for any $k \ge k_M$, $L(x, p, k) \ge M|P_{\mathcal{H}(x)}(p)|$ for any $(x, p) \in \mathbf{R}^n \times \mathbf{R}^n$,

(5) $k \to L(x, p, k)$ is nondecreasing and continuous from the right.

Proof. In view of (1.7) and (2.11), the proof is straightforward. We leave the detail to readers. ∎

Definition 2.3. For $r > 0$ and $x \in \overline{B}_r(x_0)$, a *horizontal path* from x_0 to x in $\overline{B}_r(x_0)$ is a *horizontal curve* $\xi : [0, T] \to \bar{B}_r(x_0)$ such that $\xi(0) = x_0$ and $\xi(T) = x$. The set of such horizontal paths is denoted by

$$hp(x, r) := \{\text{horizontal paths } \xi \text{ from } x_0 \text{ to } x \text{ in } \overline{B}_r(x_0)\}.$$

Now we define, for $k \ge k_0(r)$ and $x \in B_r(x_0)$,

$$C_{k,r}(x, x_0) = \inf\{\int_0^T L(\xi(t), \xi'(t), k)\, dt \mid \xi \in hp(x, r)\}. \tag{2.16}$$

Notice that $C_{k,r}(x,x_0)$ is well-defined and finite, since $(\mathbf{R}^n, d_X)$ is a length space, i.e., the distance between any two points can be realized by the length of a horizontal curve joining the two points. In particular, for any $x \in \overline{B}_r(x_0)$, there exists a horizontal curve $\gamma : [0,T] \to \mathbf{R}^n$ joining x_0 to x such that $T = d_X(x,x_0) \le r$. By Proposition 2.2 (5), we have $k \to C_{k,r}$ is nondecreasing. We set

$$C_{k-,r}(x,x_0) = \lim_{l \uparrow k} C_{l,r}(x,x_0), \quad C_{k+,r}(x,x_0) = \lim_{l \downarrow k} C_{l,r}(x,x_0). \tag{2.17}$$

Proposition 2.4. *Under the assumptions as in proposition 2.2. For any $k \ge k_0(r)$, we have (i) $C_{k,r}(x_0,x_0) \le 0$, (ii) $C_{k,2r}(x_2,x_0) \le C_{k,r}(x_1,x_0) + C_{k,r}(x_2,x_1)$ for any $x_1, x_2 \in B_r(x_0)$, and (iii) $C_{k,r}(x,x_0) \in W_X^{1,\infty}(B_r(x_0))$.*

Proof. Since $L(x,0,k) = 0$, $C_{k,r}(x_0,x_0) \le 0$. To see (ii), for $\epsilon > 0$ be arbitrarily small, let $\xi_1 : [0,T_1] \to B_r(x_0)$ be a horizontal curve connecting x_0 to x_1 and $\xi_2 : [0,T_2] \to B_r(x_1)$ be another horizontal curve connecting x_1 to x_2 such that

$$\int_0^{T_1} L(\xi_1, \xi_1', k)\, dt \le C_{k,r}(x_1,x_0) + \epsilon, \quad \int_0^{T_2} L(\xi_2, \xi_2', k)\, dt \le C_{k,r}(x_2,x_1) + \epsilon.$$

If we define $\xi_3 : [0, T_1 + T_2] \to B_{2r}(x_0)$ by letting $\xi_3(t) = \xi_1(t)$ for $0 \le t \le T_1$ and $\xi_3(t) = \xi_2(t - T_1)$ for $T_1 \le t \le T_1 + T_2$. Then ξ_3 is a horizontal curve connecting x_0 to x_2, and

$$\begin{aligned} C_{k,2r}(x_2,x_0) &\le \int_0^{T_1+T_2} L(\xi_3, \xi_3', k)\, dt \\ &= \int_0^{T_1} L(\xi_1, \xi_1', k)\, dt + \int_0^{T_2} L(\xi_2, \xi_2', k)\, dt \\ &\le C_{k,r}(x_1,x_0) + C_{k,r}(x_2,x_1) + 2\epsilon. \end{aligned}$$

This implies (ii). To see (iii), for $y, z \in B_r(x_0)$, let $\eta : [0,S] \to B_r(x_0)$ another horizontal curve connecting y to z such that $d_X(z,y) = S$. Define

$$K = \max_{x \in \overline{B}_r(x_0)} \max_{q \in \mathcal{H}(x) : H(x,q) \le k} |q|.$$

Then, similar to (ii), we have

$$\begin{aligned} C_{k,r}(z,x_0) &\le C_{k,r}(y,x_0) + \int_0^S L(\eta, \eta', k)\, dt \\ &\le C_{k,r}(y,x_0) + K \int_0^S |\eta'(t)|\, dt \\ &= C_{k,r}(y,x_0) + KS = C_{k,r}(y,x_0) + K d_X(y,z). \end{aligned}$$

This implies that $C_{k,r}(y,x_0)$ is Lipschitz continuous in $B_r(x_0)$ with respect to d_X. ■

It follows from proposition 2.4 and Rademacher's theorem on $(\mathbf{R}^n, d_X)$ (cf. [GN]) that $XC_{k,r}(x,x_0)$ exists for a.e. $x \in B_r(x_0)$.

The main result of this section is

Proposition 2.5. *Under the same assumptions as in proposition 2.2. For any $k \geq k_0(r)$, $C_{k,r}$ is a viscosity solution of*

$$H(x, XC_{k,r}(x,x_0)) = k \quad \text{in } B_r(x_0) \setminus \{x_0\}. \tag{2.18}$$

In particular, $H(x, XC_{k,r}(x,x_0)) = k$ for a.e. $x \in B_r(x_0)$.

Proof. For any $x_1 \in B_r(x_0) \setminus \{x_0\}$, Let $\phi \in C^1(B_r(x_0)$ touch $C_{k,r}(x,x_0)$ at x_1 from above. Let $\xi \in C^1([0,T], \mathbf{R}^n) \cap hp(x_1, r)$. For $0 < t_0 < T$, we have

$$\int_{t_0}^{T} \langle X\phi(\xi(t)), \xi'(t)\rangle_{\mathcal{H}(\xi(t))}\, dt = \phi(x_1) - \phi(\xi(t_0)) \leq C_{k,r}(x_1,x_0) - C_{k,r}(\xi(t_0),x_0)$$
$$\leq C_{k,2r}(x_1, \xi(t_0)) \leq \int_{t_0}^{T} L(\xi(t), \xi'(t), k)\, dt. \tag{2.19}$$

Dividing (2.19) by $T - t_0$, taking $t_0 \uparrow T$, and applying proposition 2.2(4), we obtain

$$\langle X\phi(x_1), \xi'(T)\rangle_{\mathcal{H}(x_1)} \leq L(x_1, \xi'(T), k) = \max_{\{q \in \mathcal{H}(x_1), H(x_1,q) \leq k\}} \langle q, \xi'(T)\rangle_{\mathcal{H}(x_1)}. \tag{2.20}$$

This and the quasiconvexity of $H(x_1, \cdot)$ imply $H(x_1, X\phi(x_1)) \leq k$, i.e., $C_{k,r}$ is a viscosity subsolution of (2.18).

To prove that $C_{k,r}$ is a viscosity supersolution of (2.18), let $\psi \in C^1(B_r(x_0)$ touch $C_{k,r}$ from below at $x_1 \in B_r(x_0) \setminus \{x_0\}$. Let $\xi \in C([0,T], \mathbf{R}^n) \cap hp(x_1, r)$ be such that

$$C_{k,r}(x_1, x_0) = \int_0^T L(\xi(t), \xi'(t), k)\, dt. \tag{2.21}$$

Then, for any $t_0 \in (0,T)$, we have

$$\int_{t_0}^{T} \langle X\psi(\xi(t)), \xi'(t)\rangle_{\mathcal{H}(\xi(t))}\, dt = \psi(x_1) - \psi(\xi(t_0))$$
$$\geq C_{k,r}(x_1,x_0) - C_{k,r}(\xi(t_0),x_0)$$
$$\geq \int_0^T L(\xi(t), \xi'(t), k)\, dt - \int_0^{t_0} L(\xi(t), \xi'(t), k)\, dt$$
$$= \int_{t_0}^T L(\xi(t), \xi'(t), k)\, dt = \int_{t_0}^T \max_{\{p \in \mathcal{H}(\xi(t)), H(\xi(t),p) \leq k\}} \langle p, \xi'(t)\rangle_{\mathcal{H}(\xi(t))}\, dt.$$

This implies that there exist $t_r \uparrow T$ such that $\xi'(t_r)$ exist, and

$$\langle X\psi(\xi(t_r)), \xi'(t_r)\rangle_{\mathcal{H}(\xi(t_r))} \geq \max_{\{p\in\mathcal{H}(\xi(t_r)), H(\xi(t_r),p)\leq k\}} \langle p, \xi'(t_r)\rangle_{\mathcal{H}(\xi(t_r))}. \tag{2.22}$$

Since $\langle \xi'(t_r), \xi'(t_r)\rangle_{\mathcal{H}(\xi(t_r))} = 1$, we assume that there is $q \in \mathcal{H}(x_1)$ with $\langle q, q\rangle_{\mathcal{H}(x_1)} = 1$ such that $\lim_{t_r\uparrow T} \xi'(t_r) = q$. Taking $t_r \uparrow T$, (2.22) implies

$$\langle X\phi(x_1), q\rangle_{\mathcal{H}(x_1)} \geq \max_{\{p\in\mathcal{H}(x_1): H(x_1,p)\leq k\}} \langle p, q\rangle_{\mathcal{H}(x_1)}. \tag{2.23}$$

Hence we conclude $H(x_1, X\psi(x_1)) \geq k$. The proof is complete. ■

§3. Proof of Theorem 1.4

In this section, we prove theorem 1.4. We begin with some lemmas. For $x_0 \in \Omega$, let $r > 0$ be such that $B_r(x_0) \subset \Omega$ and let $\phi \in C^2(B_r(x_0))$ be such that

$$0 = (\phi - u)(x_0) < (\phi - u)(x) \quad \text{for } x \in B_r(x_0) \setminus \{x_0\}. \tag{3.1}$$

For $k_0(r)$ give by (2.15), define

$$k_r = \inf\{k \mid k \geq k_0(r),\ u(x) \leq u(x_0) + C_{k,r}(x, x_0) \quad \text{for } x \in \partial B_r(x_0)\}. \tag{3.2}$$

Notice that it follows from Proposition 2.2 (iv) that for any $M > 0$ we have

$$C_{k,r}(x, x_0) \geq Mr \quad \text{for } x \in \partial B_r(x_0)$$

provided that $k > 0$ is sufficiently large. This implies that the quantity k_r is well-defined.

Lemma 3.1. *Let $H = H(x,p) \in C(\mathcal{H}_X)$ be quasiconvex in p-variable and satisfy (2.11). If $u \in W^{1,\infty}_X(\Omega)$ be an absolute minimizer of H, then $H(x_0, X\phi(x_0)) \leq k_r$.*

Proof. For any $k > k_r$, let $w(x) \equiv u(x_0) + C_{k,r}(x, x_0)$. Then it is easy to see that $u(x_0) \geq w(x_0)$ and

$$u(x) \leq w(x) \quad \text{for } x \in \partial B_r(x_0), \tag{3.3}$$

Hence, by proposition 2.1(b), we have

$$\begin{aligned} H(x_0, X\phi(x_0)) &\leq \lim_{s\downarrow 0} \operatorname{esssup}_{B_s(x_0)} H(x, Xu(x)) \\ &\leq \operatorname{esssup}_{B_r(x_0)} H(x, XC_{k,r}(x, x_0)) = k. \end{aligned} \tag{3.4}$$

Taking $k \downarrow k_r$, this yields the result. ■

Notice that if $H_p(x_0, X\phi(x_0)) = 0$, then $\mathcal{A}^X(\phi)(x_0) = 0$ and theorem 1.4 is proved. Hence we assume $H_p(x_0, X\phi(x_0)) \neq 0$.

Lemma 3.2. *Let $H = H(x,p) \in C^1(\mathcal{H}_X)$ be quasiconvex in p-variable and satisfy (2.11). Assume $H_p(x_0, X\phi(x_0)) \neq 0$, if $u \in W_X^{1,\infty}(\Omega)$ is an absolute minimizer of H, then, for any sufficiently small $r > 0$,*

$$H(x_0, X\phi(x_0)) > k_0(r). \tag{3.5}$$

Proof. It follows from $H_p(x_0, X\phi(x_0)) \neq 0$ that there is $p_0 \in \mathcal{H}(x_0)$ such that $H(x_0, p_0) < H(x_0, X\phi(x_0))$. By continuity of H, this implies that for a sufficiently small $r > 0$ and any $x \in B_r(x_0)$, there exists $p_x \in \mathcal{H}(x)$ such that $H(x, p_x) < H(x_0, X\phi(x_0))$. Hence $H(x_0, X\phi(x_0)) > k_0(r)$. ■

Proof of Theorem 1.4. Denote $h_0 = H(x_0, X\phi(x_0))$. For any $k < h_0 \leq k_r$ and m sufficiently large, there exist $x_m^k \in \partial B_{\frac{1}{m}}(x_0)$ such that

$$C_{k,\frac{1}{m}}(x_m^k, x_0) \leq u(x_m^k) - u(x_0). \tag{3.6}$$

For $k \uparrow h_0$, assume $x_m^k \to x_m \in \partial B_r(x_0)$. Then (3.6) yields

$$C_{h_0^-,\frac{1}{m}}(x_m, x_0) \leq u(x_m) - u(x_0). \tag{3.7}$$

Let $\epsilon_m > 0$ be sufficiently small such that $u(x_m) - u(x_0) + \epsilon_m < \phi(x_m) - \phi(x_0)$. By definition of $C_{h_0^-,\frac{1}{m}}$, there is $\xi_m \in C([0, T_m], \mathbf{R}^n) \cap hp(x_m, \frac{1}{m})$ such that

$$\begin{aligned}\int_0^{T_m} L(\xi_m(t), \xi_m'(t), h_0^-)\, dt &\leq C_{h_0^-,\frac{1}{m}}(x_m, x_0) + \epsilon_m \leq u(x_m) - u(x_0) + \epsilon_m \\ &< \phi(x_m) - \phi(x_0) = \int_0^{T_m} \langle X\phi(\xi_m(t)), \xi_m'(t)\rangle_{\mathcal{H}(\xi_m(t))}\, dt.\end{aligned} \tag{3.8}$$

Thus there are $t_m \in (0, T_m]$ such that $\xi_m'(t_m)$ exists, and

$$L(\xi_m(t_m), \xi_m'(t_m), h_0^-) < \langle X\phi(\xi_m(t_m)), \xi_m'(t_m)\rangle_{\mathcal{H}(\xi_m(t_m))}. \tag{3.9}$$

This implies that $h_0 \leq H(\xi_m(t_m), X\phi(\xi_m(t_m)))$. Assume that t_m be the largest value of $t \in (0, T_m]$ such that $h_0 \leq H(\xi_m(t), X\phi(\xi_m(t)))$. Then we have $H(\xi_m(t), X\phi(\xi_m(t))) < h_0$

for a.e. $t \in (t_m, T_m]$ and hence

$$\begin{aligned}\phi(x_m) - \phi(\xi_m(t_m)) &= \phi(\xi_m(T_m)) - \phi(\xi_m(t_m)) \\ &= \int_{t_m}^{T_m} \langle X\phi(\xi_m(t)), \xi_m'(t)\rangle_{\mathcal{H}(\xi_m(t))}\, dt \\ &\le \int_{t_m}^{T_m} L(\xi_m(t), \xi_m'(t), h_0^-)\, dt. \end{aligned} \tag{3.10}$$

Therefore, we have

$$\begin{aligned} \int_0^{t_m} L(\xi_m(t), \xi_m'(t), h_0^-)\, dt < \phi(\xi_m(t_m)) - \phi(x_0) \\ H(x_0, X\phi(x_0)) \le H(\xi_m(t_m), X\phi(\xi_m(t_m))). \end{aligned} \tag{3.11}$$

Set $y_m = \xi_m(t_m)$. It is easy to see $y_m \neq x_0$. By proposition 2.2 (4), we can find $c(h_0) > 0$ such that

$$L(\xi_m(t), \xi_m'(t), h_0^-) \ge c(h_0), \text{ for all } t \in [0, t_m]. \tag{3.12}$$

Therefore, (3.11) implies

$$c(h_0) < \frac{\phi(y_m) - \phi(x_0)}{t_m} (= \frac{1}{t_m}\int_0^{t_m} \langle X\phi(\xi_m(t)), \xi_m'(t)\rangle_{\mathcal{H}(\xi_m(t))}\, dt). \tag{3.13}$$

Set $q_m = \frac{y_m - x_0}{t_m}$. Since $\|q_m\| \le 1$, we may assume that there exist $q \in \mathbf{R}^n$, with $\|q\| \le 1$, such that $\lim_{m\to\infty} q_m = q$. Taking m to infinity, (3.13) implies obatin

$$c(h_0) \le \langle X\phi(x_0), P_{\mathcal{H}(x_0)}(q)\rangle_{\mathcal{H}(x_0)}. \tag{3.14}$$

This implies

$$X\phi(x_0) \neq 0,\ P_{\mathcal{H}(x_0)}(q) \neq 0. \tag{3.15}$$

For any $\delta > 0$, it also follows from (3.11) that

$$\begin{aligned} &\max_{\{p\in\mathcal{H}(x_0): H(x_0,p)\le h_0-\delta\}} \langle p, P_{\mathcal{H}(x_0)}(y_m - x_0)\rangle_{\mathcal{H}(x_0)} \\ &\le \int_0^{t_m} \max_{\{p\in\mathcal{H}(x_0): H(x_0,p)\le h_0-\delta\}} \langle p, \xi_m'(t)\rangle_{\mathcal{H}(x_0)}\, dt \\ &\le \int_0^{t_m} L(\xi_m(t), \xi_m'(t), h_0^-)\, dt \\ &< \phi(y_m) - \phi(x_0). \end{aligned} \tag{3.16}$$

Dividing (3.16) by t_m and sending $m \to \infty$, we have

$$\max_{\{p \in \mathcal{H}(x_0): H(x_0,p) \le h_0 - \delta\}} \langle p, P_{\mathcal{H}(x_0)}(q) \rangle_{\mathcal{H}(x_0)} \le \langle X\phi(x_0), P_{\mathcal{H}(x_0)}(q) \rangle_{\mathcal{H}(x_0)}. \tag{3.17}$$

Thus

$$\langle p, P_{\mathcal{H}(x_0)}(q) \rangle_{\mathcal{H}(x_0)} \le \langle X\phi(x_0), P_{\mathcal{H}(x_0)}(q) \rangle_{\mathcal{H}(x_0)} \tag{3.18}$$

holds for any $p \in \mathcal{H}(x_0)$ with $H(x_0, p) < h_0$. Notice that (3.18) remains true for any $p \in C$, where C is the convex set

$$C \equiv \overline{\{p \in \mathcal{H}(x_0) : H(x_0, p) < H(x_0, X\phi(x_0))\}}.$$

Since $H_p(x_0, X\phi(x_0)) \ne 0$, we have $X\phi(x_0) \in C$. Hence,(3.18) implies

$$\langle X\phi(x_0), P_{\mathcal{H}(x_0)}(q) \rangle_{\mathcal{H}(x_0)} = \max_{p \in C} \langle p, P_{\mathcal{H}(x_0)}(q) \rangle_{\mathcal{H}(x_0)}.$$

Therefore, by the Lagrange multiple theorem, we have

$$P_{\mathcal{H}(x_0)}(q) = \lambda H_p(x_0, X\phi(x_0)) \tag{3.19}$$

for some $\lambda > 0$.

Since $H(x, X\phi(x)) \in C^1(B_r(x_0))$, we have

$$\begin{aligned} 0 &\le \frac{H(y_m, X\phi(y_m)) - H(x_0, X\phi(x_0))}{t_m} \\ &= \langle X(H(x, X\phi(x)))|_{x=x_0}, P_{\mathcal{H}(x_0)}(q_m) \rangle_{\mathcal{H}(x_0)} + o(1). \end{aligned} \tag{3.20}$$

Sending $m \to \infty$ and using (3.19) lead to

$$\lambda \mathcal{A}^X[\phi](x_0) = \lambda \langle X(H(x, X\phi(x)))|_{x=x_0}, H_p(x_0, X\phi(x_0)) \rangle_{\mathcal{H}(x_0)} \ge 0.$$

Since $\lambda > 0$, we have $\mathcal{A}^X[\phi](x_0) \ge 0$ and u is a viscosity subsolution of (1.5). Similarly, one can prove that u is also a viscosity supersolution. This completes the proof of theorem 1.4. ■

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