ON UNIVALENT HARMONIC MAPPINGS AND MINIMAL SURFACES

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I. Introduction. Let $f$ be a univalent harmonic mapping of the unit disk $U$. By this it is meant not only that $f$ is $1-1$ and harmonic, but also that $f$ is sense preserving.

Harmonic univalent mappings were first studied in connection with minimal surfaces by E. Heinz [H]. However, considerable interest in their function theoretic properties, quite apart from this connection, was generated by [CS–S].

Now, the Jacobian of $f(\zeta)$ is $J = |f_\zeta|^2 - |f_\bar{\zeta}|^2$, and $f$ can be written

$$f = h + g$$

where $h$ and $g$ are analytic in $U$. If $a(\zeta)$ is defined by

$$a(\zeta) = \overline{f_\zeta(\zeta)} / f_\zeta(\zeta) = g'(\zeta) / h'(\zeta),$$

then $a(\zeta)$ is analytic and $|a(\zeta)| < 1$ in $U$. We shall refer to $a(\zeta)$ as the analytic dilatation as opposed to the usual dilatation $f_\bar{\zeta}/f_\zeta$ in the theory of quasiconformal mappings.

The case where $a(\zeta)$ is a finite Blaschke product is of special interest since this case arises in taking Fourier series of step functions [S–S]. Their function theoretic properties have been studied in [HS2] as well as in [S–S], and infinite Blaschke products have been considered in [L].

In the present paper we shall study a connection between harmonic mappings and the theory of minimal surfaces, and in §4 we use this to prove a special case of uniqueness for the Riemann mapping theorem of Hengartner and Schober [HS1]. As we have shown elsewhere, uniqueness fails in general [W].
II. Definition of the height function and conjugate height function. Using the Weierstrass representation [O; p. 63] we shall associate with \( f \), a minimal surface given parametrically in a simply connected subdomain \( N \subseteq U \) where \( a(\zeta) \) does not have a zero of odd order.

With \( g \) and \( h \) as in (1.1) we define up to an additive constant, a branch of

\[
F(\zeta) = 2i \int \sqrt{h'(\zeta)g'(\zeta)} \, d\zeta = 2i \int h'(\zeta)\sqrt{a(\zeta)} \, d\zeta = 2i \int f(\zeta)\sqrt{a(\zeta)} \, d\zeta. \tag{2.1}
\]

Then, by (1.2) it follows that a branch of \( F \) can be defined in \( N \), and for \( \zeta \in N \),

\[
\zeta \rightarrow (f(\zeta), \text{Re} \, F(\zeta)) \tag{2.2}
\]
gives a parametric representation of a minimal surface. Here we have identified \( \mathbb{R}^2 \) with \( \mathbb{C} \) by \((x, y) \leftrightarrow (\text{Re} \, f, \text{Im} \, f)\).

Let \( \hat{U} \) be the Riemann surface of the function \( \sqrt{a(\zeta)} \). Then \( \hat{U} \) has algebraic branch points corresponding to those points \( \zeta \in U \) for which \( a(\zeta) \) has a zero of odd order. Specifically, \( \hat{U} \) can be concretely described (the analytic configuration [Sp; 69–74]) in terms of function elements \((\alpha, F_\alpha)\) where \( \alpha \in U \), and \( F_\alpha \) is a power series expansion of a branch of \( F \) in a neighborhood of \( \alpha \) if \( a(\zeta) \) does not have a zero of odd order at \( \zeta = \alpha \), and \( F_\alpha \) a power series in \( \sqrt{\zeta - \alpha} \) otherwise. The mapping \( p: (\alpha, F_\alpha) \rightarrow \alpha \) is the projection of the surface so realized. The mapping \( F \) may now be lifted to a mapping \( \hat{F} \) on \( \hat{U} \).

By continuation, we may induce a mapping \( \hat{U} \rightarrow \hat{U} \) to a surface \( \tilde{U} \) with a real analytic structure defined in terms of elements \((\beta, \tilde{F}_\beta)\) with \( \beta \in f(U) \) by \( \alpha = f^{-1}(\beta) \) and \( \tilde{F}_\beta = F_\alpha \circ f^{-1} \). We again define a projection by \( \pi: (\beta, \tilde{F}_\beta) \rightarrow \beta \).

We shall refer to a point \( \hat{\zeta} \in \hat{U} \) to be over \( \zeta \), if \( p(\hat{\zeta}) = \zeta \), and \( \tilde{z} \in \tilde{U} \) to be over \( z \) if \( \pi(\tilde{z}) = z \).

The harmonic mapping \( f: U \rightarrow f(U) \) lifts to a mapping \( \hat{f}: \hat{U} \rightarrow \hat{U} \) which is \( 1-1 \), onto, and satisfies the condition \( \pi(\hat{f}(\hat{\zeta})) = f(p(\hat{\zeta})) \) for all \( \zeta \in \hat{U} \). With these notations, we shall extend the meaning of (2.2). Thus

\[
\hat{\zeta} \rightarrow (\hat{f}(\hat{\zeta}), \text{Re} \, \hat{F}(\hat{\zeta})) \tag{2.3}
\]
gives a parametric representation of a minimal surface in the sense that in a neighborhood of \( \hat{\zeta} \in \hat{U}\setminus\hat{B} \) where \( B \) is the branch set, that is, the points above the zeros of \( a \) of odd order, then (2.2) is the same as (2.3) computed in terms of local coordinates given by projection.

We may also define the surface nonparametrically on \( \hat{U}\setminus\hat{B} \), where \( \hat{B} = \hat{f}(B) \), as follows. Let \( D \) be an open disk in \( f(U) \) such that \( f^{-1}(D) \) contains no zeros of \( a \) of odd multiplicity. Let \( w = \varphi(x, y) \) be the nonparametric description of the minimal surface corresponding to (2.2), that is, for \( \zeta \in f^{-1}(0) \) (cf. [HS3; p. 87]),

\[
\begin{align*}
    x &= \Re f(\zeta) \\
    y &= \Im f(\zeta), \\
    \varphi(x, y) &= \Re F(\zeta).
\end{align*}
\]

Then, by continuation \( \varphi \) lifts to a function \( \tilde{\varphi} \) on \( \hat{U} \) which satisfies the minimal surface equation when computed in local coordinates given by projection off the branch set \( \hat{B} \). We shall call \( \tilde{\varphi}(\tilde{z}) \) a height function corresponding to \( f \). Finally, we define a conjugate height function \( \tilde{\psi}(z) \) by solving locally

\[
\begin{align*}
    \psi_y &= \varphi_x/W, \\
    \psi_x &= -\varphi_y/W \\
    (W &= \sqrt{1 + \varphi_x^2 + \varphi_y^2})
\end{align*}
\]

(cf. [F1; p. 344]) and lifting to \( \hat{U}\setminus\hat{B} \) as was done for \( \varphi \). Let \( \tilde{F} = \tilde{\varphi} + i\tilde{\psi} \). Then \( \tilde{F} \) is real analytic and locally quasiconformal on \( \hat{U}\setminus\hat{B} \), with dilatation whose magnitude is \( (W - 1)/(W + 1) \). The fact that \( \tilde{\psi} \) and \( \tilde{F} \) are well defined on \( \hat{U}\setminus\hat{B} \) follows from Theorem 1.

A glossary of terminology is given schematically in Figure 1.
Theorem 1. With the above notations, $\hat{F} = \check{F} \circ \hat{f} + C$ for some constant $C$.

Proof. Let $D$ be an open disk in $f(U)$ such that $f^{-1}(D)$ contain no zeros of odd multiplicities of $a$. We fix a branch of $\sqrt{a}$ in $f^{-1}(D)$, and consider $\hat{\phi}(\hat{\zeta}) + i\hat{\psi}(\hat{\zeta}) = \hat{F}(\hat{\zeta})$ for points in a component of $\hat{U}$ over $f^{-1}(D)$, and $\check{\phi}(\check{z}) + i\check{\psi}(\check{z}) = \check{F}(\check{z})$ for points in a component of $\check{U}$ over $D$. Since we shall compute in local coordinates given by projection, to reduce notation in this proof, we shall subsequently write $\hat{F}$, $\hat{\phi}$, $\hat{\psi}$ in place of $\hat{F} \circ p^{-1}$, $\hat{\phi} \circ p^{-1}$, $\hat{\psi} \circ p^{-1}$, and $\check{F}$, $\check{\phi}$, $\check{\psi}$ in place of $\check{F} \circ \pi^{-1}$, $\check{\phi} \circ \pi^{-1}$, $\check{\psi} \circ \pi^{-1}$ respectively. With this notation, by (2.4) we have that

$$\hat{\phi} = \check{\phi} \circ f, \quad (2.6)$$

so it suffices to show that

$$\hat{\psi} = \check{\psi} \circ f + C. \quad (2.7)$$

The result then follows from continuation.

In fact, since $\hat{\phi} + i\hat{\psi}$ is analytic in $f^{-1}(D)$, it follows from (2.6) that to prove (2.7) it suffices to show that $\hat{F} \circ f$ is analytic in $f^{-1}(D)$.

We first record the relationship between $a(\zeta)$ of (1.2) and $W(z)$ ($z = f(\zeta)$) of
This is given by [0; p. 105], [HS3; pp. 87–88] as
\[ |a| = \frac{W-1}{W+1}. \tag{2.8} \]

Now,
\[ (\tilde{F} \circ f)_{\zeta} = \tilde{F}_{\zeta} f_{\zeta} + \tilde{F}_{\zeta} \overline{f_{\zeta}} = \tilde{F}_{\zeta} f_{\zeta} + \tilde{F}_{\zeta} \overline{(f_{\zeta})}. \tag{2.9} \]

A simple computation using (2.5) gives
\[ F_{z} = \frac{W+1}{W} \varphi_{z}, \quad F_{\overline{z}} = \frac{W-1}{W} \varphi_{\overline{z}}. \]

When used in (2.9) these give
\[ (\tilde{F} \circ f)_{\zeta} = \frac{W+1}{W} \varphi_{z} f_{\zeta} + \frac{W-1}{W} \varphi_{\overline{z}} \overline{(f_{\zeta})}. \tag{2.10} \]

Again, a direct computation gives
\[ \varphi_{z} = \frac{\phi_{\zeta}(f_{\zeta}) - \phi_{-\overline{f_{\zeta}}}}{|f_{\zeta}|^2 - |f_{\overline{\zeta}}|^2}, \quad \varphi_{\overline{z}} = \frac{\phi_{\overline{\zeta}} f_{\zeta} - \phi_{\overline{\zeta}} f_{\overline{\zeta}}}{|f_{\zeta}|^2 - |f_{\overline{\zeta}}|^2}. \]

When used in (2.10) this gives
\[ (\tilde{F} \circ f)_{\zeta} = \frac{1}{W(|f_{\zeta}|^2 - |f_{\overline{\zeta}}|^2)} \left( 2 \phi_{\zeta} f_{\zeta} \overline{(f_{\overline{\zeta}})} + \phi_{\overline{\zeta}} f_{\zeta} |f_{\zeta}|^2 (W - 1 - \frac{|f_{\zeta}|^2}{|f_{\overline{\zeta}}|^2} (W + 1)) \right). \tag{2.11} \]

Now, by (1.2), (2.1), and (2.8) we have,
\[ \phi_{\zeta} = ig'/\sqrt{a}, \quad \phi_{\overline{\zeta}} = -i g'/\sqrt{a}, \quad f_{\zeta} = g'/a, \quad f_{\overline{\zeta}} = \overline{g}', \]

and
\[ W - 1 - \frac{|f_{\zeta}|^2}{|f_{\overline{\zeta}}|^2} (W + 1) = W - 1 - |a|^2 (W + 1) = 2(W - 1)/(W + 1). \]

Substituting into (2.11) we obtain
\[ (\tilde{F} \circ f)_{\zeta} = \frac{1}{W(|f_{\zeta}|^2 - |f_{\overline{\zeta}}|^2)} \left( \frac{2ig'\overline{(g')^2}}{\sqrt{a}a} - \frac{2ig'|g'|^2}{\sqrt{a}|a|^2} \left( \frac{W - 1}{W + 1} \right) \right) \]
\[ = 0. \]

Thus, \( \tilde{F} \circ f \) is analytic and (2.7) follows.
III. The height function corresponding to Poisson integrals of step functions. Let \( \mathcal{P} \) be a polygon with vertices \( c_1, \ldots, c_n \) given cyclically, and in order induced by a positive orientation of \( \partial \mathcal{P} \). Let \( f \) be the Poisson integral of a step function on \( \partial U \) having values \( c_1, \ldots, c_n \) and suppose that \( f \) is then a univalent harmonic mapping, \( f: U \to \mathcal{P} \). If \( \mathcal{P} \) is convex, for example, this will always be the case [C], [K]. The analytic dilatation \( a(\zeta) \) for such mappings were studied in [HS2] and [S-S]. In general, \( a(\zeta) \) is a Blaschke product of order at most \( n - 2 \), and of order precisely \( n - 2 \) if \( \mathcal{P} \) is convex [S-S; pp. 469, 473].

We shall now explore the boundary behavior of height functions corresponding to such mappings. The prototype for this is Scherk’s minimal surface over the square \(-\pi/2 < x < \pi/2, -\pi/2 < y < \pi/2\), given by

\[
\psi(x, y) = \log(\cos x / \cos y)
\]

which tends to \(+\infty\) and \(-\infty\) over alternate sides. It seems remarkable that this type of behavior persists in general for height functions corresponding to all such \( f \) described above.

**Theorem 2.** Let \( \mathcal{P} \) be a polygon having vertices \( c_1, \ldots, c_n \) given cyclically, and ordered by a positive orientation on \( \partial \mathcal{P} \). Let \( f \) be a univalent harmonic mapping of \( U \) such that \( f \) is the Poisson integral of a step function having the ordered sequence \( c_1, \ldots, c_n \) as its values. Then the analytic dilatation \( a(\zeta) \) of \( f \) is a finite Blaschke product of order at most \( n - 2 \), \( f(U) = \mathcal{P} \), and if \( \varphi \) is a height function for \( f \), then \( \varphi \) tends to \(+\infty\) or \(-\infty\) at points over the open segments making up the sides of \( \mathcal{P} \). If \( \mathcal{P} \) is convex, then \(+\infty\) and \(-\infty\) alternate on adjacent sides.

**Proof.** That \( a(\zeta) \) is a Blaschke product of order at most \( n - 2 \) and \( f(U) = \mathcal{P} \) follow from general properties of Poisson integrals [S–S; p. 469], [HS2; p. 203].

Let \( f = h + \overline{g} \) as in (1.1). Then we may write \( h' \) and \( g' \) in the form [S–S; pp. 460–461]

\[
h'(\zeta) = \sum_{k=1}^{n} \frac{\alpha_k}{\zeta - \zeta_k}, \quad g'(\zeta) = -\sum_{k=1}^{n} \frac{\bar{\alpha}_k}{\zeta - \bar{\zeta}_k},
\]
where $\alpha_k \neq 0, \ k = 1, \ldots, n$.

With $F$ as in (2.1), we are then interested in the branches of

$$F(\zeta) = 2i \int \sqrt[n]{\sum_{k=1}^{n} \frac{\alpha_k}{\zeta - \zeta_k} \sum_{k=1}^{n} \frac{-\alpha_k}{\zeta - \zeta_k}} \, d\zeta \quad (3.2)$$

as $\zeta \to \zeta_k, \ k = 1, \ldots, n$. The cluster sets for the nontangential approaches to points over the $\zeta_k$ give the points lying over the open segments making up the sides of $\mathcal{P}$.

Thus, take a vertex $\zeta_j$, and an open segment $l_j$ of $\partial \mathcal{P}$ corresponding to it. Then, as $\zeta \to \zeta_j$,

$$\sum_{k=1}^{n} \frac{\alpha_k}{\zeta - \zeta_k} \sum_{k=1}^{n} \frac{-\alpha_k}{\zeta - \zeta_k} = -\frac{|\alpha_j|^2}{(\zeta - \zeta_j)^2}(1 + o(1)),$$

and hence, by (3.2), a branch of $F$ satisfies

$$F(\zeta) = \pm 2|\alpha_j| \log(\zeta - \zeta_j) + o(1) \quad (3.3)$$

as $\zeta \to \zeta_j$, for a fixed branch of the log. Suppose the fixed branch of (3.3) has minus sign, and let $\phi(z) = \text{Re} \ F \circ f^{-1}(z)$ be a corresponding branch in $\mathcal{P}$ for points near the corresponding side $l_j$. Now suppose $\mathcal{P}$ is convex and $F(\zeta)$ is analytically continued to an adjacent point, say $\zeta_{j+1}$, so that $\phi$ is then continued to a corresponding side $l_{j+1}$ having common endpoint $c_j$ with $l_j$. Since $\phi \to -\infty$ as $z \to l_j$, it remains to show that $\phi \to +\infty$ as $z \to l_{j+1}$. This effect has been noted for minimal surfaces [JS], and can be accomplished by a simple barrier argument. I thank Professor Finn for pointing this out.

Let $0 < \beta < \pi$ be the angle in $\mathcal{P}$ between $l_j$ and $l_{j+1}$. Suppose that $\phi \to -\infty$ on both open segments $l_j$ and $l_{j+1}$. Since $\phi$ satisfies the minimal surface equation, $\phi$ can only tend to $-\infty$ over line segments [O; p. 102]. Since we make no assumption at the common endpoint $c_j$, in order to get a contradiction we must show that $\phi \to -\infty$ at $c_j$ as well. We may assume that $c_j = (\pi/2, 0)$, and $l_j, l_{j+1}$ make the angle $\beta$ symmetrically with respect to the $x$ axis, opening toward the origin. Let $0 < \varepsilon < (\pi/2) \cot(\beta/2)$ be small enough so that the isosceles triangle $N$ formed by
the sector and the line \( x = \pi/2 - \varepsilon \) has the given branch of \( F \) single valued. Then, two of the sides of \( N \) are contained in the segments \( l_j \) and \( l_{j+1} \), and the third is \( x = \pi/2 - \varepsilon, -\delta < y < \delta \), where \( \delta = \varepsilon \tan(\beta/2) \). If \( \psi \) is the height function for Scherk’s surface given by (3.1), then for any \( M > 0 \), clearly

\[
\phi(x, y) < -\psi(x - \pi + \varepsilon, y) - M \tag{3.4}
\]

on \( \partial N \setminus \{c_j\} \). By the extended maximum principle [F1; pp. 342-343], it follows that (3.4) holds throughout \( N \). Since \( M > 0 \) was arbitrary, it follows that \( \phi \equiv -\infty \) on \( N \), a contradiction. Thus \( \phi = +\infty \) on \( l_{j+1} \).

**IV. An application to the Riemann mapping theorem.** One of the most basic results in the theory of univalent harmonic mappings is the Riemann mapping theorem of Hengartner and Schober [HS1].

**Theorem A.** Let \( D \) be a bounded simply connected domain whose boundary is locally connected. Fix \( w_0 \in D \), and let \( a(\zeta) \) be analytic in \( U \), with \( a(U) \subseteq U \). Then there exists a univalent harmonic mapping \( f \) with the following properties.

a) \( f \) maps \( U \) into \( D \) and \( f(0) = w_0, f_z(0) > 0 \).

b) \( f \) satisfies the equation \( \overline{(f \zeta)} = af \zeta \).

c) Except for a countable set \( E \subseteq \partial U \), the unrestricted limit \( f^*(e^{it}) = \lim_{\zeta \to e^{it}} f(\zeta) \) exists and belongs to \( \partial D \).

d) The one sided limits \( \lim_{\tau \to t^+} f^*(e^{i\tau}) \) and \( \lim_{\tau \to t^-} f^*(e^{i\tau}) \) through values of \( e^{i\tau} \not\in E \) exist and belong to \( \partial D \); for \( e^{i\tau} \not\in E \) they are equal and for \( e^{i\tau} \in E \) they are different.

e) The cluster set of \( f \) at \( e^{it} \in E \) is the straight line segment joining the left and right limits in d).

If in Theorem A, the set \( D \) is convex, and \( a(\zeta) \) is a finite Blaschke product, one can say more [HS2; p. 203], [S–S; p. 473].
**Theorem B.** Let $f$ be as in Theorem A with $D$ bounded and convex, and $a(\zeta)$ a Blaschke product of order $n-2$. Then $f(U)$ is a polygon with $n$ vertices all of which lie on $\partial D$.

We shall prove uniqueness in the case $a(\zeta) = \zeta^n$ and $D$ convex. The case of uniqueness when $D = U$ and $a(\zeta) = \zeta$ was done in [HS2; p. 204].

The proof involves a combinatorial argument with the level sets of the height function. Such arguments are often useful in the theory of partial differential equation, and in particular the minimal surface equation [F1], [FO], [JS], [Se].

**Theorem 3.** The solution $f(\zeta)$ to the Riemann mapping theorem above with $D$ convex and

$$a(\zeta) = \zeta^{n-2} \quad (4.1)$$

is unique for each $n = 3, 4, \ldots$

*Proof.* Let $f_1$ and $f_2$ be Riemann mappings corresponding to $D$. We may assume $f_1(0) = f_2(0) = 0$. Let $\Delta$ be a disk centered at 0, and contained in $f_1(U) \cap f_2(U)$.

If $n$ is even, then $\tilde{U} = U$ and if $n$ is odd $\tilde{U}$ is a two sheeted cover of $U$ with branch point over 0. Similarly, if $\tilde{U}_1$ corresponds to $f_1(U)$ and $\tilde{U}_2$ to $f(U_2)$, then $\tilde{U}_1$ and $\tilde{U}_2$ are one or two sheeted according as $n$ is even or odd. We consider the case where $n$ is odd. The even case goes the same way, but is simpler since one can bypass discussion of Riemann surfaces.

Let $\varphi_j, \psi_j, \tilde{\varphi}_j, \tilde{\psi}_j, \tilde{F}_j, \tilde{U}_j, \pi_j$, $j = 1, 2$ be the quantities of §2 defined for $f_1$ and $f_2$ respectively. We may assume that $\tilde{F}_1(\tilde{0}) = \tilde{F}_2(\tilde{0}) = 0$. If $\tilde{\Delta}$ represents the Riemann surface of $\sqrt{z}$ over $\Delta$, then we may consider $\tilde{\Delta} \subseteq \tilde{U}_1$ and $\tilde{\Delta} \subseteq \tilde{U}_2$, so that $\tilde{F}_1$ and $\tilde{F}_2$ may both be considered as defined for all $\tilde{z} \in \tilde{\Delta}$. For brevity of notation, we shall write $\tilde{F}$ for $\tilde{F} \circ \pi^{-1}$.

Since the analytic dilatation for $f_1(\zeta)$ and $f_2(\zeta)$ is 0 when $\zeta = 0$, it follows from (1.2), (4.1), and a) of Theorem A, that

$$f_j(\zeta) = c_j \zeta (1 + o(1)) \quad (\zeta \to 0, \ c_j > 0, \ j = 1, 2). \quad (4.2)$$
Then, from (2.1), (4.1), (4.2), and Theorem 1 we may take determinations of $\tilde{F}_1$ and $\tilde{F}_2$ in $\tilde{\Delta}$ so that

$$\tilde{\varphi}_j(z) + i\tilde{\psi}_j(z) = \tilde{F}_j(z) = d_j z^{n/2} (1 + o(1)) \quad (j = 1, 2 \quad z \to 0)$$

(4.3)

with $d_1, d_2 > 0$ and $z^{n/2}$ is some fixed branch.

Having thus fixed branches in (4.3) we may then take a constant $\lambda > 0$ such that

$$\tilde{F}_1(z) - \lambda \tilde{F}_2(z/\lambda) = C z^{p+2} (1 + o(1)) \quad (z \to 0)$$

(4.4)

for some constant $C$ and integer $p \geq n$. We suppose $\lambda \geq 1$; otherwise we interchange $\tilde{F}_1$ and $\tilde{F}_2$. Now, the change from $F(z)$ to $\lambda F(z/\lambda)$ corresponds to replacing $f$ by $\lambda f$. Then the analytic dilatation is unchanged, and following the change in (2.1) it gives the parametrization $\zeta \to (\lambda f(\zeta), \text{Re} \lambda F(\zeta))$.

Let $\varphi_3, \psi_3, \tilde{\varphi}_3, \tilde{\psi}_3$ correspond to $f_3 = \lambda f_2$ so that $f_3(U)$, is nothing more than $f_1(U)$ dilated by the constant $\lambda \geq 1$, and (4.5) becomes

$$\tilde{F}_1(\tilde{z}) - \tilde{F}_3(\tilde{z}) = C z^{n+2} (1 + o(1)) \quad (z \to 0).$$

(4.5)

Case 1. $C = 0$ for every $p$. Since $\tilde{F}_1(z^2) - \tilde{F}_3(z^2)$ is real analytic, then $\tilde{F}_1 \equiv \tilde{F}_3$.

Thus, in particular $\lambda = 1$ and $f_1(U) = f_3(U) = P$. In order to show that $f_1 \equiv f_3$ we use the subordination principle of [BHH; p. 170]. Briefly, since $P$ is a convex polygon by Theorem B, and $(f_1)_z(0), (f_3)_z(0) > 0$, we may apply the argument principle in [BHH; p. 170] to

$$G(z) = (f_3)_z(0)f_1(z) - (f_1)_z(0)f_3(z)$$

to deduce that $(f_1)_z(0) = (f_3)_z(0)$. Then, another application of the argument principle as in [BHH] to $G_\varepsilon(z) = (1 + \varepsilon)f_1(z) - f_3(z) \quad (\varepsilon \to 0)$ shows that $f_1 \equiv f_3$.

Case 2. $C \neq 0$ for some $p \geq n$. In this case, near the origin on $\tilde{\Delta}$, by (4.5) there are $2p + 4$ level curves $\tilde{\varphi}_1 - \tilde{\varphi}_3 = 0$ emanating from $0$. Between the level curves, $\tilde{\varphi}_1 - \tilde{\varphi}_3$ alternates in sign. In order to analyze the component sets between the level sets, we must modify $f_3$. 
Let $\eta_1, \eta_2, \ldots$ be homeomorphisms of $|\zeta| = 1$ onto the boundary of $\lambda D$, which converge to the (step function) boundary values of $f_3$, and let $f_3^{(n)}$, $n = 1, 2, \ldots$ their corresponding Poisson integrals so that $f_3^{(n)} \to f_3$ uniformly on compact subsets of $U$.

The level sets of $\tilde{\varphi}_1 - \tilde{\varphi}_3 = 0$ create $2p + 4$ disjoint component open sets $O_1, O_2, \ldots, O_{2p+4}$ where $\tilde{\varphi}_1 - \tilde{\varphi}_3 > 0$ in $O_{2j-1}$ and $\tilde{\varphi}_1 - \tilde{\varphi}_3 < 0$ in $O_{2j}$ for $j = 1, \ldots, p + 2$. These components alternate in position around the origin.

For $\varepsilon > 0$ we can find nonempty components at $O_1(\varepsilon), O_2(\varepsilon), \ldots, O_{2p+4}(\varepsilon)$ where $\tilde{\varphi}_1 - \tilde{\varphi}_3^{(n)} > \varepsilon$ in $O_{2j-1}(\varepsilon)$, $\tilde{\varphi} - \tilde{\varphi}_3^{(n)} = \varepsilon$ on $\partial O_{2j-1}(\varepsilon)$, $\tilde{\Delta} \cap O_{2j-1}(\varepsilon) \subseteq O_{2j-1}$, $j = 1, \ldots, 2p$, and analogous statements hold for $O_{2j}(\varepsilon)$, $j = 1, \ldots, p + 2$.

Now, $f_3^{(j)}(U) = \lambda D$, so by the maximum principle for solutions to the minimal surface equation, the level sets forming the boundaries of the $O_j(\varepsilon)$'s must extend to points over the boundary of $P = f_1(U)$. As in [FO; pp. 357-358], we observe that since $\tilde{F}_1$ is $\pm \infty$ over the sides of $P$ by Theorem 2, if a component $O_j(\varepsilon)$ has a boundary point over an interior point of a side of $P$, then the boundary must contain that side. Since, by Theorem B, $P$ has $n$ sides, then $\tilde{P} = \pi_1^{-1}(P)$ has $2n$ sides. This implies that there are at most $2n$ sets $O_j(\varepsilon)$ whose boundaries have interior points over $\partial P$. If $O_j(\varepsilon)$ were a component whose boundary contained no points over $\partial P$, then its boundary could only be interior points over $P$, or vertices. As pointed out in [FO; p. 358], this is impossible by a theorem of Finn [F1; pp. 342-343]. Thus, $2p + 4 \leq 2n$. Since $p \geq n$, we obtain a contradiction and the theorem is proved.
References


