Math 266 Second Examination, Nov. 9, 1999

Wilkerson Section, 9:00 AM.

Show all work on these pages for complete credit. No books, notes, or calculators. Read problems carefully to avoid extra work.

I. (10 points) Pick two linearly independent solutions $\{y_1, y_2\}$ of the second order constant coefficient linear homogeneous equation

$$y'' + y = 0$$

and compute the Wronskian $W(y_1, y_2)$. Finally, choose a solution $y(t)$ such that $y(0) = 5$ and $y'(0) = 7$.

ANS: The characteristic equation is $r^2 + 1 = 0$, with roots $\pm i$. One choice is the pair of complex exponentials $\{e^{it}, e^{-it}\}$. The Wronskian $W(e^{it}, e^{-it}) = -2ie^0 = -2i$.

A second choice is $\{cos(t), sin(t)\}$ with Wronskian $cos^2(t) + sin^2(t) = 1$.

To use the initial conditions, $y(t) = acos(t)+bsin(t)$ and $y'(t) = -asin(t)+bcos(t)$, so $a+b0 = a = 5$ and $-a0 + b = b = 7$. So $y(t) = 5cos(t) + 7sin(t)$.

II: (20 points) Give all solutions of these linear homogeneous differential equations:

1) $y'' - 16y = 0$

ANS: The characteristic equation is $r^2 - 16 = 0$, so $r = \pm 4$ are two distinct roots. Thus any solution is a linear commbination of e^{4t} and e^{-4t}.

2) $y'''' - y = 0$ or $y^{(4)} - y = 0$.

The characteristic equation is $r^4 - 1 = (r^2 - 1)(r^2 + 1)$, which has four distinct roots $\{\pm i, \pm 1\}$. This gives the four LI solutions $\{e^t, e^{-t}, e^{it}, e^{-it}\}$. Another choice is $\{e^t, e^{-t}, cos(t), sin(t)\}$. Any solution is a linear combination of these solutions.

3) $(D^2 - 3D + 2)(D + 3)(D + 3)(D - 3)(D - 2)y = 0$.

The characteristic polynomial is $(r^2-3r+2)(r+3)^2(r-3)(r-2) = 0$ or $(r-1)(r-2)^2(r+3)^2(r-3) = 0$. Thus the roots are $\{1, 2, 2, 3, -3, -3\}$. So linear combinations of $\{e^t, e^{2t}, te^{2t}, e^{-3t}, te^{-3t}, e^{3t}\}$ give the solution set.

4) $y'' - 2y' + y = 0$

The characteristic polynomial is $r^2 - 2r + 1 = (r-1)^2 = 0$ with roots $\{1, 1\}$. Two LI solutions are e^t and te^t, so the linear combinations $Ae^t + Bte^t = e^t(A + Bt)$ describe all solutions.

III: (10 points) Give all solutions to $y'' + 2y' + 5y = 0$, and the only solution for which $y(0) = 0$ and $y'(0) = 1$. If we view this as the DE describing the motion of a spring-mass system, does the amplitude of the oscillation get larger with time?

The characteristic equation is $r^2 + 2r + 5 = (r^2 + 2r + 1) + 4 = (r + 1)^2 + 4 = 0$, so the roots are $\{-1 \pm 2i\}$. Thus the solutions have the form $y(t) = e^{-t}(Acos(2t) + Bsin(2t))$. Thus $y(0) = 0$ implies that $A = 0$, so $y(t) = Be^{-t}sin(2t)$. But then $y'(t) = Be^{-t}2cos(2t) + Be^{-t}(-1)sin(2t)$ and $y'(0) = 1 = 2B - B(0)(-1) = 2B$. That is, $B = 1/2$. So $y(t) = (e^{-t}cos(2t))/2$ is the solution. As $t \to \infty$, $y(t) \to 0$.

IV: (10 points) Inhomogeneous equations. Find a particular solution to the equation

$$y'' - y = t + e^{2t}$$

using the method of undetermined coefficients. Start by finding an appropriate trial solution. Then solve for the unknown coefficients.

The characteristic equaiton of the homogeneous equation is $r^2 - 1 = 0$ with roots ± 1, corresponding to solutions $\{e^t, e^{-t}\}$. Hence we can take our initial guess for the particular solution to be $y_p(t) = At + Be^{2t}$. $(D^2 - 1)y_p = A(D^2 - 1)(t) + B(D^2 - 1)(e^{2t}) = -tA + B(2^2 - 1)e^{2t} = (?)t + e^{2t}$. Hence we should take $A = -1$ and $3B = 1$ or $B = 1/3$. That is the particular solution is

$$y_p(t) = -t + e^{2t}/3 + c_1e^t + c_2e^{-t}$$

for arbitrary constants $\{c_1, c_2\}$.

V: (10 points) The linear inhomogeneous differential equation

$$t^2y'' + ty' - 4y = 5t^3$$

has $y_1(t) = t^2$ and $y_2(t) = t^{-2}$ as solutions to the homogenous version. Use the method of variation of parameters to set up a formula for a solution of the inhomogeneous equation. That is, try to find funtions u_1 and u_2 so that $y = u_1 y_1 + u_2 y_2$ where $u_1' y_1 + u_2' y_2 = 0$ (Correction to typo on original exam). Solve for u_1' and u_2'.

ANS: Divide by t^2 to change to the standard form

$$y'' + y'/t - 4t^{-2}y = 5t = y'' + p(t)y' + q(t)y = g(t)$$

One can check that t^2 and t^{-2} are solutions to the homogeneous problem. Given this, the method of variation of parmaters states that we can solve for u_1' and u_2' from the equations $u_1' y_1 + u_2' y_2 = 0$ and $u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(2)$. That is, $t^2 u_1' + t^{-2}u_2' = 0$ and $2tu_1' - 2t^{-3}u_2' = 5t$. Solving, we find that $u_1' = 5/4$ and $u_2' = -(5/4)t^4$. Finally, one could finish to find (although not asked) $y_p = (5/4)(t)(t^2) - (1/4)t^5(t^{-2}) = t^3((5/4) - (1/4)) = t^3$.

VI: (15 points) A mass M hangs from a perfect spring with constant K in a uniform gravitational field with constant G. Assume that all these constants are 1. Write the equation of motion for this system, and determine the angular frequency of oscillation. What is the equilibrium position? If the initial position is 0 and the initial velocity is 1, what is the maximum amplitude of the movement of the mass?

ANS: $My'' = -Ky - GM$ is the basic equation, where y points upward. At equilibrium y_{eq}, the downward gravitation force $-GM = K(y_{eq})$ is balanced by the upward force exerted by the spring. Thus $y_{eq} = -GM/K = -1$. Substitute $x(t) = y(t) - y_{eq} = y + 1$. Then $x' = y'$ and $x'' = y''$ and the rewritten equation reads $x'' + x = 0$. The angular frequency is $\sqrt{K/M} = 1$. Finally, $x(t) = Acos(t) + Bsin(t)$, so $x'(t) = -Asin(t) + Bcos(t)$. Now $x(0) = y(0) + 1 = 1 = A$. Also, $x'(0) = y'(0) = 1 = -A(0) + B(1)$, so $B = 1$. Hence $x(t) = cos(t) + sin(t)$, and $y(t) = x(t) + y_{eq} = 1 + cos(t) + sin(t)$. The max amplitude is thus 1 about the equilibrium point y_{eq}.

VII (10 points) The inhomogeneous DE

$$y''' - y'' + y' - y = t$$

has a particular solution $y_p = -t - 1$. (Correction of typo on exam : was listed as $-t + 1$.) First, check that y_p is a solution. Second, the roots to the characteristic equation

$$r^3 - r^2 + r - 1 = (r - 1)(r^2 + +1) = 0$$

are $\{+1, i, -i\}$. Give solutions to the homogeneous problem corresponding to these roots and use them to find another particular solution z_p such that $z_p(0) = 0$ and $z_p'(0) = 0$

ANS: For some constants a,b, and c, $z_p = y_p + ae^t + bcos(t) + csin(t)$. Hence $z_p' = -1 + ae^t - bsin(t) + c(cos(t))$. So $z_p(0) = -1 + a + b = 0$ and $z_p'(0) = -1 + a + c = 0$. We have only 2 equations to use to find 3 constants, so there is no unique solution. For example, $a = -3$, $b = 2$ and $c = 0$ work.

VIII. (5 points) Suppose that $ay'' + by' + cy = g(t)$ is an inhomogeneous linear differential equation. If $y_1(t) = e^{t^2}$ and $y_2(t) = e^{t^2} - t$, find a solution to the homogeneous problem

$$ay'' + by' + cy = 0.$$

ANS: For any two solutions to the non-homogeneous problem, y_p and z_p, their difference must be a solution to the corresponding homogeneous problem. Therefore, $y_1 - y_2 = e^{t^2} - (e^{t^2} - t) = t$ must be a solution to the homogeneous equation $ay'' + by' + cy = 0$.

IX. (10 points total) Fill in the blanks:

a) $exp(i\pi) = e^{i\pi} = cos(\pi) + isin(\pi) = -1 + i0 = -1$

b) $1 + i = e^{ln(2)/2 + i(\pi/4)} = e^{ln(2)/2}e^{i(\pi/4)} = 2^{1/2}(cos(\pi/4) + isin(\pi/4)) = \sqrt{2}(\sqrt{2}/2 + i\sqrt{2}/2) = 1 + i$

c) Give an example of a 5-th order linear homogeneous differential equation.

ANS: $D^5y = y''''' = 0$

d) Give an example of a nice function that oscillates between positive and negative values and approaches 0 as $t \to \infty$.

ANS: y(t) $= e^{-t}sin(t)$.