I. Find the two eigenvalues of the matrix

$$A = \begin{bmatrix} -5 & 12 \\ -4 & 9 \end{bmatrix}.$$

II. For the matrix

$$B = \begin{bmatrix} 11 & -18 \\ 6 & -10 \end{bmatrix},$$

the two eigenvalues are 2 and -1. Find a non-zero eigenvector for each of these eigenvalues.

III. Suppose that a matrix C has the eigenvalues are 1 and -5, with corresponding eigenvectors

$$\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\}.$$

Find the general solution to

$$\vec{x'} = C\vec{x}.$$

Then find a solution so that $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Finally, find all solutions $\vec{x}(t)$ that approach $\vec{0}$ as $t \to +\infty$.

IV. Suppose that a matrix E has eigenvalues $1 \pm i$ and eigenvectors $(1, \pm i)$ respectively. Give a real valued solution to

$$\vec{x'} = E\vec{x}.$$

V. Solve the system of D.E.

$$x_1' = x_2, x_2' = x_1.$$

Hint. Differentiate the first equation to obtain $x_1'' = x_2' = x_1$ by substitution. Now solve $x_1'' = x_1$ for x_1 and hence x_2.

VI. The system

$$\vec{x'} = H\vec{x}, \quad H = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

has general solution

$$Ae^{-t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + Be^{2t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Find a particular solution to

$$\vec{x'} = H\vec{x} + e^t(1,1)$$

by the method of undetermined coefficients.

VII. Convert the second order linear constant coefficient equation

$$y'' + 5y' + 4y = 0$$

into a system of two first order linear constant coefficient equations. Find the eigenvalues of the matrix you get this way.

VII. Problem 1, page 396 of textbook.

I. $\{1, 3\}$.

II. $(2, 1)$ for 2, and $(3, 2)$ for -1.

III. $\vec{x}(t) = Ae^t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + Be^{-5t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Then $A + 2B = 1$ and $2A + B = 0$, so $A = -1/3$ and $B = 2/3$. The only solutions that approach 0 as $t \to +\infty$ must have $A = 0$.

IV. One complex solution is

$$\vec{z}(t) = e^{(1+i)t} \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

The complex conjugate of $\vec{z}(t)$ is another linearly independent complex solution. Also the real and imaginary parts of $\vec{z}(t)$ give real valued solutions. $Re(\vec{z}(t)) = e^t \begin{bmatrix} cos(t) \\ -sin(t) \end{bmatrix}$ and $Im(\vec{z}(t)) = e^t \begin{bmatrix} sin(t) \\ cos(t) \end{bmatrix}$.

V. $x_1'' = x_1$ has solution $x_1(t) = Ae^t + Be^{-t}$. Hence $x_2 = x_1' = Ae^t - Be^{-t}$. That is,

$$\vec{x}(t) = Ae^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + Be^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

VI. Try the solution $\vec{x_p}(t) = e^t \vec{v}$ for unknown v. Then $\vec{x_p'} = e^t \vec{v} ? =? e^t H\vec{v} + e^t(1, 1)$. Thus $\vec{v} = H\vec{v} + (1, 1)$, or $(H - Id)\vec{v} = -(1, 1)$. That is $-2a = -1$ and $1b = -1$ for $\vec{v} = (a, b)$.

VII. $x_1 = y$, $x_2 = y' = x_1'$. Hence $y'' = x_2' = -5y' - 4y = -5x_2 - 4x_1$.That is

$$x_1' = x_2, \quad x_2' = -4x_1 - 5x_2$$

or

$$\vec{x'}(t) = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \vec{x}(t).$$

The eigenvalues are -1 and -4, the same as the roots of the characteristic equation of the second order DE we started with.

VII. Repeated eigenvalue $+1$. Only eigenvectors are multiples of $(2, 1)$. So one solution is $e^t(2, 1)$. The second solution has the general form $\vec{x} = e^t(v_0 + t(2, 1))$, where v_0 is a solution (there are many) to $(A - 1)v_0 = (2, 1)$. For example, $v_0 = (1, 0)$ works.