A SHARP BOUND FOR THE GROWTH OF MINIMAL GRAPHS

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Dedicated with gratitude to the memory of Walter Hayman.

Abstract. We consider minimal graphs $u = u(x, y) > 0$ over unbounded domains $D \subset R^2$ bounded by a Jordan arc γ on which $u = 0$. We prove a sort of reverse Phragmén-Lindelöf theorem by showing that if D contains a sector

$$S_\lambda = \{(r, \theta) = \{-\lambda/2 < \theta < \lambda/2\} \quad (\pi < \lambda \leq 2\pi),$$

then the rate of growth is at most $r^{\pi/\lambda}$.

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1. Introduction

Let D be an unbounded plane domain. In this paper we consider the boundary value problem for the minimal surface equation

$$\text{div}\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = 0 \tag{1.1}$$

with

$$u = 0 \quad \text{on } \partial D \quad \text{and } u > 0 \quad \text{in } D. \tag{1.2}$$

We shall study the constraints on growth of nontrivial solutions to (1.1) and (1.2) as determined by the maximum

$$M(r) = \sup\, u(x, y),$$

where the sup is taken over the values $r = \sqrt{x^2 + y^2}$ and $(x, y) \in D$.

The methods of this paper extend the results of [6], where the following is proved.

Theorem A. *Suppose D is a simply connected domain whose boundary is a Jordan arc, and D contains a sector $S_\lambda = \{z : |\arg z| < \lambda/2\}$, with $\pi < \lambda \leq 2\pi$. With $M(r)$ defined as above, if u satisfies (1.1) and (1.2) in D, then there exist positive constants K and R such that*

$$M(r) \leq Kr, \quad r > R. \tag{1.3}$$

As in Theorem A above, throughout this paper we shall use complex notation for convenience.

Results regarding upper and lower bounds for the growth of solutions to (1.1) and (1.2) are rather scarce and fragmented. To begin with, the first relevant theorem in this direction was proved by Nitsche [8, p. 256] who observed that there are no solutions to (1.1) and (1.2) with D being contained in a sector of opening strictly less than π. For domains contained in a half plane, but not contained in any such sector, there are solutions to (1.1) and (1.2) with differing growth rates given in [6].

For angles $\lambda \geq \pi$, in terms of the order ρ of u defined by

$$\rho = \limsup_{r\to\infty} \frac{\log M(r)}{\log r},$$

it follows by using the module estimates of Miklyukov [7] as in [11] that if D omits a sector of opening $2\pi - \alpha$, $(\pi \leq \alpha \leq 2\pi)$, the omitted set in the case $\alpha = 2\pi$ being a line, then the order ρ of any nontrivial solution to (1.1) and (1.2) is at least π/α, More precisely, the results in [11] are phrased in terms of the asymptotic angle β defined as follows.

Let $\Theta(r)$ be the angular measure of the set $D \cap \{|z| = r\}$ and

$$\beta = \limsup_{r\to\infty} \Theta(r).$$

With this definition, the lower bound is given by

Theorem B. *Let D be an unbounded domain whose boundary ∂D is a piecewise differentiable arc, and suppose that u satisfies (1.1) and (1.2). If $\beta \geq \pi$, then $\rho \geq \pi/\beta$.*

Regarding upper bounds, it was conjectured [12] that solutions to (1.1) and (1.2) in general have at most exponential growth, and this is achieved by the horizontal catenoid. In [12] the following is proved.

Theorem C. *If u satisfies (1.1) and (1.2) in a domain D contained in a half plane and bounded by an unbounded Jordan arc, then*

$$Cr \leq M(r) \leq e^{Cr} \quad (r > r_0)$$

for some positive constants C and r_0.

The main result of this paper is the following bound for the order ρ of solutions when D contains a large sector.

Theorem 1.1. *Let D be a simply connected domain whose boundary is a Jordan arc, and D contains a sector $S_\lambda = \{z : |\arg z| < \lambda/2\}$, with $\pi < \lambda \leq 2\pi$. If u satisfies (1.1) and (1.2) in D, then $\rho \leq \pi/\lambda$.*

The examples given in [6] show that the theorem is sharp. Further details regarding those prototypes can be found in [13].

Note that Theorem B and Theorem 1.1 taken together imply that if D is bounded by a piecewise differentiable arc and is asymptotic to a sector S_λ with $\lambda > \pi$, then the order of u will in fact be equal to π/λ.

2. PRELIMINARIES

Let $u(z)$ be a solution to (1.1) and (1.2) over a simply connected domain D. We shall make use of the parametrization of the surface given by u in isothermal coordinates using Weierstrass functions $(x(\zeta), y(\zeta), U(\zeta))$ with ζ in the right half plane H and $U(\zeta) = u(f(\zeta))$, where

$$z = f(\zeta) = x(\zeta) + iy(\zeta), \quad \zeta \in H. \tag{2.1}$$

Then $f(\zeta)$ is univalent and harmonic, and since D is simply connected it can be written in the form

$$f(\zeta) = h(\zeta) + \overline{g(\zeta)} \tag{2.2}$$

where $h(\zeta)$ and $g(\zeta)$ are analytic in H,

$$|h'(\zeta)| > |g'(\zeta)|,$$

and

$$U(\zeta) = 2\Re e\, i \int \sqrt{h'(\zeta)g'(\zeta)}\, d\zeta. \tag{2.3}$$

(cf. [3]).

Now, $U(\zeta)$ is harmonic and in (2.3) can be taken as positive in H and vanishing on ∂H. Thus (cf. [10, p. 151]),

$$U(\zeta) = C\, \Re e\, \zeta,$$

where C is a positive constant. This with (2.3) gives

$$h'(\zeta)g'(\zeta) = -C^2/4.$$

We may reparametrize for convenience, replacing $f(\zeta)$ and $U(\zeta)$ by $f(2\zeta/C)$ and $U(2\zeta/C)$. Continuing to use ζ as the preferred variable, this means we may assume that

$$h'(\zeta)g'(\zeta) = -1 \quad \text{and} \quad U(\zeta) = 2\Re e\, \zeta. \tag{2.4}$$

In order to estimate the function $f(\zeta)$ in (2.2), we shall use the following lemma on quasiconformal mappings from [2] (see [1, Lemma 5.8]).

Lemma A. *Suppose φ is quasiconformal in the plane such that $\varphi(\infty) = \infty$, and the dilatation*

$$\mu(z) = \varphi_{\overline{z}}(z)/\varphi_z(z)$$

satisfies

$$\int_0^{2\pi} |\mu(re^{i\theta})|\, d\theta \to 0 \qquad (r \to \infty). \tag{2.5}$$

Then, in any fixed annulus $A(R) = \{R^{-1} \le |z| \le R\}$ $(R > 1)$,

$$\frac{\varphi(tz)}{\varphi(t)} \to z$$

uniformly in $A(R)$ as $0 < t \to \infty$. In particular,

$$|\varphi(z)| = |z|^{(1+o(1))} \quad (z \to \infty).$$

Remark. For our later applications of Lemma A, note that if $r > 0$ and a and b are in $(0, 2\pi)$, then

$$\frac{\varphi(tre^{ia})}{\varphi(tre^{ib})} = \frac{\varphi(tre^{ia})}{\varphi(t)} \frac{\varphi(t)}{\varphi(tre^{ib})} \to \frac{re^{ia}}{re^{ib}} = e^{i(a-b)}$$

as $t \to \infty$, uniformly in $A(R)$.

At the last stage we shall need a barrier argument based on the following [4, p.827].

Lemma B. Let $u(z)$ be a solution to the minimal surface equation over a domain Ω of the form $S_\lambda \backslash E$ $(\lambda < \pi)$ with $u(z) = 0$ on ∂E and $u(z) \le ax^m + b$ $(0 < m < 1, a,\ b \ge 0)$ for $z = x + iy \in \partial S_\lambda$. Then $u(z) \le ax^m + b$ in Ω.

Proof. Let $T_1 = S_\lambda \cap \{z : \Re e\, z < 1\}$ and $A > 0$. Then, there exists [5, p.322] a solution $V_{1,A}(z)$ to the minimal surface equation over T_1 with values Ax on $\partial S_\lambda \cap \{z : \Re e\, z < 1\}$, and $V_{1,A}(z) \to +\infty$ uniformly if $\Re e\, z \to 1$ and $|\arg z| \le \lambda'/2 < \lambda/2$ in T_1. The dilations $V_{R,A}(z) = RV_{1,A}(z/R)$ have corresponding properties for $T_R = S_\lambda \cap \{z : \Re e\, z < R\}$. Now, by the max/min principle [9, p.94], $V_{R,A}(z) > Ax$ for $z \in S_\lambda \cap \{z : \Re e\, z < R\}$ and $\{V_{R,A}\}$ decreases with R on compact subsets of S_λ. Thus, by the monotone convergence theorem [5, p.329], $V_{R,A} \to V_A$ on S_λ, where V_A is a solution to the minimal surface equation with boundary values Ax in a sector of opening less than π. Therefore [8, p.256], $V_A(z) \equiv Ax$ for $z = x + iy \in S_\lambda$.

Let $u(z)$ be as in the statement of the lemma and take a fixed $x_0 > 0$. Then, for $z \in \partial S_\lambda$,

$$u(z) \le a\left(x_0^m + mx_0^{m-1}(x - x_0)\right) + b. \tag{2.6}$$

Since $u(z) = 0$ on ∂E, it follows that

$$u(z) \le V_{R,A}(z) + B \quad z \in \Omega \cap \{\Re e\, z < R\}, \tag{2.7}$$

where $A = amx_0^{m-1}$, and $B = ax_0^m(1-m)+b$. Letting $R \to \infty$ in (2.7), it then follows that (2.6) holds in Ω. Thus for any $z = x_0 + iy$ in Ω we have

$$u(x_0 + iy) \le ax_0^m + b,$$

and since x_0 was arbitrary, the lemma is proved. □

As a final preliminary lemma, we need the following qualitative growth estimate.

Lemma 2.1. *Let $u(z)$ be a solution to (1.1) and (1.2) over a domain D containing a sector S_λ with $\lambda > \pi$, and $f(\zeta)$, $h(\zeta)$, $g(\zeta)$, and $U(\zeta)$ be as in (2.2) and (2.3) corresponding to $u(z)$. Then, for any proper subsector $S_{\lambda'}$ with $\pi < \lambda' < \lambda$ and $D_{\lambda'} = f^{-1}(S_{\lambda'})$,*

$$h'(\zeta) \to \infty \text{ as } \zeta \to \infty$$

uniformly for $\zeta \in D_{\lambda'}$.

Proof. Let $f(\zeta)$, $U(\zeta)$ be as above. So, $u(f(\zeta)) = U(\zeta) = 2\Re e\,\zeta$.

Let $P_\alpha = \{\zeta : \Re e\ e^{i\alpha} f(\zeta) > 0\}$ $(|\alpha| < \lambda/2 - \pi/2)$ and introduce a new variable $\tilde{\zeta}$, and let $\zeta = \psi_0(\tilde{\zeta})$ be a conformal map from the right half plane $H = \{\tilde{\zeta} : \Re e\,\tilde{\zeta} > 0\}$ onto P_0 with $\psi_0(\infty) = \infty$.

Define

$$\begin{cases} \tilde{f}(\tilde{\zeta}) = f(\psi_0(\tilde{\zeta})) \\ \tilde{g}(\tilde{\zeta}) = g(\psi_0(\tilde{\zeta})) \\ \tilde{h}(\tilde{\zeta}) = h(\psi_0(\tilde{\zeta})) \end{cases}$$

Then $\tilde{f}$ is a harmonic map

$$\tilde{f}(\tilde{\zeta}) = \tilde{h}(\tilde{\zeta}) + \overline{\tilde{g}(\tilde{\zeta})}, \quad (\tilde{\zeta} \in H)$$

satisfying

$$|\tilde{h}'(\tilde{\zeta})| > |\tilde{g}'(\tilde{\zeta})|, \quad (\tilde{\zeta} \in H). \tag{2.8}$$

Note $\tilde{F}(\tilde{\zeta}) = \tilde{h}(\tilde{\zeta}) + \tilde{g}(\tilde{\zeta})$ is an analytic function with the same real part as $\tilde{f}$. Then $\Re e \tilde{F}$ is positive in H and vanishes on ∂H, and therefore (see [10, p. 151])

$$\tilde{F}(\tilde{\zeta}) = k\tilde{\zeta} + ik_0 \implies \tilde{F}'(\tilde{\zeta}) = k, \qquad k > 0,$$

that is,

$$\tilde{h}'(\tilde{\zeta}) + \tilde{g}'(\tilde{\zeta}) = k > 0. \tag{2.9}$$

Now,

$$\tilde{h}'(\tilde{\zeta}) = h'(\psi_0(\tilde{\zeta})) \cdot \psi_0'(\tilde{\zeta}), \tag{2.10}$$

and by (2.4),

$$\tilde{g}'(\tilde{\zeta}) = -\frac{\psi_0'(\tilde{\zeta})}{h'(\psi_0(\tilde{\zeta}))} = -\frac{\psi_0'(\tilde{\zeta})^2}{\tilde{h}'(\tilde{\zeta})}. \tag{2.11}$$

Combining this with (2.9)) we have

$$k = \tilde{h}'(\tilde{\zeta}) - \frac{\psi_0'(\tilde{\zeta})^2}{\tilde{h}'(\tilde{\zeta})}$$

which implies

$$\tilde{h}'(\tilde{\zeta})^2 - k\tilde{h}'(\tilde{\zeta}) - \psi_0'(\tilde{\zeta})^2 = 0.$$

Thus,

$$\tilde{h}'(\tilde{\zeta}) = \frac{k \pm \sqrt{k^2 + 4\psi_0'(\tilde{\zeta})^2}}{2}, \quad \tilde{g}'(\tilde{\zeta}) = \frac{-2\psi_0'(\tilde{\zeta})^2}{k \pm \sqrt{k^2 + 4\psi_0'(\tilde{\zeta})^2}}. \tag{2.12}$$

Since $\psi_0(\tilde{\zeta})$ is a conformal map with $\Re e\, \psi_0(\tilde{\zeta}) > 0$ in H, there exists a real constant $0 \le c < \infty$ such that in any sector $S_\beta = \{\tilde{\zeta} : |\arg \tilde{\zeta}| \le \beta < \pi/2\}$, $\psi_0'(\tilde{\zeta}) \to c$ uniformly as $\tilde{\zeta} \to \infty$ in S_β (see [10, p. 152]). Thus,

$$\tilde{h}'(\tilde{\zeta}) \to \frac{k \pm \sqrt{k^2 + 4c^2}}{2}, \quad \tilde{g}'(\tilde{\zeta}) \to \frac{-2c^2}{k \pm \sqrt{k^2 + 4c^2}}, \qquad \zeta/\tilde{\zeta} \to c. \tag{2.13}$$

If $c > 0$, a simple calculation with (2.13) shows that if the minus sign in (2.13) were to hold, this would contradict (2.8). In case $c = 0$, with a minus sign in (2.13), this would imply that $\tilde{h}'(\tilde{\zeta}) \to 0$. However, (2.8) and (2.9) show that this is not possible. Thus, (2.13) becomes

$$\tilde{h}'(\tilde{\zeta}) \to \frac{k + \sqrt{k^2 + 4c^2}}{2}, \quad \tilde{g}'(\tilde{\zeta}) \to \frac{-2c^2}{k + \sqrt{k^2 + 4c^2}}, \qquad \zeta/\tilde{\zeta} \to c. \tag{2.14}$$

Case 1: $\psi_0'(\tilde{\zeta}) \to c > 0$ as $\tilde{\zeta} \to \infty$ in S_β.

Using (2.14) we have

$$\tilde{h}(\tilde{\zeta}) + \overline{\tilde{g}(\tilde{\zeta})} = \left[k\Re e\, \tilde{\zeta} + i\sqrt{k^2 + 4c^2}\, \Im m\, \tilde{\zeta}\right](1 + o(1)) \tag{2.15}$$

as $\tilde{\zeta} \to \infty$ uniformly in S_β. From this it follows that

$$f(\zeta) = \left[k\Re e\, \zeta/c + i\sqrt{k^2 + 4c^2}\, \Im m\, \zeta/c\right](1 + o(1))) \tag{2.16}$$

uniformly as $\zeta \to \infty$ in proper subsectors of H. Therefore, P_0 is asymptotically the half plane H, in the sense that for any $0 < \delta < \pi$, we have $P_0 \supset S_\delta \cap \{|\zeta| > R\}$ for large R.

By (2.1) and (2.4), the graph of the minimal surface is given parametrically by $(\Re e\, f(\zeta), \Im m\, f(\zeta), 2\Re e\, \zeta)$. Using (2.16) we then have that the surface is asymptotic to a plane as $\zeta \to \infty$ in proper subsectors of H.

Consider now P_α as above with $\alpha \neq 0$ and let $\psi_\alpha(\tilde{\zeta})$ be a conformal map from the right half plane $H = \{\tilde{\zeta} : \Re e\, \tilde{\zeta} > 0\}$ onto P_α with $\psi_\alpha(\infty) = \infty$. In this case we define

$$\left\{ \begin{array}{l} \tilde{f}_\alpha(\tilde{\zeta}) = e^{i\alpha} f(\psi_\alpha(\tilde{\zeta})) \\ \tilde{g}_\alpha(\tilde{\zeta}) = e^{-i\alpha} g(\psi_\alpha(\tilde{\zeta})) \\ \tilde{h}_\alpha(\tilde{\zeta}) = e^{i\alpha} h(\psi_\alpha(\tilde{\zeta})) \end{array} \right. \tag{2.17}$$

Proceding in analogy with (2.9)-(2.16), we have

$$\tilde{h}'_\alpha(\tilde{\zeta})^2 - k_\alpha \tilde{h}'_\alpha(\tilde{\zeta}) - \psi'_\alpha(\tilde{\zeta})^2 = 0 \tag{2.18}$$

and with the principal branch of the square root,

$$\tilde{h}'_\alpha(\tilde{\zeta}) = \frac{k_\alpha + \sqrt{k_\alpha^2 + 4\psi'_\alpha(\tilde{\zeta})^2}}{2}, \quad \tilde{g}'_\alpha(\tilde{\zeta}) = \frac{-2\psi'_\alpha(\tilde{\zeta})^2}{k_\alpha + \sqrt{k_\alpha^2 + 4\psi'_\alpha(\tilde{\zeta})^2}} \tag{2.19}$$

in H, where the minus sign in the roots of (2.18) is eliminated as before.

With $S_\beta = \{\tilde{\zeta} : |\arg \tilde{\zeta}| \leq \beta < \pi/2\}$, again $\psi'_\alpha(\tilde{\zeta}) \to c_\alpha \geq 0$ as $\tilde{\zeta} \to \infty \in S_\beta$. We wish to show that when $c_0 > 0$ then $c_\alpha > 0$.

Suppose that $c_\alpha = 0$. We reflect $\tilde{f}_\alpha$ to the left half $\tilde{\zeta}$ plane and note that the β corresponding to the sectors S_β can approach $\pi/2$. It then follows from (2.19) that Lemma A applies to $\tilde{f}_\alpha(\tilde{\zeta})$. To apply Lemma A, we note that the dilatations in S_β tend to 0, and since $\tilde{f}_\alpha$ is sense preserving, the dilatations are less than 1 outside. So (2.5) applies.

Then for $\epsilon > 0$ (see Remark following Lemma A), the image $\tilde{f}(S_\beta)$ covers $S_{\pi-\varepsilon} \cap \{z : |z| > R\}$ if β is sufficiently close to $\pi/2$ and R sufficiently large. From this and (2.15) it follows that if $Q = Q_{\alpha,\beta} = \tilde{f}(S_\beta) \cap e^{-i\alpha}\tilde{f}_\alpha(S_\beta)$, then for β close to $\pi/2$ and all large R, the intersection $Q \cap \{z : |z| = R\}$ is nonempty.

From the original analysis of P_0, we find that for points $\zeta \in f^{-1}(Q)$, (2.10) and (2.14) imply that $h'(\zeta)$ remains bounded.

On the other hand, from the analysis of P_α, it follows from (2.17) and (2.19) that $g'(\zeta)/h'(\zeta) \to 0$ as $\zeta \to \infty$ and $\zeta \in f^{-1}(Q)$. This with (2.4) implies that $h'(\zeta)$ is unbounded, a contradiction.

Therefore, it must be that $c_\alpha > 0$ also, and as in the case of P_0 above, the graph above $f(P_\alpha)$ must be asymptotically a plane. Since $f(P_0)$ and $f(P_\alpha)$ overlap, these graphs must be asymptotically the same plane, and since $f(P_0) \cup f(P_\alpha)$ extends outside a half plane with $u(z) > 0$, we obtain a contradiction. We conclude that Case 1 cannot occur.

Case 2: $\psi_0'(\tilde\zeta) \to 0$ as $\tilde\zeta \to \infty$ in S_β.

As in Case 1 above, Lemma A can be used to show that for $\epsilon > 0$, the image $\tilde f(S_\beta)$ covers $S_{\pi-\varepsilon} \cap \{z : |z| > R\}$ for large R if β is sufficiently close to π. Using an argument similar to Case 1, we can then deduce that the c_α corresponding to each P_α $(|\alpha| < \lambda/2 - \pi/2)$ must also be 0. Since $S_{\lambda'}$, with $\pi < \lambda' < \lambda$, can be covered by the union of the $\tilde f(S_\beta)$ corresponding to $P_0,\ P_\alpha,\ P_{-\alpha}$ for some $0 < \alpha < \lambda/2 - \pi/2$ and large R, it follows from (2.17), and (2.19) that $g'(\zeta)/h'(\zeta) \to 0$. This with (2.4) implies that $h'(\zeta) \to \infty$ uniformly as $\zeta \to \infty$ with $\zeta \in f^{-1}(S_{\lambda'})$. □

3. Proof of Theorem 1.1

Proof. For fixed λ, let $f_1(\zeta)$ denote the function in (2.1) corresponding to a solution to (1.1) and (1.2) over a domain D containing S_λ. Then for λ' such that $\pi < \lambda' < \lambda$ we define $f_2(\zeta) = \zeta^{\lambda'/\pi} + 1$. Let $\tilde S_{\lambda'} = f_2(H)$ and $\tilde H = f_1^{-1}(\tilde S_{\lambda'})$. Then if $\psi(\zeta)$ is a $1-1$ conformal mapping of H onto $\tilde H$ with $\psi(\infty) = \infty$, it follows that $f_1(\psi(H)) = f_2(H)$ and there exists an orientation preserving homeomorphism $\varphi : H \to H$ with $\varphi(\infty) = \infty$ such that

$$f_1(\psi(\zeta)) = f_2(\varphi(\zeta)), \quad \zeta \in H. \tag{3.1}$$

Differentiating (3.1) with respect to ζ and $\overline{\zeta}$, and using the first equality in (2.4) we obtain

$$\psi'(\zeta)h_1'(\psi(\zeta)) = \varphi_\zeta(\zeta)f_2'(\varphi(\zeta)) \tag{3.2}$$

and

$$-\overline{\frac{\psi'(\zeta)}{h_1'(\psi(\zeta))}} = \varphi_{\overline\zeta}(\zeta)f_2'(\varphi(\zeta)). \tag{3.3}$$

Dividing (3.3) by (3.2) we have

$$\frac{1}{|h_1'(\psi(\zeta))|^2} = \left|\frac{\varphi_{\overline\zeta}(\zeta)}{\varphi_\zeta(\zeta)}\right|. \tag{3.4}$$

Now, $\psi(\zeta) \to \infty$ as $\zeta \to \infty$ in H, so by Lemma 2.1 it follows that the left side of (3.4) tends to 0.

It therefore follows from (3.4) and the fact that φ is a sense preserving differentiable homeomorphism, that φ is quasiconformal in H and that the dilatation of φ satisfies

$$\left|\frac{\varphi_{\overline\zeta}(\zeta)}{\varphi_\zeta(\zeta)}\right| \to 0, \qquad (\zeta \to \infty,\ \ \zeta \in H). \tag{3.5}$$

The mapping φ can then be extended by reflection to a quasiconformal mapping of the complex plane onto the complex plane with (3.5) still in force. As in the proof of

Lemma 2.1, Lemma A is applicable to φ. Further, by the symmetry of the reflection, the conclusion of Lemma A can be improved to

$$\varphi(re^{i\theta}) = r^{(1+o(1))}e^{i(\theta+o(1))}$$

so that

$$f_1(\psi(re^{i\theta})) = f_2(\varphi(re^{i\theta})) = r^{(\lambda'/\pi+o(1))}e^{i(\lambda'\theta/\pi+o(1))}, \quad (\zeta = re^{i\theta} \to \infty, \quad \zeta \in H).$$

From this we see that, given any λ'' such that $\pi < \lambda'' < \lambda'$, there is a proper sector $\Sigma_{\lambda''}$ in H such that $f_1(\psi(\Sigma_{\lambda''}))$ covers $S_{\lambda''} \cap \{|z| > R\}$ for large R. But $\psi(\zeta)$ is a conformal mapping of H into H, so $\psi'(\zeta) \to k$ as $\zeta \to \infty$ in $\Sigma_{\lambda''}$ for some $k \geq 0$ (cf. [10, p. 151]). Combining this with (2.4) we conclude that for sufficiently large z,

$$u(z) < |z|^{(\pi/\lambda'+o(1))}, \quad z \in S_{\lambda''}. \tag{3.6}$$

The boundary of the sector $S_{\lambda''}$ on which (3.6) holds forms an angle in the left half plane of opening less than π. On the boundary of D in the left half plane $u(z) = 0$. Therefore, Lemma B applied to the sector of opening $2\pi - \lambda''$ centered on the negative real axis, with E being the complement of D, tells us that (3.6) holds in $D \backslash S_{\lambda''}$. Thus we see that (3.6) holds on all of D. Since λ' can be taken arbitrarily close to λ in (3.6), the proof is complete. □

References

1. D. Drasin, *On Nevanlinna's inverse problem*, Complex Variables **37** (1999), 123-143
2. D. Drasin, A. Weitsman, *Meromorphic functions with large sums of deficiencies*, Advances in Math. **15** (1974), 93-126.
3. P. Duren, *Harmonic mappings in the plane*, Cambridge Tracts in Mathematics, 2004.
4. J-F Hwang, *Phragmén Lindelöf theorem for the minimal surface equation*, Proc. Amer. Math. Soc. **104** (1988), 825-828.
5. H. Jenkins, j. Serrin, *Variational problems of minimal surface type II. Boundary value problems for the minimal surface equation*, Arch. Rat. Mech. Anal., **21** (1965/66), 321-342.
6. E. Lundberg, A. Weitsman, *On the growth of solutions to the minimal surface equation over domains containing a half plane*, Calc. Var. Partial Differential Equtions **54** (2015), 3385-3395.
7. V. Miklyukov, *Some singularities in the behavior of solutions of equations of minimal surface type in unbounded domains*, Math. USSR Sbornik **44** (1983), 61-73.
8. J.C.C. Nitsche,*On new results in the theory of minimal surfaces*, Bull. Amer. Mat. Soc. **71** (1965), 195-270.
9. R. Osserman, *A survey of minimal surfaces.* Dover Publications Inc. (1986).
10. M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen Co., Ltd., Tokyo (1959).
11. A. Weitsman, *On the growth of minimal graphs*, Indiana Univ. Math. J. **54** (2005), 617-625.
12. A. Weitsman, *Growth of solutions to the minimal surface equation over domains in a half plane*, Communications in Analysis and Geometry **13** (2005), 1077-1087.
13. A. Weitsman, *Level curves of minimal graphs.* Arxiv:2008.10197, To appear in Communications in Analysis and Geometry

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