Circular means of fine Green's functions and the longest arc relation.

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Abstract

An inequality of the first author involving circular means of classical Green's functions is extended to fine Green's functions on fine domains. The inequality leads to a more natural proof of the longest arc relation recently proved by the authors.

Key words: fine topolgy, Green's function, symmetrization

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1 Introduction

Let D be an open set in the plane. We define the Green's function of D with pole at $\zeta \in D$ as follows. Express D as the disjoint union of its countably many components D_i. Let $g_i(z,\zeta)$ be the Green's function of D_i, extending g to be zero if either argument lies outside D_i. We define $G(z,\zeta) = \sum g_i(z,\zeta)$ and note that at most one summand in the sum is not zero. For a fixed ζ, $G(z,\zeta)$ is subharmonic in the plane except at the point ζ, in a neighborhood of which it is superharmonic.

We will say that the open set D does not contain arcs of opening greater than $2l$ if for every $r > 0$ the intersections

$$D \cap \{z : |z| = r\}, \qquad r > 0,$$

do not contain arcs of angular opening greater than $2l$.

In [11] the first author proved

Theorem 1.1 *Let D be an open set which does not contain arcs of opening greater than $2l$ and let D_0 be the angle $\{z : |\arg z| < l\}$. Denote by $G(z,\xi)$ and $G_0(z,\xi)$ their respective Green's functions. Then*

$$\int_0^{2\pi} G(z, Re^{it})\, dt \leq \int_0^{2\pi} G_0(|z|, Re^{it})\, dt \tag{1.1}$$

and

$$\max_t G(z, Re^{it}) \leq \max_t G_0(|z|, Re^{it}). \tag{1.2}$$

Recently [12] the authors applied Theorem 1.1 on components of the open set where $u > 0$ to prove the longest arc relation for continuous δ-subharmonic functions u. The discontinuous case was handled by an *ad-hoc* modification [12, Theorem 1.2]. Since the set where a discontinuous δ-subharmonic function is positive is finely open, it became apparent that to deal effectively with such problems, it would be advantageous to have Theorem 1.1 in the generality of the fine topology. In this paper we shall give such a version of Theorem 1.1 and use it to give a more natural proof of the aforementioned longest arc relation.

The authors are deeply indebted to Bent Fuglede who pointed out useful references concerning the fine topology and gently guided us in the right direction.

2 Fine Potential Theory - Preliminaries

We give some background material on the fine topology. The reader may wish to consult the paper of Eremenko, Fuglede and Sodin [8]. Their notation and use of fine potential theory parallels our own. For a detailed account of the fine topology, we refer the reader to [6, Part 1,Chapter 11]. Much of what follows comes from this source.

The fine topology of classical potential theory in $\mathbf{C}$ is defined as the coarsest topology on $\mathbf{C}$ making every superharmonic function continuous. A set open in this topology is called *finely open.* It follows that a fine neighborhood base about a point ζ is the class of sets of the form

$$\bigcap_1^n \{\eta \in B : u_j < c_j\},$$

where B is a ball containing ζ, u_j is an upper-bounded superharmonic function on B vanishing at ζ and c_j is a strictly positive constant. Thus a point ζ is a fine limit point of a set A if and only if it is a Euclidean limit point of A and if each superharmonic function u defined on an open neighborhood D of A has $u(\zeta)$ as a cluster value at ζ along A. Since u is superharmonic, we already have that $u(\zeta) = \liminf_{D \ni \eta \to \zeta} u(\eta)$. Thus ζ is a fine limit point of A if and only if this limit inferior is attained through values of A.

A set A is said to be *thin* at a point ζ if ζ is not a fine limit point of A. It is fairly easy to prove the following thinness criterion [6, p.168]

Proposition 2.1 *If a set A has a finite (Euclidean) limit point ζ, then A is thin at ζ if and only if there is a superharmonic function u defined on an open neighborhood of ζ such that*

$$\liminf_{A \ni \eta \to \zeta} u(\eta) > u(\zeta).$$

Let us fix a Euclidean open set Ω in the complex plane. We consider all topological and potential theoretic notions relative to Ω. For $E \subset \Omega$, we define the *base* $b(E)$ as the set of all fine limit points of E in Ω. (This is written as E^f in [6].) This set is *finely closed* relative to Ω. The classical definition of a set E being *polar* is equivalent to $b(E)$ being empty, i.e. a polar set consists only of finely isolated points. Also two subsets of Ω, which differ by a polar set, have the same base. As is known [6, p.177], $b(E)$ is a perfect (i.e. $b(b(E)) = b(E)$) Euclidean G_δ-set with the property that $E \setminus b(E)$ is polar.

To generalize Theorem 1.1 to finely open sets, we need a generalization of the classical Green's function. We assume that the open set Ω has a (classical) Green's function, $G(\cdot, \cdot)$. Let $s(z) \geq 0$ be a superharmonic function on Ω and $E \subset \Omega$ be an arbitrary set. The function R_s^E defined for $z \in \Omega$ as

$$R_s^E(z) = \inf\{u(z) : \ u \geq s \text{ on } E; u \geq 0 \text{ and superharmonic on } \Omega\}$$

is called the reduced function of s and its *lower semicontinuous regularization* $\widehat{R}_s^E$ called the *swept-out* function of s.

Let $\mu \geq 0$ be a measure on Ω and $G\mu$ be its Green's potential, that is

$$G\mu(z) = \int G(z, \xi)\, d\mu(\xi) \tag{2.1}$$

The swept-out function $\widehat{R}^E_{G\mu}$ is then the Green's potential of a measure μ^E [6, p.52, p.155]. The measure μ^E is called the *swept-out measure* of μ on E (relative to Ω). Let ε_ζ be the probability measure supported at $\zeta \in \Omega$ (namely the Dirac measure). For $E \subseteq \Omega$, and $\zeta \in \Omega$, the swept-out measure of ε_ζ will be denoted by ε_ζ^E. Classically ([6, p.157]) if E is relatively closed in Ω, $\varepsilon_\zeta^E(F)$ coincides with the harmonic measure of Borel subsets F of $\Omega \cap \partial E$, evaluated at ζ, with respect to the domain $\Omega \setminus E$. Hence if E is finely closed, we can think of ε_ζ^E as a *generalized harmonic measure.* We have the following identity [6, p.160] for any measure $\mu \geq 0$ on Ω and any subsets E and B in Ω, B Borel:

$$\mu^E(B) = \int_\Omega \varepsilon_\zeta^E(B)\, d\mu(\zeta). \tag{2.2}$$

We are now in a position to define the generalized Green's function. (See [8] and [9].) If $D \subset \Omega$ is a finely open set and $E = \Omega \setminus D$, then the *generalized Green's function* for D with pole at ζ is defined as

$$G^D(z,\zeta) = G(z,\zeta) - G\varepsilon_\zeta^E(z) = G\varepsilon_\zeta(z) - \widehat{R}^E_{G\varepsilon_\zeta}(z). \tag{2.3}$$

The function $G^D(\cdot\,,\zeta)$ is non-negative, subharmonic on $\Omega \setminus \{\zeta\}$ and its Riesz measure on $\Omega \setminus \{\zeta\}$ is the (harmonic) measure ε_ζ^E supported on $b(E) \cap \Omega$ [6, p.183]. We have $G^D(\cdot\,,\zeta) = 0$ *quasi-everywhere* on E, that is except on a polar set, and everywhere on E if $E = b(E)$ (that is if E contains no finely isolated points) [6, p 70]. Moreover if D is relatively compact in Ω, then by [6, p.186] and the fact that E is finely closed, ε_ζ^E is supported on the *fine boundary* of D denoted $\partial^f D$. This parallels exactly the classical situation.

If μ is a measure for which $G\mu$ is superharmonic (i.e. $G\mu$ is not identically zero on any component of Ω, then we may define the *generalized Green's potential* $G^D\mu$ by replacing ε_ζ on the right hand side of (2.3) by the measure μ.

It follows from (2.1) that

$$G^D\mu(z) = \int G^D(z,\zeta)\, d\mu(\zeta) \tag{2.4}$$

and $G^D\mu(z) = 0$ on $b(E)$.

One should note, that this fine Green's potential (and the fine Green's function) when extended to be 0 outside $b(E)$, is a δ-subharmonic function on the whole complex plane.

Recall that a function $u(z)$ defined in an open subset G of the complex plane is called δ-subharmonic if it may be represented as a difference of two subharmonic functions

$$u(z) = u_+(z) - u_-(z) \ .$$

This representation is not unique, but among all such representations, there exist representations where u_+ and u_- have no common Riesz mass. In what follows, these representations will be called *canonical*. It is evident that the canonical representations are uniquely defined up to a harmonic summand. Note that u is well defined except on the set where both u^+ and u^- are $+\infty$. This set is polar and hence the set $E = \{z \in \Omega : \, u(z) = 0\}$ differs by a polar set from its fine closure F, that is $b(F) = b(E)$.

3 The fine version of Theorem 1.1

Theorem 3.1 *Theorem 1.1 is true for D a finely open set and G=G^D.*

We have extended the definition of the fine Green's function and the fine Green's potentials to the whole complex plane. Note however that these definitions seemingly depend on the superset Ω. It can be proved that this is not the case. Alternatively, our longest arc hypothesis on the finely open set D in Theorem 3.1 ensures that the complement of D is non-polar. In fact the capacity of the complement of D in the circle of radius R centered at the origin is comparable to R. Hence the theory of Choquet capacitability [13, Chapter 5] guarantees the existence of a classical open set $\Omega \supset D$ which possesses a classical Green's function and whose complement is asymptotically as large as that of D. We keep this set Ω fixed throughout this section and extend our fine Green's functions and potentials to the whole plane by setting them to be zero outside $b(\Omega \setminus D)$.

Define

$$u_1(z) = \int_0^{2\pi} G^D(z, Re^{i\lambda})\, d\lambda,$$

and

$$u_2(z) = G^D(z, 1),$$

and extend them as before to be δ-subharmonic in the plane.

Both these functions are nonnegative and δ-subharmonic in the whole complex plane and vanish on $b(E)$, where $E = \Omega \setminus D$. Since D does not contain arcs of angular opening greater that $2l$ ($l < \pi$), we claim that

$$\operatorname*{ess\,inf}_{\theta \in I} u(re^{i\theta}) = 0 \tag{3.1}$$

for any $r > 0$ and any interval I of length $2l$.

To see this note that $u_i(z) = u_+(z) - u_-(z)$ may be also represented as a monotonically increasing limit of upper semicontinuous δ-subharmonic functions

$$u_\alpha(z) = u_+(z) - (u_-)_\alpha(z)\,, \qquad \alpha \downarrow 0\,,$$

where $(u_-)_\alpha(z)$ is continuous, $(u_-)_\alpha(z) > u_-(z)$ and $(u_-)_\alpha(z) \searrow u_-(z)$. Since u_i vanishes in $b(E)$ and $u_\alpha < u_i$, we have by upper semicontinuity that

$$u_\alpha(z) < 0 \quad z \in N, \tag{3.2}$$

where N is a neighborhood of $b(E)$. Clearly (3.2) implies that if $|z| = r$ does not intersect $E \setminus b(E)$ then

$$\operatorname*{ess\,inf}_{\theta \in I} u_\alpha(re^{i\theta}) \le 0 \tag{3.3}$$

But $E \setminus b(E)$ is a polar set and (3.2) holds in a neighborhood of $b(E)$. Thus (3.3) holds for *every* $r > 0$. Letting α go to zero, gives (3.1).

Let us begin by recalling the method of [11]. For a δ-subharmonic function u defined in an annulus $\{z : |z| \in (r_1, r_2)\}$, let

$$u(z) = u_+(z) - v(z) \tag{3.4}$$

be one of its representations as a difference of two subharmonic functions which may have common Riesz mass. Let

$$S_l = \{z = re^{i\theta} : \theta \in (0, l)\}$$

and let u_l^* be defined in the sector by

$$u_l^*(re^{i\theta}) = \sup\{\int_E u(re^{i\phi})d\phi : E \in \Gamma(\theta, l)\}. \tag{3.5}$$

Here $\Gamma(\theta, l)$ is the family of measurable sets of the real axis satisfying the conditions

(i) $|E| = 2\theta$,
(ii) $\mathrm{diam}(E) \leq 2l$,
where (ii) means that there exists an arc I such that $|I| = 2l$ and $E \subseteq I$. We remark that the supremum in the definition of u_l^* is attained [11].

Proof of Theorem 3.1: Let us first prove (1.1). In (3.4) we may take

$$v(z) = \int_{-\pi}^{\pi} \log|z - Re^{it}|dt. \tag{3.6}$$

We shall then define

$$T_l^*(re^{i\theta}, u_1) = (u_1)_l^*(re^{i\theta}) + \int_{-\theta}^{\theta} v(re^{it})dt \tag{3.7}$$

which is subharmonic is S_l and continuous in its closure [11, p. 513]. We define $T_l^*(re^{i\theta}, G_0)$ similarly, where G_0 is as in the statement of Theorem 1.1.

Now let $\Psi(z) = T_l^*(z, u_1) - T_l^*(z, G_0)$. Then, as in [11], we observe that $\Psi(z)$ is subharmonic in S_l and vanishes on the positive real axis.

By (3.1),

$$\frac{\partial T_l^*(z, u_1)(re^{i\theta})}{\partial\theta} = 0 \tag{3.8}$$

for $\theta = l$ and every r. A calculation shows that the same is true for Ψ.

Now extend the function $\Psi(z)$ onto the domain $S_{2l} = \{z : 0 < \arg z < 2l\}$ setting

$$\Psi(z) = \Psi(z^*)$$

where z^* is symmetric to z relative to the ray $\arg z = l$. The extended function is obviously subharmonic due to the relation (3.8) and vanishes on the edges of the sector S_{2l}.

Thus,

$$\Psi \leq 0 \tag{3.9}$$

in S_l.

Since both sides of (3.9) are 0 when $\theta = 0$, the inequality is preserved when one differentiates with respect to θ and evaluates the derivative at $\theta = 0$. As in [12] we obtain (1.1).

The proof of (1.2) involves a similar analysis on u_2.

4 The longest arc relation

Let u be a δ-subharmonic function in the complex plane. To simplify our further considerations we can assume that $u_+(0) = u_-(0) = 0$. Nevanlinna's functions $N(r,u)$, $m(r,u)$ and $T(r,u)$ are defined by

$$N(r,u) = \frac{1}{2\pi}\int_0^{2\pi} u_-(re^{i\varphi})\,d\varphi\ ,$$

$$m(r,u) = \frac{1}{2\pi}\int_0^{2\pi} \max(u,0)(re^{i\varphi})\,d\varphi\ ,$$

$$T(r,u) = \frac{1}{2\pi}\int_0^{2\pi} \max(u_+,u_-)(re^{i\varphi})\,d\varphi\ .$$

Since the representation $u(z) = u_+(z) - u_-(z)$ is unique up to a harmonic summand, then the characteristic $T(r,u)$ may be defined as

$$T(r,u) = m(r,u) + N(r,u)$$

as well.

The lower order ρ of the function $u(z)$ is defined by

$$\rho = \liminf_{r\to\infty} \frac{\log T(r,u)}{\log r}.$$

Nevanlinna's deficiency of infinity is defined as

$$\delta = \delta(\infty,u) = \liminf_{r\to\infty} \frac{m(r,u)}{T(r,u)} = 1 - \limsup_{r\to\infty} \frac{N(r,u)}{T(r,u)}$$

If $f(z)$ is a meromorphic function defined in the whole complex plane, then the function $u(z) = \log|f(z)|$ is δ-subharmonic and the conventional Nevanlinna characteristic of $f(z)$ coincides with that given above for u.

The celebrated spread relation of A. Baernstein [4] states that if f is a meromorphic function of finite order ρ and positive Nevanlinna deficiency $\delta = \delta(\infty)$, then

$$\limsup_{r\to\infty} |E(r)| \geq \min\left(2\pi, 4\rho^{-1}\arcsin\sqrt{\delta/2}\right). \tag{4.1}$$

Here,

$$E(r) = \{\theta : |f(re^{i\theta})| > 1\}, \tag{4.2}$$

and $|E(r)|$ refers to the angular Lebesgue measure.

In [5], Baernstein proved that if f is entire and we denote the longest arc in the set $E(r)$ by $L(r)$, then (4.1) is true with $|E(r)|$ replaced by $|L(r)|$ (See also [2]).

In [12] we proved the analogue of (4.1) with $E(r)$ replaced by $L(r)$ for $\delta-$subharmonic functions. Namely, we proved

Theorem 4.1 *Let u be a $\delta-$subharmonic function of order ρ. If*

$$L(r) = \text{longest arc of } \{z : \, u(z) > 0\} \cap \{z : \, |z| = r\},$$

then

$$\limsup_{r\to\infty} |L(r)| \geq \min\left(2\pi, 4\rho^{-1} \arcsin\sqrt{\delta/2}\right). \tag{4.3}$$

In the next section, using Theorem 3.1 and the notion of limit functions, we will prove Theorem 4.1, replacing the definition of $L(r)$ by

$$L(r) = \text{longest arc of}\{z : \, u(z) > \varepsilon(z)T(|z|, u)\} \cap \{z : \, |z| = r\}, \tag{4.4}$$

where $\varepsilon(z)$ is an arbitrary positive function approaching zero as $|z|$ approaches ∞.

5 Proof of Theorem 4.1-Preliminaries

We prove this theorem by contradiction. So let l be such that

$$\limsup_{r\to\infty} |L(r)| < 2l < \min\left(2\pi, 4\rho^{-1} \arcsin\sqrt{\delta/2}\right) \tag{5.1}$$

We may assume that

$$u \geq 0, \tag{5.2}$$

since replacing u by $u^+ = \max(u, 0)$ we preserve the left-hand side of inequality (4.3) and by a lemma of Grishin [7] (see also [10]), which says that a nonnegative δ-subharmonic function v has no negative measure on any subset of the level set $\{v = 0\}$, we do not reduce its right-hand side.

It is possible to further reduce the theorem to its so-called limit version. Let $\{r_k\}$ be a Pólya peak sequence of order ρ of the characteristic $T(r,u)$ (c.f. [14]). Consider a sequence of δ-subharmonic functions defined by

$$u_k(z) = \frac{u(r_k z)}{T(r_k,u)} = u_{k+} - u_{k-}$$

Due to the properties of Pólya peaks, the sequences of subharmonic functions u_{k+} and u_{k-} have a common majorant on any compact subset of the complex plane and are bounded from below at the origin. So we can find a subsequence of $\{r_k\}$ which we continue to call $\{r_k\}$ such that both u_{k+} and u_{k-} converge to subharmonic functions u_{0+} and u_{0-} respectively in the distributional sense (cf. [3]).

It is well known that this weak convergence of subharmonic functions provides a uniform convergence of the corresponding characteristics $T(r,u_k)$, $m(r,u_k)$ and $N(r,u_k)$ on any closed interval. [3]. We represent the limit function of the u_k as $u_0 = u_{0-} - u_{0+}$, a possibly non-canonical representation of u_0, since the summands may have common Riesz masses. In what follows there is no loss of generality in considering such a non-canonical representation of a δ-subharmonic function since defining the Nevanlinna deficiency in terms of the ratio $m(r,u)/T(r,u)$ as before, the resulting "deficiency" is no greater than the correct deficiency.

Thus the limit function u_0 of the sequence $\{u_k\}$ satisfies the following properties:

a) $u_0(z) \geq 0, \qquad T(r,u_0) \leq r^\rho, \quad$ and $\quad T(1,u_0) = 1;$

b) $m(r,u_0)/T(r,u_0) \geq \delta$ for some positive δ and every $r > 0$;

c) the essential infimum of the function $u_0(z)$ is zero on any arc of opening at least $2l$ on $|z| = r$.

Properties a) and b) follow from (5.2) and the continuity of the characteristics $T(r,\cdot)$, $m(r,\cdot)$, $N(r,\cdot)$ and the function $T_l^*(z,\cdot)$ with respect to convergence in the distributional sense of the u_k.

To prove c) we note that there exists an R_0 such that ess inf$u_k(z) = 0$ on any arc of opening at least $2l$ on any circle $|z| = r$, $r > R_0/r_k$. (The analysis is similar to that used in obtaining (3.1).)

We first prove that the essential infimum of the limit function on $|z| = r$ is zero. To prove this we recall the definition of the star function $T^*(re^{i\theta})$ introduced by A. Baernstein [5]. Let $u(z) = u_+(z) - u_-(z)$ be a difference of two subharmonic functions. The star function T^* is defined by

$$T^*(re^{i\theta}, u) = u^*(re^{i\theta}) + \int_{-\pi}^{\pi} u_-(re^{i\varphi})\, d\varphi\,, \quad \theta \in (0, \pi)\,,$$

where $u^* := u^*_\pi$. In [5] it is proven as well that $T^*(z, u)$ is a continuous subharmonic function and if u_{k+} and u_{k-} converge weakly to a pair u_{0+} and u_{0-} then the corresponding T^* functions converge uniformly on any compact set contained in the upper half-plane.

According to the definition we have

$$\frac{\partial T^*(re^{i\theta}, u_k)}{\partial \theta} = 0\,, \quad \theta = \pi\,, \quad r \geq R_0/r_k\,.$$

Hence, the function $T^*(z, u_k)$ initially defined on the upper half-plane may be extended by symmetry to be subharmonic on the entire complex plane minus the segment $[-R_0/r_k, 0]$. It is not difficult to see that the function $T^*(z, u_0)$ is subharmonic and symmetric in the whole plane. Thus

$$\frac{\partial T^*(re^{i\theta}, u_0)}{\partial \theta} = 0\,, \quad \theta = \pi\,, \quad r > 0\,,$$

and hence, the essential infimum of the limit function u_0 is equal to zero on $|z| = r$.

To finish the proof of property c), apply the latter argument to the functions $u_k(z) + B_I(z)$ where $B_I(z) = r^\rho h_I(\theta)$. Here $h_I(\theta)$ is a smooth function vanishing on an interval I of opening $2l$ and positive outside I. The functions $u_k(z) + B_I(z)$ defined this way are obviously δ-subharmonic.

Remark. We have just shown that it suffices to prove Theorem 4.1 for delta-subharmonic functions u satisfying properties a)-c).

We need the following theorem involving representations of fine Green's potentials, which may be of interest in its own right.

Theorem 5.1 *Let $u(z)$ be a nonnegative δ-subharmonic function in the plane of order ρ and let D be the finely open set where $u > 0$. If D contains no arcs of length l and $2l < \pi/\rho$ then*

$$u(z) = -\int G_D(z, \xi)\, d\nu(\xi) \tag{5.3}$$

where ν is the signed Riesz measure of the function u in the plane.

Proof. We first note that the integral on the right hand side of (5.3) converges, since u is of finite order ρ and hence,

$$\int_{|\xi|\le r} |\,d\nu(\xi)| = O(r^{\rho+\varepsilon}), \qquad r\to\infty.$$

By Theorem 3.1 and a simple calculation

$$G_D(z,\xi) \le G_0(|z|,|\xi|) = O(|\xi|^{-\pi/(2l)}).$$

Denote by $\tilde{u}$ the function defined by the right hand side of (5.3). Both u and $\tilde{u}$ vanish on $b(C\setminus\Omega)$ and the difference $u-\tilde{u}$ has no Riesz mass in the complement of $b(C\setminus\Omega)$. By Grishin's Lemma a nonnegative δ-subharmonic function can have no negative mass on any set where the function is zero. Therefore, the function $v=(u-\tilde{u})^+$ is *subharmonic* on C.

For quasi-every r, the set $\{z : v(z)>0, |z|=r\}$ does not contain arcs of length $2l$. Then by the arguments used in Section 3, we find that $v_l^*(z)=0$ on the ray $\theta=0$ and its normal derivative vanishes *everywhere* on $\theta=l$. Thus we may extend v_l^* by symmetry to the sector $\{z : \arg(z)\le 2l\}$ with $v_l^*(z)=0$ on $\theta=2l$. The hypothesis that $2l<\pi/\rho$ and the Phragmén - Lindelöf principle then imply that $v_l^*(z)=0$ in $\{z : \arg(z)\le 2l\}$. Thus $v(z)=0$ in the whole complex plane.

Applying the same arguments to the function $\tilde{u}-u$, we have that $u-\tilde{u}$ vanishes identically. The Theorem is proved.

6 Proof of Theorem 4.1-completion

As mentioned before we may assume that our function u satisfies properties a)-c) in section 4. Let D be the set where $u>0$ and let the Riesz measure ν of u have Jordan decomposition $\nu=\nu_+-\nu_-$ and let the fine Green's functions G_D and G_0 be as in Theorem 3.1.

By Theorem 5.1

$$u(z) = -\int G_D(z,\xi)d\nu, \tag{6.1}$$

and thus

$$u(z) \le \int G_D(z,xi)d\nu_-\,. \tag{6.2}$$

Consider the function u_0 defined by

$$u_0(z) = \int G_0(z, |\xi|)\, d\nu_-(\xi) = \int G_0(z,\xi)\, d\nu_0(\xi),$$

where ν_0 is supported on the positive ray and defined by

$$\nu_0\{E\} = \nu_-\{z : |z| \in E\}, \qquad E \in \mathbf{R}^+,$$

Clearly

$$N(r,u) = N(r,u_0)\,.$$

It follows from Theorem 3.1, (6.1), and (6.2) that

$$m(r,u_0) \geq m(r,u).$$

Hence,

$$\frac{m(r,u_0)}{T(r,u_0)} \geq \frac{m(r,u)}{T(r,u)} \geq \delta. \tag{6.3}$$

So the function u_0 satisfies properties b) and c)with the same δ and l. The order of the function does not exceed ρ, since its negative Riesz mass lies on the positive x-axis and u_0 vanishes outside the angle $\{z : \arg z| < l\}$.

Without loss of generality we may suppose that the function u_0 satisfies the condition a) as well, otherwise we may replace it with an appropriate limit function of the sequence $u_0(r_k z)/T(r_k,u_0)$ preserving all the properties of the original function u_0 listed above.

We now define the function

$$v_a(z) = \frac{1}{I(a)} \int_{a^{-1}}^{a} \frac{u_0(tz)}{t^{\rho+1}} dt\ ,$$

where

$$I(a) = \int_{a^{-1}}^{a} \frac{T(t,u_0)}{t^{\rho+1}} dt\ .$$

A simple calculation shows that passing to the limit along some suitable subsequence $a_k \to \infty$ gives us a new function $v(z)$ which is ρ-homogeneous, i.e., satisfies the relation

$$v(rz) = r^\rho v(z) \tag{6.4}$$

for any $r > 0$. Indeed, first note that the function $I(a)$ is a function of slow growth, i.e., it satisfies the relation

$$\lim_{a\to\infty} \frac{I(ra)}{I(a)} = 1 \tag{6.5}$$

for any positive r. (This follows easily from property a)). This relation implies immediately that the sequence v_a, $a \to \infty$, has a weakly convergent subsequence since all the functions are bounded from below at a common point (say the origin) and the corresponding sequence of the Nevanlinna characteristics are bounded from above on any compact set (see for example [1]). Let $a \to \infty$ be a sequence for which v_a tends to some limit function. Note from (6.5) that the sequence v_{ra} has the same limit. Note now that $v_{ra}(z)$ differs from $r^\rho v_a(z/r)$ only by a multiplier which tends to 1 as $a \to \infty$; thus (6.4) is true.

One should note that the limit function $v(z) \not\equiv 0$ is harmonic inside the angles $\arg z \in (-l, 0)$ and $\arg z \in (0, l)$ and vanishes outside these angles. Hence, it may be represented as

$$v(re^{i\theta}) = A_0 r^\rho \sin|\rho\theta - \rho l|, \qquad |\theta| < l.$$

By (6.3) the deficiency δ_0 of the limit function v is greater or equal to δ, but a calculation easily gives that

$$\delta \le \delta_0 = 2\sin^2\left(\frac{\rho}{2}\right).$$

Thus,

$$l \ge \min\left(\pi, 2\rho^{-1} \arcsin\sqrt{\frac{\delta}{2}}\right).$$

The theorem is proven.

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