

## STUDY GUIDE FOR MA 265 LINEAR ALGEBRA

This study guide describes briefly the course materials to be covered in MA 265. In order to be qualified for the credit, one is expected not only to “know” these materials but also to demonstrate the skills to solve the quantitative and numerical problems.

The current textbook used in MA 265 is (as of April 16, 2001):

◦ Introduction to Linear Algebra with Applications (6th Edition) by Bernard Kolman

where most of the materials discussed below can be found and studied.

### 1. Linear Systems

- Definition of a linear system
- Method of elimination
  - Elementary Operations
  - Description of the set of solutions

One should know how to solve a linear system of equations by the method of elimination (Gauss-Jordan elimination method), using elementary operations, which leads to the complete description of the set of solutions.

### 2. Matrices

- Definition of a matrix
- Matrix Operations
  - Addition
  - Scalar multiplication
  - Transpose
  - Multiplication of matrices
- Properties of Matrix Operations
- Inverse Matrix
  - How to find the inverse via Gauss-Jordan method
  - The formula  $A^{-1} = \frac{1}{\det A} \text{adj}(A)$ .

One should know the basic operations of matrices: addition, scalar multiplication, transpose and matrix multiplication. We observe the analogy with the usual operations of numbers, while the matrix multiplication is not commutative, i.e.,  $AB \neq BA$  in general. The inverse  $A^{-1}$  of a matrix  $A$ , analogous to the notion of the reciprocal  $1/a$  of a number  $a$ , is studied. One should know how to compute the inverse of a matrix  $A$ , when it exists: One is to apply the Gauss-Jordan

reduction to the augmented matrix  $[A; I] \rightarrow [I; A^{-1}]$ . Another is to apply the formula  $A^{-1} = \frac{1}{\det A} \text{adj}(A)$ , which can be obtained as an easy application of the basic properties of the determinant of a matrix.

### 3. Solutions of Linear Systems in terms of Matrices

- Augmented Matrix
- Reduced Row Echelon Form (RREF)
- Gauss-Jordan reduction
  - Description of the set of solutions via RREF
- Homogeneous System
  - Dimension of the set of solutions (the null space)
- Non-Homogeneous System
  - Reduction to the associated homogeneous system via a particular solution

One should know how the problem of solving a linear system of equations can be translated into the problem of transforming a given augmented matrix into Reduced Row Echelon Form via the Gauss-Jordan reduction using elementary row operations. One should be able to read off, by looking at RREF, such information as the dimension of the null space (the space of the set of solutions) for a homogeneous system and be able to describe the whole set of solutions for a nonhomogeneous system given a particular solution and the description of the null space.

### 3. Determinants

- Definition of a determinant
  - Permutation
  - The number of inversion
- Properties of determinants
- Cofactor Expansion
  - $A \cdot \text{adj}(A) = \det A \cdot I$
  - Cramer's rule

One should know how to compute the determinant of a matrix. Starting with the very definition using the language of permutation (odd and even permutations and the number of inversions etc.), via the study of its basic properties, one is led to the cofactor expansion. Cofactor expansions can be collectively described as the formula  $A \cdot \text{adj}(A) = \det A \cdot I$ , which then provides the formula for the inverse of a matrix  $A$ . An important application is Cramer's rule.

### 4. Linear (Affine) Geometry of $\mathbb{R}^n$

- Vectors in  $\mathbb{R}^n$
- Operations among vectors
  - Addition
  - Scalar multiplication
  - Dot Product
    - Orthogonality in terms of dot product
- Lines and planes in  $\mathbb{R}^2$  and  $\mathbb{R}^3$ 
  - Parametric and symmetric equations of a line
  - Equation for a plane with a given normal vector and a point on it

One should be able to describe the linear (affine) geometry of  $\mathbb{R}^n$  in terms of vectors. Aside from the basic operations of addition and scalar multiplication, the dot product is the most important one to study, providing, e.g., the orthogonality criterion of two vectors. The emphasis is put on how to find the parametric and symmetric equations of a line in  $\mathbb{R}^2$  and  $\mathbb{R}^3$  as well as how to find the equation of a plane in  $\mathbb{R}^3$  satisfying certain conditions.

## 5. Vector spaces

- Definition of an abstract vector space
- Subspaces
- Linear Independence and Dependence
- Basis and Dimension
  - Column and row space of a matrix
  - Rank of a matrix
- Orthonormal basis
  - Gram-Schmidt process
  - Orthogonal complements
  - Applications to the method of least squares

This material is the most abstract and hardest of all, though, once understood, it will be the powerful ally of yours in analysing a wide varieties of mathematical and physical phenomena, ranging from differential equations to optimization. One should know the definition of a vector space over  $\mathbb{R}$  and subspaces, abstracting the notion of  $\mathbb{R}^n$ . The notion of linear independence and dependence clarifies, e.g., what we mean by independent (or free) variables in describing the set of solutions for a linear system. One should be able to find a basis (a set of linearly independent vectors which span) for a given (sub)space, e.g., the column space or row space of a matrix. Through the Gram-Schmidt process, one should be able to compute and obtain an orthonormal basis out of a given basis. One should be able to compute the projection of a vector onto a subspace and onto its orthogonal complement. As an important application, we discuss the least square fit solutions, notably how to draw the least square fit line for given data points.

## 6. Linear Transformations

- Definition of a linear transformation
- Kernel and range of a linear transformation
- Matrix representing a linear transformation
- Coordinates and change of basis

A linear transformation between  $\mathbb{R}^n$  and  $\mathbb{R}^m$  is simply a map defined by multiplication of an  $m \times n$  matrix, called the standard matrix of the linear transformation. One should know how to find the standard matrix for rotation, reflection or combination of these. A linear combination between vector spaces is similarly defined, once we identify them with  $\mathbb{R}^n$  and  $\mathbb{R}^m$ , by choosing bases. The choice of bases corresponds to certain transformation of the standard matrix.

## 7. Eigenvalues and Eigenvectors

- Definition of eigenvalues and eigenvectors
  - Characteristic polynomial

- Diagonalization
  - Diagonalization of a matrix (similar to a diagonal matrix)
  - Diagonalization of a symmetric matrix

Eigenvalues  $\lambda$  and eigenvectors  $v$  of a square matrix  $A$  is characterized as the solutions to the equation  $Av = \lambda v$  with  $v \neq 0$ . On a more concrete level, one should know how to compute the eigenvalues as roots of the characteristic polynomial  $\det(\lambda I - A)$  and compute then the eigenvectors as solutions to  $(\lambda I - A)v = 0$ . One of the main applications one should know is the diagonalization problem: Given a square matrix  $A$ , find an invertible matrix  $P$  such that  $P^{-1}AP = D$  is a diagonal matrix. Though this is not always possible, when  $A$  has distinct eigenvalues or when  $A$  is symmetric, we can give a specific algorithm to find  $P$  (in the case of a symmetric  $A$ , we may require further for  $P$  to be orthogonal).

### 8. Linear Differential Equations

- Solutions to linear differential equations (with diagonalizable matrices)
  - real eigenvalues
  - real solutions with complex eigenvalues
    - Complex numbers and Euler's formula
- Solution of the form  $X = e^{tA}$  for  $\frac{dX}{dt} = AX$ 
  - exponential of a matrix

As an application of the diagonalization problem (and its solution), we discuss how to solve linear differential equations. While the more detailed treatment should be given in MA 266, we give a complete discussion when the representation matrix is diagonalizable. One should know how to write down the general REAL solution when the representation matrix has COMPLEX eigenvalues via Euler's formula. The solution in the case when the representation matrix is not diagonalizable can be found via the discussion of the exponentials of matrices.

### 9. Matlab (computer)

- Understanding of basic commands
- Matrix operations in Matlab
- Elementary Row Operations in Matlab
- Vectors in Matlab

The use of Matlab is an essential and active part of MA 265, though the real use of computers in the exams is not required or practiced under the current curriculum. One should know the basic commands, such as *rref* and *inv*, and be able to interpret the output depending on the situation. The skills to read the necessary information off the output of Matlab will be tested.

# Math 265 Linear Algebra

# Credit Exam Practice

Student Name (print):
-----------------------

Student ID:
-------------

Do not write below this line.

Please be neat and show all work.  
Write each answer in the provided box.  
Use the back of the sheets and the last 3 pages for extra scratch space.  
Return this entire booklet to your instructor.  
**No books. No notes. No calculators.**

Problem #	Max pts.	Earned points
1	20	
2	8	
3	8	
4	8	
5	8	
6	8	
7	8	
8	8	
9	8	
10	8	
11	8	
<b>Section I</b>	<b>100</b>	

13	10	
14	10	
15	10	
<b>Section II</b>	<b>30</b>	
16	20	
17	15	
18	15	
19	20	
<b>Section III</b>	<b>70</b>	
<b>TOTAL</b>	<b>200</b>	

## Section I: Short problems

No partial credit on this part, but show all your work anyway. It might help you if you come close to a borderline. Please be neat. Write your answer in the provided box.

1. It is given that  $A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$ ,  $\text{rref}(A) = \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$  and

$$\text{rref}(A^T) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find the rank of  $A$ .

(b) Find a basis for the null space of  $A$ .

(c) Find a basis for the column space of  $A$ . We require that you choose the vectors for the basis from the column vectors of  $A$ .

(d) Find a basis for the row space of  $A$ . We require that you choose the vectors for the basis from the row vectors of  $A$ .

2. Determine the value(s) of  $a$  so that the following linear system has no solution.

$$\begin{cases} x_1 + 2x_2 + x_3 = a \\ x_1 + x_2 + ax_3 = 1 \\ 3x_1 + 4x_2 + (a^2 - 2)x_3 = 1. \end{cases}$$

3. Find the standard matrix for the linear transformation  $L : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  such that

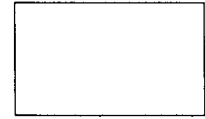
$$L \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, L \left( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, L \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ -2 \end{bmatrix}.$$

4. Determine the value(s) of  $a$  so that the line whose parametric equations are given by

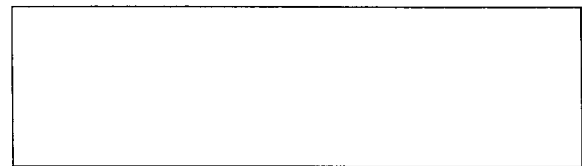
$$\begin{cases} x = -3 + t \\ y = 2 - t \\ z = 1 + at \end{cases}$$

is parallel to the plane

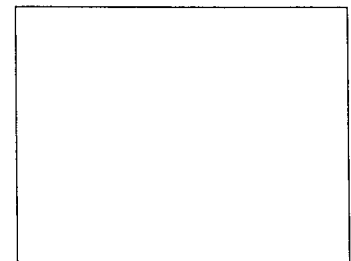
$$3x - 5y + z + 3 = 0.$$



5. Find the symmetric equations of the line which is the intersection of the following two planes:  $x + y - z = 2$ ,  $3x + 4y + z = 5$ .



6. Compute the inverse of the matrix  $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ .

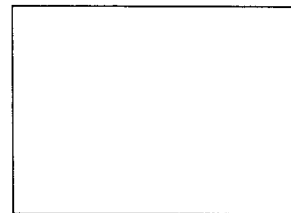


7.  $E$  is a  $3 \times 3$  matrix of the form

$$E = \begin{bmatrix} 1 & 8 & 3 \\ x & y & z \\ -3 & 7 & 2 \end{bmatrix}.$$

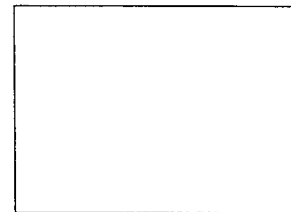
Given  $\det(E) = 5$ , compute the determinant of the following matrix

$$F = \begin{bmatrix} x & y & z \\ 1 & 8 & 3 \\ -3 + 4x & 7 + 4y & 2 + 4z \end{bmatrix}$$



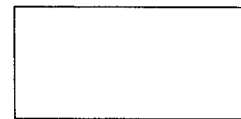
8. Find the matrix  $G$  such that

$$\text{adj}(G) = \begin{bmatrix} 2 & 4 \\ -5 & 7 \end{bmatrix}.$$



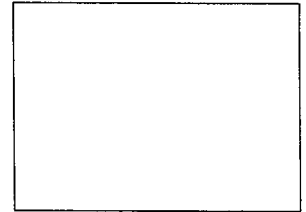
9. Find the dimension of the subspace  $V = \text{span}\{v_1, v_2, v_3, v_4\}$  in  $\mathbb{R}^3$  where

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$



10. Find the projection  $\text{Proj}_W v$  of the vector  $v = \begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix}$  onto the subspace  $W$  spanned by

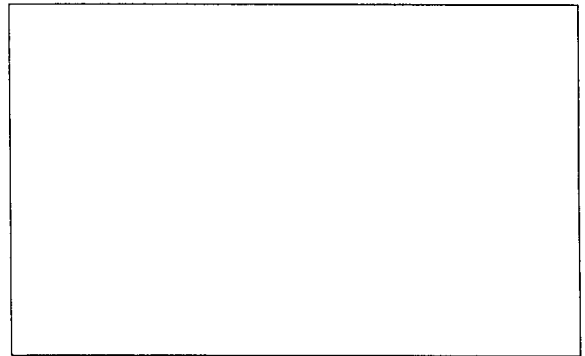
$$\left\{ v_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$



11. We have a subspace  $W$  in  $\mathbb{R}^4$  spanned by the following three linearly independent vectors

$$\left\{ u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, u_3 = \begin{bmatrix} 3 \\ -3 \\ 0 \\ -2 \end{bmatrix} \right\}.$$

Find an orthonormal basis of  $W$ .



## Section II: Multiple choice problems

For Problems 12 through 15, circle only one (the correct) answer for each part.  
No partial credit.

12. Let  $A$  be a  $3 \times 3$  matrix with  $\det(A) = 0$ . Determine if each of the following statements is true or false.

- |   |              |
|---|--------------|
| (a) $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.   | True   False |
| (b) $A\mathbf{x} = \mathbf{b}$ has at least one solution for every $\mathbf{b}$ .                                 | True   False |
| (c) For every $3 \times 3$ matrix $B$ , we have $\det(A + B) = \det(B)$ .   | True   False |
| (d) For every $3 \times 3$ matrix $B$ , we have $\det(AB) = 0$ .  | True   False |
| (e) There is a vector $\mathbf{b}$ in $\mathbb{R}^3$ such that $\text{rank}([A \ \mathbf{b}]) > \text{rank}(A)$ . | True   False |

13. For each of the following sets, determine if it is a vector (sub)space:

(a) The set of all vectors  $(x_1, x_2, x_3, x_4)$  in  $\mathbb{R}^4$  with the property  $2x_1 - x_2 = 0, 3x_3 - x_4 = 0$ ;  
Yes   No

(b) The set of all vectors  $(x_1, x_2, x_3)$  in  $\mathbb{R}^3$  with the property  $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$ ;  
Yes   No

(c) The set of all vectors  $(x_1, x_2, x_3, x_4)$  in  $\mathbb{R}^4$  with the property  $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$ ;  
Yes   No

(d) The set of all vectors of the form  $(a + b - 1, 2a + 3c - 1, b - c, a + b + c + 2)$  in  $\mathbb{R}^4$  where  $a, b$  and  $c$  are arbitrary real numbers;  
Yes   No

(e) The set of all solutions to the linear system of differential equations  $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$   
 where  $A = \begin{bmatrix} 5 & -4 \\ 1 & 1 \end{bmatrix}$ ;  
Yes   No

14. For the problems (a), (b) and (c), determine if the given set of vectors is linearly independent or linearly dependent:

$$(a) \left\{ \begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}; \quad \text{Independent} \quad \text{Dependent}$$

$$(b) \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}; \quad \text{Independent} \quad \text{Dependent}$$

$$(c) \left\{ \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 11 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 5 \end{bmatrix} \right\}; \quad \text{Independent} \quad \text{Dependent}$$

For the problems (d) and (e), determine if the given set of vectors spans  $\mathbb{R}^3$ :

$$(d) \left\{ \begin{bmatrix} \pi \\ 2\pi \\ -1\pi \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}; \quad \text{span} \quad \text{not span}$$

$$(e) \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}; \quad \text{span} \quad \text{not span}$$

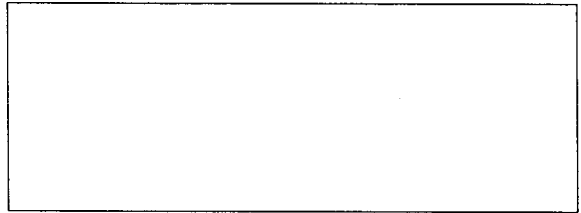
### Section III: Multi-Step problems

Show all work (no work - no credit!) and display computing steps. Write clearly.

15. Let

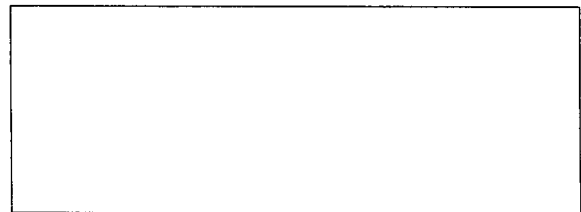
$$A = \begin{bmatrix} -15 & 28 \\ -8 & 15 \end{bmatrix}.$$

(a) Find the eigenvalues and compute an eigenvector for each eigenvalue.

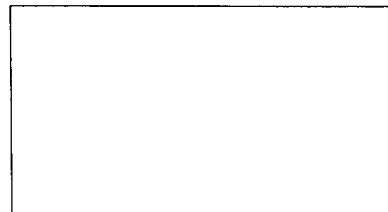


(b) Find an invertible matrix  $P$  and a diagonal matrix  $D$  such that

$$P^{-1}AP = D.$$



(c) Compute  $A^{37}$ .



16. Find the least squares fit line for the points

$$(-2, 1), (-1, 3), (0, 2), (1, 3), (2, 1).$$

17. Let

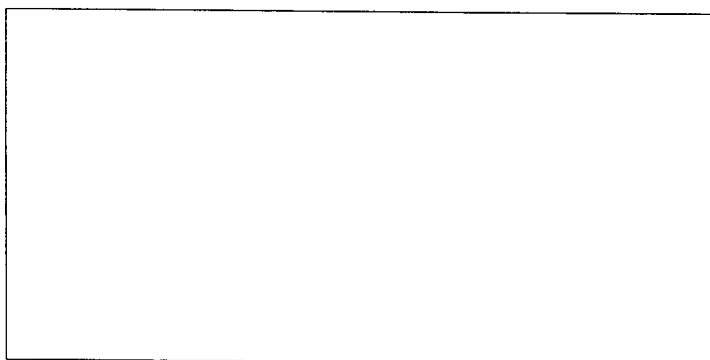
$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

be the linear system of differential equations where

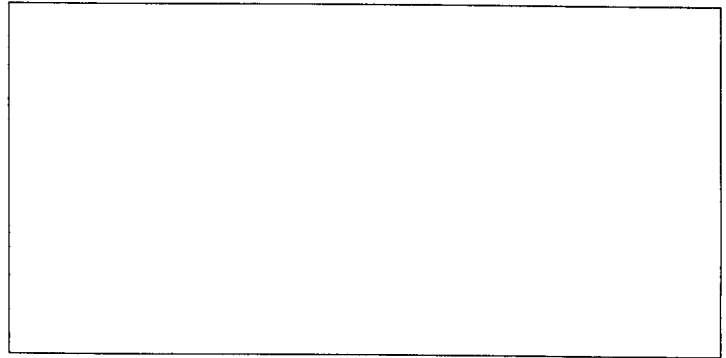
$$A = \begin{bmatrix} 3 & -5 \\ 5 & 3 \end{bmatrix}.$$

(a) Find the eigenvalues and find an eigenvector for each eigenvalue for  $A$ .

Note: The eigenvalues are COMPLEX-valued.



(b) Find the general REAL solution to the linear system of differential equations.

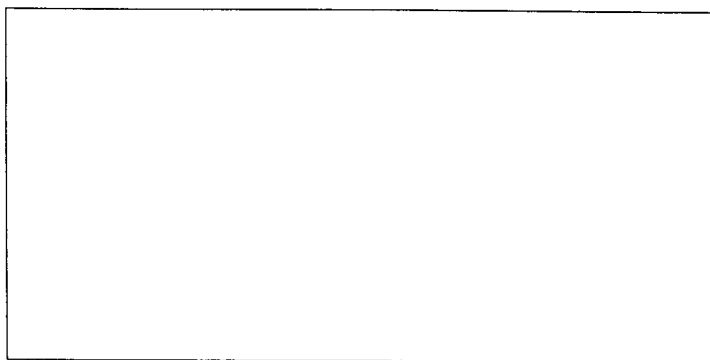


18. Let

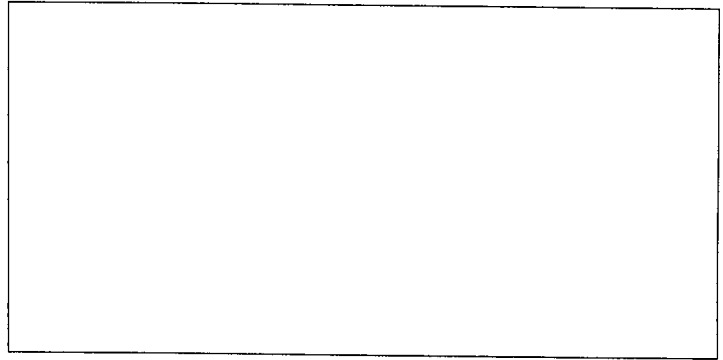
$$\begin{cases} \frac{dx_1}{dt} = 2x_1 + 5x_2 \\ \frac{dx_2}{dt} = 3x_1 + x_2 + 3x_3 \\ \frac{dx_3}{dt} = -x_1 \end{cases}$$

be a linear system of differential equations.

- (a) Find the eigenvalues and find an eigenvector for each eigenvalue for the coefficient matrix of the linear system of differential equations.

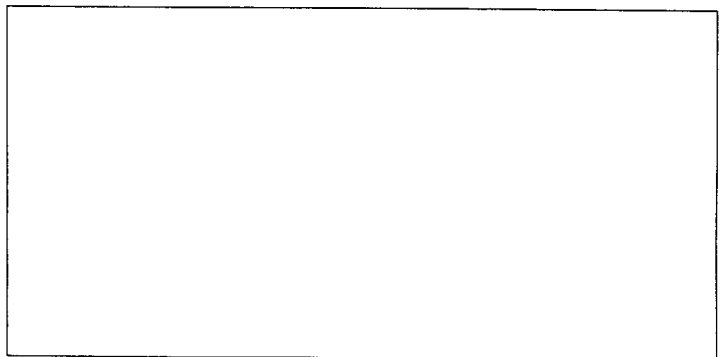


(b) Find the general solution to the linear system of differential equations.



(c) Find the solution to the initial value problem

$$x_1(0) = 4, x_2(0) = 16, x_3(0) = 0.$$



1. (a) 3. (b)  $\begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} -5 \\ 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ . (c)  $A_1, A_2, A_4$ . (d)  $A^1, A^2, A^3$ .

2.  $a = 3$ . 3.  $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -2 \end{bmatrix}$ . 4.  $a = -8$ . 5.  $\frac{x-3}{5} = \frac{y+1}{-4} = z$ .

6.  $\begin{bmatrix} 0 & 1 & 0 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{bmatrix}$ . 7.  $-5$ . 8.  $\begin{bmatrix} 7 & -4 \\ 5 & 2 \end{bmatrix}$ . 9. 3.

10.  $\begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}$ . 11.  $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ .

12. (a) True. (b) False. (c) False. (d) True. (e) True.

13. (a) Yes. (b) No. (c) No. (d) Yes. (e) Yes.

14. (a) Independent. (b) Dependent. (c) Dependent. (d) not span. (e) span.

15. (a)  $\lambda_1 = -1$ ,  $P_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ .  $\lambda_2 = 1$ ,  $P_2 = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$ . (b)  $P = \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix}$ ,  $\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ .

16.  $y = 2$ .

17. (a)  $\lambda_1 = 3 + 5i$ ,  $P_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ .  $\lambda_2 = 3 - 5i$ ,  $P_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$ .

(b)  $\mathbf{x}(t) = c_1 e^{3t} \begin{bmatrix} -\sin 5t \\ \cos 5t \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} \cos 5t \\ \sin 5t \end{bmatrix}$ .

18. (a) (a)  $\lambda_1 = -3$ ,  $P_1 = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}$ .  $\lambda_2 = 1$ ,  $P_2 = \begin{bmatrix} -5 \\ 1 \\ 5 \end{bmatrix}$ .  $\lambda_3 = 5$ ,  $P_3 = \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}$ .

(b)  $\mathbf{x}(t) = c_1 e^{-3t} \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} -5 \\ 1 \\ 5 \end{bmatrix} + c_3 e^{5t} \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}$ .

(c)  $\mathbf{x}(t) = -2e^{-3t} \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} + e^t \begin{bmatrix} -5 \\ 1 \\ 5 \end{bmatrix} + -3e^{5t} \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}$ .