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On the Poisson Integral of Step Functions and Minimal Surfaces

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Abstract. Applications of minimal surface methods are made to obtain information about univalent harmonic mappings. In the case where the mapping arises as the Poisson integral of a step function, lower bounds for the number of zeros of the dilatation are obtained in terms of the geometry of the image.

1 Introduction

Let f be a univalent harmonic mapping of the unit disk U. By this it is meant not only that f is $1-1$ and harmonic, but also that f is sense preserving. Then f can be written

$$f = h + \bar{g} \tag{1.1}$$

where h and g are analytic in U. If $a(\zeta)$ is defined by

$$a(\zeta) = \overline{f_{\bar{\zeta}}(\zeta)}/f_{\zeta}(\zeta) = g'(\zeta)/h'(\zeta), \tag{1.2}$$

then $a(\zeta)$ is analytic and $|a(\zeta)| < 1$ in U. We shall refer to $a(\zeta)$ as the *analytic dilatation* of f. General function theoretic properties of univalent harmonic mappings may be found in [CS-S]. The case where $a(\zeta)$ is a finite Blaschke product is of special interest since this case arises in taking the Poisson integrals of step functions [S-S]. This connection has been studied in [HS] and [S-S]. In [W], a method was developed using the theory of minimal surfaces to study univalent harmonic mappings. We shall continue this study.

In Section 2 we shall review the definitions of the height function and conjugate height function introduced in [W], along with their relevant properties. In Section 3 we shall prove the comparison principle for the height function. In Section 4 we collect some results from the theory of minimal surfaces which enable us to use the conjugate height function as a combinatorial tool. In Section 5 we give some applications to the Poisson integrals of step functions.

2 The Height Function and Conjugate Height Function

Using the Weierstrass representation [O, p. 63], we associate with f a minimal surface given parametrically in a simply connected subdomain $N \subseteq U$ where $a(\zeta)$ does not have a zero of odd order.

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With g and h as in (1.1) we define up to an additive constant, a branch of

(2.1)
$$F(\zeta) = 2i \int \sqrt{h'(\zeta)g'(\zeta)}\, d\zeta = 2i \int h'(\zeta)\sqrt{a(\zeta)}\, d\zeta = 2i \int f_\zeta(\zeta)\sqrt{a(\zeta)}\, d\zeta.$$

Then, by (1.2) it follows that a branch of F can be defined in N, and for $\zeta \in N$,

$$\zeta \to \big(f(\zeta), \operatorname{Re} F(\zeta)\big) \tag{2.2}$$

gives a parametric representation of a minimal surface. Here we have identified $\mathbb{R}^2$ with $\mathbb{C}$ by $(x, y) \leftrightarrow (\operatorname{Re} f, \operatorname{Im} f)$.

Let $\hat{U}$ be the Riemann surface of the function $\sqrt{a(\zeta)}$. Then $\hat{U}$ has algebraic branch points corresponding to those points $\zeta \in U$ for which $a(\zeta)$ has a zero of odd order. Specifically, $\hat{U}$ can be concretely described (the *analytic configuration* [S, pp. 69–74]) in terms of function elements (α, F_α) where $\alpha \in U$, and F_α is a power series expansion of a branch of F in a neighborhood of α if $a(\zeta)$ does not have a zero of odd order at $\zeta = \alpha$, and F_α a power series in $\sqrt{\zeta - \alpha}$ otherwise. The mapping $p\colon (\alpha, F_\alpha) \to \alpha$ is the *projection* of the surface so realized. The mapping F may now be lifted to a mapping $\hat{F}$ on $\hat{U}$.

By continuation, we may induce a mapping $\hat{U} \to \tilde{U}$ to a surface $\tilde{U}$ with a real analytic structure defined in terms of elements $(\beta, \tilde{F}_\beta)$ with $\beta \in f(U)$ by $\alpha = f^{-1}(\beta)$ and $\tilde{F}_\beta = F_\alpha \circ f^{-1}$. We again define a projection by $\pi\colon (\beta, \tilde{F}_\beta) \to \beta$.

We refer to a point $\hat{\zeta} \in \hat{U}$ to be over ζ, if $p(\hat{\zeta}) = \zeta$, and $\tilde{z} \in \tilde{U}$ to be over z if $\pi(\tilde{z}) = z$.

The harmonic mapping $f\colon U \to f(U)$ lifts to a mapping $\hat{f}\colon \hat{U} \to \tilde{U}$ which is $1-1$, onto, and satisfies the condition $\pi\big(\hat{f}(\hat{\zeta})\big) = f\big(p(\hat{\zeta})\big)$ for all $\zeta \in \hat{U}$. With these notations, we extend the meaning of (2.2). Thus

$$\hat{\zeta} \to \big(\hat{f}(\hat{\zeta}), \operatorname{Re} \hat{F}(\hat{\zeta})\big) \tag{2.3}$$

gives a parametric representation of a minimal surface in the sense that in a neighborhood of $\hat{\zeta} \in \hat{U} \setminus \mathcal{B}$ where $\mathcal{B}$ is the branch set, that is, the points above the zeros of a of odd order, then (2.2) is the same as (2.3) computed in terms of local coordinates given by projection.

We may also define the surface nonparametrically on $\tilde{U} \setminus \tilde{\mathcal{B}}$, where $\tilde{\mathcal{B}} = \hat{f}(\mathcal{B})$, as follows. Let D be an open disk in $f(U)$ such that $f^{-1}(D)$ contains no zeros of a of odd multiplicity. Let $w = \varphi(x, y)$ be the nonparametric description of the minimal surface corresponding to (2.2), that is, for $\zeta \in f^{-1}(0)$ (*cf.* [HS3, p. 87]),

$$\begin{gathered} x = \operatorname{Re} f(\zeta) \quad y = \operatorname{Im} f(\zeta), \\ \varphi(x, y) = \operatorname{Re} F(\zeta). \end{gathered} \tag{2.4}$$

Then, by continuation φ lifts to a function $\tilde{\varphi}$ on $\tilde{U}$ which satisfies the minimal surface equation when computed in local coordinates given by projection off the branch set $\tilde{\mathcal{B}}$. We call $\tilde{\varphi}(\tilde{z})$ a *height function* corresponding to f. We define a *conjugate height function* $\tilde{\psi}(z)$ by solving locally

$$\psi_y = \varphi_x/W, \psi_x = -\varphi_y/W \quad \left(W = \sqrt{1 + \varphi_x^2 + \varphi_y^2}\right) \tag{2.5}$$

(*cf.* [JS, p. 326]), and lifting to $\tilde{U} \setminus \tilde{\mathcal{B}}$ as was done for φ. Let $\tilde{F} = \tilde{\varphi} + i\tilde{\psi}$. Then, as shown in [W], $\tilde{F} = \hat{F} + c$ is well defined on $\tilde{U} \setminus \tilde{B}$. Finally, we extend $\hat{F}$ and $\tilde{F}$ to $\hat{U}$ and $\tilde{U}$ respectively by continuity to the branch points.

A glossary of terminology is given schematically in Figure 1.

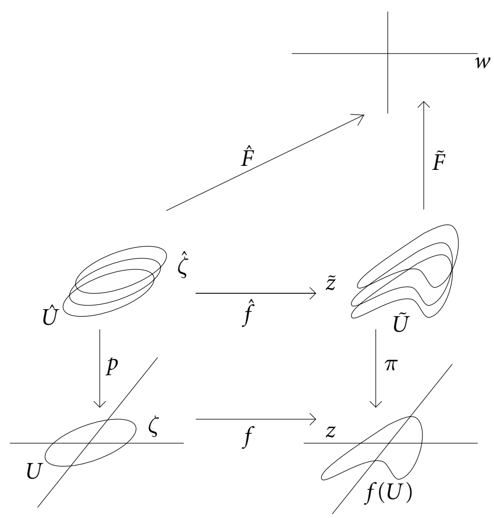

Figure 1

3 The Comparison Principle for the Height Function

In this section we point out that the comparison principle for solutions to the minimal surface equation

$$\operatorname{div} \frac{\nabla u}{W} = 0 \quad \left(W = \sqrt{1 + |\nabla u|^2}\right) \tag{3.1}$$

carries over to the height function corresponding to univalent harmonic mappings.

Let f_1 and f_2 be univalent harmonic mapping in U, and suppose $U_0 \subseteq f_1(U) \cap f_2(U) \neq \phi$, where U_0 is open, connected and bounded. Let $z_0 \in U_0$ not be the image of a branch point of f_1 or f_2, and in a neighborhood N of z_0 define $\varphi_1(x, y)$ and $\varphi_2(x, y)$ corresponding to f_1 and f_2 respectively, as is done for $\varphi(x, y)$ in (2.4). Let

$\Phi(x, y) = \varphi_1(x, y) - \varphi_2(x, y)$. We now consider continuations of Φ along all curves in U_0 emanating from N.

Theorem 1 *Suppose that* $\limsup \Phi \leq 0$ *for any continuation along curves from N in U_0, tending to points on ∂U_0. Then, the supremum M of any continuation of Φ along curves in U_0 also satisfies $M \leq 0$.*

Proof We may assume that f_1 and f_2 are univalent harmonic in $\bar{U}$ by considering $f_1(rz)$ and $f_2(rz)$ where $r < 1$, and letting $r \to 1$. Thus we may assume the analytic dilatations for f_1 and f_2 have finitely many zeros, so their height functions have only a finite number of different function elements over each point. Let $z = z(t), 0 \leq t < 1$, $z(0) = z_0$ be a curve along which the continuation of Φ tends to its supremum on some sequence. We assume, contrary to the theorem, that M is positive. By hypothesis there exists $z^* \in U_0$ such that $z(t_k) \to z^*$ and $\Phi\big(z(t_k)\big) \to M$ for some sequence $t_k \to 1$. In order to analyze this, we consider the individual continuations of φ_1 and φ_2 along the curve and recall that φ_1 and φ_2 have only a finite number of function elements over each point of U_0. Thus, at least for some subsequence $t_{k_n} \to 1$, the continuation of Φ corresponds to fixed function elements of φ_1 and φ_2 over z^*. For these branches we then consider $\Phi = \varphi_1 - \varphi_2$, and show that this determination of Φ cannot have a negative relative maximum at z^*. The theorem will then follow.

If φ_1 and φ_2 do not have a branch point at z^*, then the result follows from the usual comparison principle for solutions to (2.1) (*cf.* [O, p. 91]).

If φ_1 or φ_2 do have a branch point at z^*, then we analyze Φ on a two sheeted surface $\tilde{D}$ over a disk $D = \{|z - z^*| < \rho$ where we have assumed ρ is small enough so that z^* is the only branch point in D, and $\bar{D} \subseteq U_0$.

Following Nitsche [N], let m and ϵ be positive constants, with m large and ϵ small. In particular $\epsilon < \rho$. On $\tilde{D}$ define

$$\tau = \begin{cases} m - \epsilon & \text{if } \varphi_1 - \varphi_2 \geq m \\ \Phi - \epsilon & \text{if } \epsilon < \varphi_1 - \varphi_2 < m \\ 0 & \text{if } \varphi_1 - \varphi_1 \leq \epsilon \end{cases}. \tag{3.2}$$

Let $\tilde{C}_\epsilon = \partial \tilde{D}_\epsilon$ be oriented positively, where $\tilde{D}_\epsilon$ is the subset of $\tilde{D}$ over $D \setminus \{|z - z^*| < \epsilon\}$. Then, using coordinates given by projection on $\tilde{D}$ and W_1, W_2 as defined in (3.1) for φ_1 and φ_2 in place of u, we have by (3.1) and the divergence theorem that

$$\oint_{\tilde{C}_\epsilon} \tau\Big(\frac{\nabla\varphi_1}{W_1} \cdot \nu - \frac{\nabla\varphi_2}{W_2} \cdot \nu\Big)\, ds = \iint_{\tilde{D}_\epsilon} \nabla\tau \cdot \Big(\frac{\nabla\varphi_1}{W_1} - \frac{\nabla\varphi_2}{W_2}\Big)\, dA, \tag{3.3}$$

where ν is the outward unit normal.

It follows from (3.3) that

$$8\pi m\epsilon \geq \iint_{\tilde{D}_\epsilon} \nabla\tau \cdot \Big(\frac{\nabla\varphi_1}{W_1} - \frac{\nabla\varphi_2}{W_2}\Big)\, dA \geq 0.$$

Here we have also used the observation that with (3.2), the integrand is nonnegative. As $\epsilon \to 0$ the sets $\tilde{D}_\epsilon$ expand and we obtain the fact that $\nabla\varphi_1 = \nabla\varphi_2$ in the set $\{0 <$

$\varphi_1 - \varphi_2 < m\}$. Thus $\varphi_1 \equiv \varphi_2+$ constant in any component of $\tilde{D}$ where $\varphi_1 > \varphi_2$. Since φ_1 and φ_2 are real analytic off the branch set, we obtain a contradiction unless the set where $\varphi_1 > \varphi_2$ is empty.

4 Combinatorial Properties of the Conjugate Height Function

In this section we shall collect the relevant facts which enable us to give combinatorial arguments using the conjugate height functions.

Let $\tilde{\psi}$ be a conjugate height function for a univalent harmonic mapping f in U. We assume in Theorems A, B, C below that $f(U)$ is bounded, and its analytic dilatation has finitely many zeros of odd order so that $\tilde{U}$ is finite sheeted. In the present context, [JS, p. 327] gives

Theorem A *Let $\tilde{C}$ be a simple piecewise smooth curve in the closure of $\tilde{U}$. Then, using the coordinates given by projection,*

$$\Big| \int_{\tilde{C}} d\tilde{\psi} \Big| \leq \text{length}\,(\tilde{C})$$

with strict inequality if any portion of $\tilde{C}$ lies over points in $\tilde{U}$.

Moreover [JS, Lemma 2], we have

Theorem B *If $\tilde{C}$ is a simple, piecewise smooth closed curve in the closure of $\tilde{U}$, then $\int_{\tilde{C}} d\tilde{\psi} = 0$.*

The companion to Theorems A and B is provided by Lemma 4 of [JS]. Again paraphrasing, in the current setting we have

Theorem C *Suppose that $\partial f(U)$ contains a line segment T, and $\tilde{T}$ is a segment over T in $\tilde{U}$. If $\tilde{T}$ is oriented so that the right hand normal to $\tilde{T}$ is the outer normal to $\tilde{U}$, and $\tilde{\varphi} = +\infty$ on $\tilde{T}$, then in the coordinates given by projection,*

$$\int_{\tilde{T}} d\tilde{\psi} = \text{length}\,(T).$$

The value of Theorems A, B, and C when applied to Poisson integrals of step functions stem from [W, Theorem 2]

Theorem D *Let $\mathcal{P}$ be a polygon having vertices $c_1, \dots, c_n$ given cyclically on $\partial\mathcal{P}$, and ordered by a positive orientation on $\partial\mathcal{P}$. Let f be a univalent harmonic mapping of U such that f is the Poisson integral of a step function having the ordered sequence $c_1, \dots, c_n$ as its values. Then the analytic dilatation $a(\zeta)$ of f is a finite Blaschke product of order at most $n-2$, $f(U) = \mathcal{P}$, with equality if $\mathcal{P}$ is convex. If $\tilde{\varphi}$ is a height function for f, then $\tilde{\varphi}$ tends to $+\infty$ or $-\infty$ at points over the open segments making up the sides of $\mathcal{P}$. At any vertex c_j of $\mathcal{P}$ at which the interior angle is less than π, then $+\infty$ and $-\infty$ alternate on adjacent sides having c_j as the common vertex.*

The proofs of Theorems A, B, and C are just as given in [JS]. The statement of Theorem D differs in the last sentence in [W, Theorem 2], but the statement given above is actually what is proved there.

5 Poisson Integrals of Step Functions

Throughout this section we use the notations of Theorem D. The classical Scherk surface arises from taking $c_1 = 0$, $c_2 = 1$, $c_3 = 1 + i$, $c_4 = i$ as vertices, and with the Poisson integral f taking those values on respective intervals of equal length, and ordered positively around ∂U. Then $a(\zeta) = c\zeta^2$, and the height function can be taken as a saddle with heights $\pm\infty$ alternately over the sides of the square. As forecast by Theorem D, $a(\zeta)$ has two zeros (counting multiplicity). In order to motivate the general phenomenon, suppose we extend the top and bottom sides of the square to make a rectangle R, and stretch or shrink the corresponding intervals for the new f in ∂U in any fashion. Still, since R is convex, the analytic dilatation will have two zeros ζ_1, ζ_2. As we shall see in the proof of Theorem 2, the images $z_1 = f(\zeta_1)$, $z_2 = f(\zeta_2)$ of these points lie cannot both be to the left or right of the center of R, regardless of the relative sizes of the intervals in ∂U corresponding to the vertices of R.

In general, if f comes from the Poisson integral of a step function mapping U onto a polygon $\mathcal{P}$ with the values of the step function being vertices $c_1, \ldots, c_n$, then its analytic dilatation $a(\zeta)$ has at most $n - 2$ zeros in U. Theorem 2 shows that if we have some knowledge of $\partial\mathcal{P}$, we can say more.

Theorem 2 *With f, a, $\mathcal{P}$, and $c_1, \ldots, c_n$ as above, suppose that the interior angles at c_j and c_{j+1} are less than π. Let d_j, d_{j+1} be points on the segments $c_{j-1}c_j$ and $c_{j+1}c_{j+2}$ respectively, and assume that the open quadrilateral $\mathcal{P}_j$ with vertices d_j, c_j, c_{j+1}, d_{j+1} is contained in $f(U)$. If* length (d_jd_{j+1}) + length (c_jc_{j+1}) < length (d_jc_j)+ length $(c_{j+1}d_{j+1})$, *then $\mathcal{P}_j$ contains the image of at least one zero of $a(\zeta)$ of odd order. Thus, in particular, if $\mathcal{P}$ has k disjoint such quadrilaterals, then $a(\zeta)$ has at least k zeros.*

Proof Suppose that $\mathcal{P}_j$ were such a quadrilateral without a branch point. Then there would exist a single valued branch of the height function $\tilde{\varphi}$ over $\mathcal{P}_j$ whose values over d_jc_j, c_jc_{j+1}, $c_{j+1}d_{j+1}$ would alternate $\pm\infty$ by Theorem D. Let $\tilde{\gamma}$ be the boundary of a quadrilateral $\tilde{\mathcal{P}}_j$ over $\mathcal{P}_j$ in $\tilde{U}$, oriented positively. We have from Theorem B that

$$\int_{\tilde{\gamma}} d\tilde{\psi} = 0. \tag{5.1}$$

By the alternation of signs, if $\tilde{\gamma}_1$ is the edge over d_jc_j, and $\tilde{\gamma}_3$ is over $c_{j+1}d_{j+1}$, then by Theorem C,

$$\left| \int_{\tilde{\gamma}_1} d\tilde{\psi} + \int_{\tilde{\gamma}_3} d\tilde{\psi} \right| = \text{length}\,(d_jc_j) + \text{length}\,(c_{j+1}d_{j+1}). \tag{5.2}$$

From (5.1) and (5.2) we then obtain

$$\text{length}\,(d_jc_j) + \text{length}\,(c_{j+1}d_{j+1}) \le \left| \int_{\tilde{\gamma}_2} d\tilde{\psi} \right| + \left| \int_{\tilde{\gamma}_4} d\tilde{\psi} \right|, \tag{5.3}$$

where $\tilde{\gamma}_2$ is over c_jc_{j+1}, and $\tilde{\gamma}_4$ is over d_jd_{j+1}.

Again, by Theorem C,

$$\left| \int_{\tilde{\gamma}_2} d\tilde{\psi} \right| = \text{length}\,(c_j c_{j+1}), \tag{5.4}$$

and by Theorem A,

$$\left| \int_{\tilde{\gamma}_4} d\tilde{\psi} \right| \leq \text{length}\,(d_j d_{j+1}). \tag{5.5}$$

Combining (5.3)–(5.5), we contradict the hypothesis of the theorem. Thus, $\mathcal{P}_j$ must contain the image of at least one zero of odd order of $a(\zeta)$.

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