MA 266 Fall 2000 REVIEW 3

FIRST ORDER DIFFERENTIAL EQUATIONS

You should be able to recognize and know how to solve **first** order differential equations that are either separable, linear, exact, or homogeneous.

You should be able to evaluate integrals of the following types:

$\int$(polymonial) dx,

$\int e^u\,du$,

$\int u^r\,du$, (including $r = -1$)

$\int \frac{ax+b}{(x-r_1)(x-r_2)}\,dx$, (partial fractions)

You should be able to use given values $y(x_0) = y_0$ to determine unknown constants in **a** solution.

You should know the relation of the graph of the solution of an initial value problem to the corresponding direction field.

HOMOGENEOUS EQUATIONS

$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$

Let $y = xv$, so $\frac{dy}{dx} = x\frac{dv}{dx} + v$ and $\frac{y}{x} = v$.

Substitute to obtain $x\frac{dv}{dx} + v = F(v)$.

Solve the above separable equation for v in terms of x.

Substitute $v = \frac{y}{x}$ to obtain a formula for the solution y of the original homogeneous **equation.**

EXACT EQUATIONS

$M(x,y) + N(x,y)\frac{dy}{dx} = 0$ is exact if $M_y(x,y) = N_x(x,y)$.

Find a function $\psi(x, y)$ such that $\psi_x(x,y) = M(x, y)$ and $\psi_y(x,y) = N(x,y)$.

$(\psi(x,y) = \int M(x,y)\,dx + h(y)$; solve $\frac{\partial}{\partial y}\left(\int M(x,y)\,dx\right) + h'(y) = N(x,y)$ for $h(y)$.)

A solution $y = y(x)$ of the exact equation then satisfies

$$\frac{d}{dx}(\psi(x,y)) = \psi_x(x,y) + \psi_y(x,y)\frac{dy}{dx} = M(x,y) + N(x,y)\frac{dy}{dx} = 0,$$

so the general solution is *of* the **form** $\psi(x,y) = c$.

Numerical Methods For Solving $y' = f(t,y), y(t_0) = y_0$:

Create an **M-file** to define the function $f(t, y)$. The function name and the **file** name should be the same. Note that M-f&s are not entered in the **matlab** ***command* window,** but are external text files that are created with a text editor.

EXAMPLE If $y' = \sqrt{t+y}$, create an M-file named **f11.m:**

```
function z=f11(t,y)
z=sqrt(t+y);
```

The general syntax for the Euler tangent line method is

```
>> [t,y]=eul('dfile',t0,tfinal,y0,stepsize);
```

Note that **stepsize** = $(\text{tfinal} - \text{t0})/n$, where **$n$ is the number of steps.**

EXAMPLE To find the Euler tangent line approximation of the solution of the initial **value** problem $y' = \sqrt{t+y}, y(1) = 3$, where $t = 2$ using **stepsize** $h = 0.5$:

```
>> [t,y]=eul('f11',1,2,3,0.5);
>> [t,y]
ans=
    1.0000    3.0000
    1.5000    4.0000
    2.0000    5.1726
```

The syntax is the same for the improved Euler method (use rk2 in place of eul) and the **runge-kutta** method (use rk4 in place of eul.)

To obtain the graph of an approximate solution on a direction field, enter

```
>>plot(t,y,C)
```

where **C='o','x','+'. Omit C** for a connected graph.

APPROXIMATE SOLUTIONS

The matlab commands eul, rk2, and rk4 can be used to obtain approximate **solutions** of **the** initial value problem $y' = f(x,y), y(x_0) = y_0$.

You should **be able** to use the formula $y_n = y_{n-1} + f(x_{n-1}, y_{n-1})h$ **to evaluate** Euler tangent line values by hand.

Approximation methods may not give good **approximations** of **the solution of the initial** value problem $y' = f(x,y), y(x_0) = y_0$, **if:**

- The **initial value** problem does not have a unique solution, because either f **or** f_y **is not** continuous at the initial point.
- The **approximation** extends beyond the interval where the solution is valid, because either $y'(t)$ or $y(x)$ becomes unbounded.
- The solution is unstable, because solutions that have slightly **different** initial values *diverge* **from** the desired solution.

PROP&TIES OF SOLUTIONS

If f *has* continuous first partial derivatives, then solutions of the **differential** equation $y' = f(x,y)$ satisfy

$$y'' = f_x(x,y) + f_y(x,y)\frac{dy}{dx}(x) = f_x(x,y) + f_y(x,y)f(x,y).$$

If $y(x)$ is **a** solution of the differentiable equation $y' = f(x,y)$:

If $y' > 0$ at **a point,** then y is **increasing** near the point.

If $y' < 0$ **at a point,** then y is **decreasion** near the point.

If $y'' > 0$ at a point, then y is concave upward and the Euler tangent line **approximations** are less than or equal to the solution near the point.

If $y'' < 0$ at a point, then y is concave downward and the Euler tangent line approximations are greater than or equal to the solution near the point.

The Taylor **expansion** of f **about** $x = c$ is

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \frac{f''''(c)}{4!}(x-c)^4 + \cdots$$

MA 266 Fall 2000 REVIEW 3 PRACTICE QUESTIONS

1. Determine whether each of the following differential equations is separable, homogeneous, linear, or exact. Briefly justify your answers.

(a) $2x + y + (x+3y)\frac{dy}{dx} = 0$

(b) $x + 3y + (2x+y)\frac{dy}{dx} = 0$

(c) $x + 3y + 1 + (2x+y+1)\frac{dy}{dx} = 0$

(d) $2xy + 1 + (x^2+1)\frac{dy}{dx} = 0$

(e) $x^2 + 1 + (y^2+1)\frac{dy}{dx} = 0$

2. Find the explicit solution of the initial value problem $y' = y^2 - 1, y(0) = 0$.

3. Find the general solution of the differential equation $xy' + 2y = x^2$.

4. Use the formula $y = xv$ to express the differential equation $\frac{dy}{dx} = \frac{x+y}{x-y}$ in terms of x, v, and $\frac{dv}{dx}$.

5. Find an implicit form of the general solution of the **differential** equation $\frac{dy}{dx} = \frac{x^2+y^2}{xy}$.

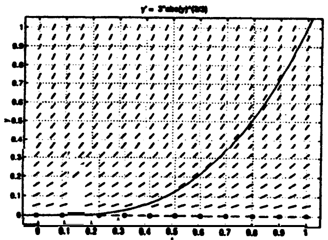
6. Find an implicit solution of the initial value problem

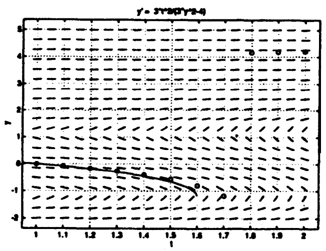
$$2xy + 1 + (x^2 + 2y)\frac{dy}{dx} = 0, \quad y(1) = -1.$$

7. Determine approximate values at $x = 0.5$ of the solution of the initial value problem $y' = 3x + y, y(0) = 1$ by using the Euler tangent line method with $h = 0.25$.

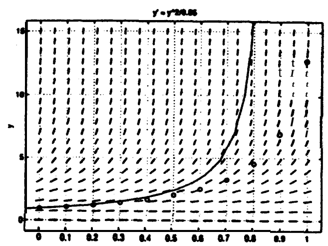
8. Use the given direction fields and the graph of an Euler tangent line approximation of a solution of an initial value problem to explain why the approximation is not a good approximation of the solution.

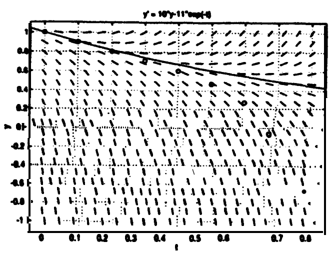
(a) $y' = 3y^{2/3}, y(0) = 0$ (b) $y' = \dfrac{3t^2}{3y^2 - 4}, y(1) = 0$





(c) $y' = y^2/0.85, y(0) = 1$ (d) $y' = 10y - 11e^{-t}, y(0) = 1$





9. Consider the initial value problem $y' = xy + y^2$, y(3) = -1.

(a) Is the solution increasing or decreasing near $x = 3$?

(b) Is the solution concave upward or downward near x = 3?

(c) Are the Euler tangent line approximations of the solution near x = 3 greater than or less than the solution?

10. Find the first four nonzero terms of the Taylor series about $c = 1$ of the solution of the initial value problem $y' = x^2y, y(1) = 2$.

MA 266 Fall 2000 REVIEW 3 PRACTICE QUESTION ANSWERS

1. (a) homogeneous, exact (b) homogeneous (c) none of these types (d) linear, exact (e) separable, exact

2. $y = \dfrac{1 - e^{2x}}{1 + e^{2x}}$

3. $y = \dfrac{x^2}{4} + \dfrac{c}{x^2}$

4. $x\dfrac{dv}{dx} + v = \dfrac{1+v}{1-v}$

5. $\dfrac{1}{2}\left(\dfrac{y}{x}\right)^2 = \ln|x| + C$

6. $x^2y + x + y^2 = 1$

7. $y_2 = 1.75$

8. (a) The initial value problem does not have a unique solution near x_0. The functions $y(t) = 0$ and $y(t) = t^3$ are both solutions.

(b) The approximation extends beyond where the solution is valid . The solution approaches a point where y'(x) becomes unbounded.

(c) The approximation extends beyond where the solution is valid. The solution approaches a vertical asymptote.

(d) The solution is unstable. Solutions that have slightly different initial values diverge from the desired solution.

9. (a) $(x, y) \approx (3, -1)$ implies $y' = xy + y^2 \approx -2 < 0$, so y is decreasing.

(b) $(x, y) \approx (3, -1)$ implies $y'' = xy' + y + 2yy' \approx -3 < 0$, so y is concave downward.

(c) The Euler approximations of the solution near x = 3 are greater than the solution.

10. $y = 2 + 2(x - 1) + 3(x - 1)^2 + 3(x - 1)^3 + \ldots$