MA 266
First Examination, Wilkerson 9AM Section
70 minutes, Oct. 5, 1999

Do all your work on the question sheets. Calculators are NOT allowed. NO BOOKS OR PAPERS ARE ALLOWED. Use the back of the test pages for scrap paper. Problems are 15 points each unless noted. NAME Solutions

1. Suppose that $ty' - 2y = t$ for $t > 0$. Solve for $y(t)$. Then find the solution such that $y(1) = 0$. Finally, for this y, calculate $y(e)$.

ANS: Divide by t to get $y' - 2t^{-1}y = 1$. Thus $p(t) = -2t^{-1}$ has indefinite integral $\int p(t)dt = -2ln(t)$. The integrating factor for this equation is $e^{-2ln(t)} = t^{-2}$ for $t > 0$. Thus $(t^{-2}y)' = t^{-2}1 = t^{-2}$ and $y(t)t^{-2} = \int t^{-2}dt = t^{-2+1}/(-2+1) + c$. Hence $y(t) = t^2(-t^{-1} + c)$. Using $y(1) = 0$, this gives $0 = 1(-1 + c)$, so $c = 1$. Hence $y(t) = t^2 - t$ and $y(e) = e^2 - e^1$.

2. Give the general solution of the differential equation

$(x^2 + y)dy + (2xy + 1)dx = 0$. If $y(0) = 1$, give the solution with this initial value.

ANS: $M = (2xy + 1)$, $N = (x^2 + y)$. Hence $\partial M/\partial y = 2x = \partial N/\partial x$, so the equation is exact. $\phi(x, y) = \int Mdx + g(y) = x^2y + x + g(y)$. But $\partial\phi/\partial y = N = x^2 + y = x^2 + g'(y)$. Thus $g' = y$, and $g(y) = y^2/2$ works. That is, $\phi(x, y) = x^2y + x + y^2/2 = C$. Plug in $y(0) = 1$ yields $C = 1$.

3. Suppose that a tank initially contains 20 gallons of a salt water solution with a concentration of 2 lbs/gallon. A salt water solution with a concentration of $\frac{1}{2}$ lb/gallon is poured into the tank at a rate of 2 gallons/min and the well stirred solution exits the tank at the rate of 2 gallons/min. First write a first order differential equation that predicts the concentration of salt in the tank at time t. Solve this differential equation and calculate the concentration of the solution in the tank after 10 minutes.

ANS: Let $A(t)$ be the amount of salt in the tank at time t. Let $C(t)$ be the concentration, and $V(t)$ be the volume of fluid in the tank. Then $C(t) = A(t)/V(t)$. Since the rate of flows of fluid into and out of the tank are the same, $V(t) = V(0) =$ 20 gallons, and $C(t) = A(t)/20$. Now analysing the salt input-output gives

$$dA/dt = (2)(0.5) - 2C(t) = 1 - A(t)/10$$

or $dA/dt + A(t)/10 = 1$. This has solution $A(t) = be^{-t/10} + 10$. The initial condition is that $A(0) = 2(20) = 40$, so $b = 40 - 10 = 30$. That is, $A(t) = 10(1 + 3e^{-t/10})$. $A(10) = 10(1 + 3e^{-1}) \approx 10(1 + 1.1) \approx 21$ pounds.

4. If the population of a bacterial colony equals 400 after 2 hours and equals 500 after 3 hours, what was the initial population? (10 points)

ANS: $P(t) = P(0)e^{kt}$ models the growth of the colony. Thus $P(2) = 400 = P(0)e^{k2}$ and $P(3) = 500 = P(0)e^{k3}$, so $P(3)/P(2) = 500/400 = 5/4 = e^k$. Hence $P(t) = P(0)(5/4)^t$ is another way of writing the growth equation. Put $t = 2$ to get $400 = P(0)(5/4)^2 = P(0)(25)/(16)$, or $P(0) = 256$.

5. Convert the differential equation $y'' + ty' = 0$ into a first order linear equation by the substituition $v = y'$, then give the new equation, and solve for $v(t)$. (10 points)

ANS: $v' + tv = 0$. Separate variables to obtain $v'/v = -t$. So $ln(|v|) = C - t^2/2$, and $v = e^{-t^2/2}B$, for some constants C and B. The remaining work, to recover y from v does not have an elementary anti-derivative.

6. Use the homogeneous equation method to solve the differential equation $y' = (x)/(y - x)$ near the point $(1, 0)$
ANS: Let $y = xv$. Then $y' = xv' + v$ and $v = y/x$ for $x \neq 0$. In this case $y' = xv' + v = (x/x)/(y/x - x/x) = 1/(v - 1)$. Hence $xv' = -v + (1/(v - 1)) = (-v^2 + v + 1)/(v - 1)$. Now separate the variables to get $dv(v - 1)/(1 - v^2 + v) = dx/x$. This can be explicitly found by partial fractions, but it is somewhat messy.

7. Use the Euler tangent line method to estimate $y(1.1)$ and $y(1.2)$, for $y(t)$ the solution to the differential equation

$$y' = (y + x).$$

Assume that $y(1) = 5$. Hint: make a table of values. (10 points)
ANS: Let $f(x, y) = x + y$, and $h = 0.1$. Make the table

x	$y(x)$	$f(x - y) = (x + y)$
1	5	6
1.1	5.6	6.7
1.2	12.27	13.47

The answers are 5.6 and 12.27.

8. Two solutions to

$$y'' - 4y = 0$$

are $y_1 = e^{2t}$ and $y_2 = e^{-2t}$. Find a solution $\phi(t)$ for which $\phi(0) = 0$ and $\phi'(0) = 1$. (10 points)
ANS: The general solution has the form $y_g = ay_1 + by_2 = ae^{2t} + be^{-2t}$ for constants a and b. To fit the initial conditions, one must choose the correct constants. $y_g(t) = ae^{2t} + be^{-2t}$ so $y_g' = 2ae^{2t} - 2be^{-2t}$. So at $t = 0$, $y_g(0) = 0 = a + b$ and $y_g' = 2a - 2b = 1$. This gives that $b = -1/4$ and $a = 1/4$. The answer can also be written as $sinh(2t)/2$.