MA 266 SPR 00 REVIEW 3

FIRST ORDER DIFFERENTIAL EQUATIONS

You should be able to recognize and know how to solve first order differential equations that are either separable, linear, exact, or homogeneous.
You should be able to evaluate integrals of the following types:

$\int$(polymonial) dx,

$\int e^u\, du$,

$\int u^r\, du$, (including $r = -1$)

$\int \frac{ax+b}{(x-r_1)(x-r_2)}\, dx$, (partial fractions)

You should be able to use given values $y(x_0) = y_0$ to determine unknown constants in a solution.
You should know the relation of the graph of the solution of an initial value problem to the corresponding direction field.

HOMOGENEOUS EQUATIONS

$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$

Let $y = xv$, so $\frac{dy}{dx} = x\frac{dv}{dx} + v$ and $\frac{y}{x} = v$.

Substitute to obtain $x\frac{dv}{dx} + v = F(v)$.

Solve the above separable equation for v in terms of x.

Substitute $v = \frac{y}{x}$ to obtain a formula for the solution y of the original homogeneous equation.

EXACT EQUATIONS

$M(x,y) + N(x,y)\frac{dy}{dx} = 0$ is exact if $M_y(x,y) = N_x(x,y)$.

Find a function $\psi(x,y)$ such that $\psi_x(x,y) = M(x,y)$ and $\psi_y(x,y) = N(x,y)$.

($\psi(x,y) = \int M(x,y)\, dx + h(y)$; solve $\frac{\partial}{\partial y}\left(\int M(x,y)\, dx\right) + h'(y) = N(x,y)$ for $h(y)$.)

A solution $y = y(x)$ of the exact equation then satisfies

$\frac{d}{dx}(\psi(x,y)) = \psi_x(x,y) + \psi_y(x,y)\frac{dy}{dx} = M(x,y) + N(x,y)\frac{dy}{dx} = 0$,

so the general solution is of the form $\psi(x,y) = c$.

Numerical Methods For Solving $y' = f(t,y),\ y(t_0) = y_0$:

Create an M–file to define the function $f(t,y)$. The function name and the file name should be the same. Note that M–files are not entered in the matlab *command* window, but are external text files that are created with a text editor. .

EXAMPLE If $y' = \sqrt{t+y}$, create an M–file named f11.m:

```
function z=f11(t,y)
z=sqrt(t+y);
```

The general syntax for the Euler tangent line method is

```
>> [t,y]=eul('dfile',t0,tfinal,y0,stepsize);
```

Note that stepsize = (tfinal − t0)/n, where n is the number of steps.

EXAMPLE To find the Euler tangent line approximation of the solution of the initial value problem $y' = \sqrt{t+y},\ y(1) = 3$, where $t = 2$ using stepsize $h = 0.5$:

```
>> [t,y]=eul('f11',1,2,3,0.5);
>> [t,y]
 ans=
        1.0000    3.0000
        1.5000    4.0000
        2.0000    5.1726
```

The syntax is the same for the improved Euler method (use rk2 in place of eul) and the runge–kutta method (use rk4 in place of eul.)

To obtain the graph of an approximate solution on a direction field, enter

```
>>plot(t,y,C)
```

where C='o','x','+'. Omit C for a connected graph.

APPROXIMATE SOLUTIONS

The matlab commands eul, rk2, and rk4 can be used to obtain approximate solutions of the initial value problem $y' = f(x,y),\ y(x_0) = y_0$.

You should be able to use the formula $y_n = y_{n-1} + f(x_{n-1}, y_{n-1})h$ to evaluate Euler tangent line values by hand.

Approximation methods may not give good approximations of the solution of the initial value problem $y' = f(x,y),\ y(x_0) = y_0$, if:

- The initial value problem does not have a unique solution, because either f or f_y is not continuous at the initial point.
- The approximation extends beyond the interval where the solution is valid, because either $y'(t)$ or $y(x)$ becomes unbounded.
- The solution is unstable, because solutions that have slightly different initial values diverge from the desired solution.

PROPERTIES OF SOLUTIONS

If f has continuous first partial derivatives, then solutions of the differential equation $y' = f(x,y)$ satisfy

$$y'' = f_x(x,y) + f_y(x,y)\frac{dy}{dx}(x) = f_x(x,y) + f_y(x,y)f(x,y).$$

If $y(x)$ is a solution of the differentiable equation $y' = f(x,y)$:

If $y' > 0$ at a point, then y is increasing near the point.

If $y' < 0$ at a point, then y is decreasion near the point.

If $y'' > 0$ at a point, then y is concave upward and the Euler tangent line approximations are less than or equal to the solution near the point.

If $y'' < 0$ at a point, then y is concave downward and the Euler tangent line approximations are greater than or equal to the solution near the point.

The Taylor expansion of f about $x = c$ is

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \frac{f''''(c)}{4!}(x-c)^4 + \cdots$$

MA 266 SPR 00 REVIEW 3 PRACTICE QUESTIONS

1. Determine whether each of the following differential equations is separable, homogeneous, linear, or exact. Briefly justify your answers.

(a) $2x + y + (x + 3y)\dfrac{dy}{dx} = 0$

(b) $x + 3y + (2x + y)\dfrac{dy}{dx} = 0$

(c) $x + 3y + 1 + (2x + y + 1)\dfrac{dy}{dx} = 0$

(d) $2xy + 1 + (x^2 + 1)\dfrac{dy}{dx} = 0$

(e) $x^2 + 1 + (y^2 + 1)\dfrac{dy}{dx} = 0$

2. Find the explicit solution of the initial value problem $y' = y^2 - 1,\ y(0) = 0$.

3. Find the general solution of the differential equation $xy' + 2y = x^2$.

4. Use the formula $y = xv$ to express the differential equation $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$ in terms of x, v, and $\dfrac{dv}{dx}$.

5. Find an implicit form of the general solution of the differential equation $\dfrac{dy}{dx} = \dfrac{x^2+y^2}{xy}$.

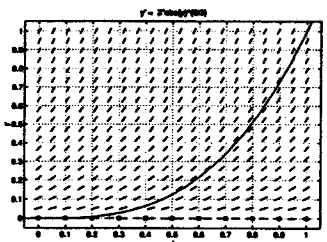
6. Find an implicit solution of the initial value problem

$$2xy + 1 + (x^2 + 2y)\frac{dy}{dx} = 0, \quad y(1) = -1.$$

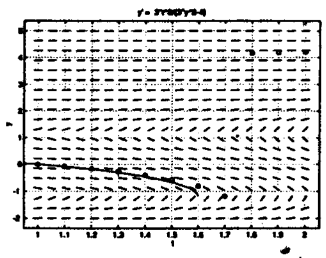
7. Determine approximate values at $x = 0.5$ of the solution of the initial value problem $y' = 3x + y,\ y(0) = 1$ by using the Euler tangent line method with $h = 0.25$.

8. Use the given direction fields and the graph of an Euler tangent line approximation of a solution of an initial value problem to explain why the approximation is not a good approximation of the solution.

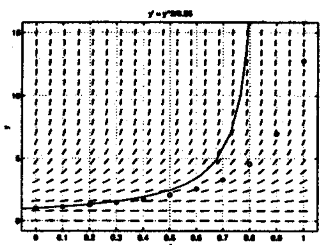
(a) $y' = 3y^{2/3},\ y(0) = 0$



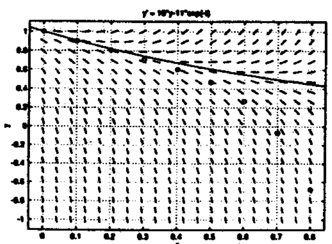
(b) $y' = \dfrac{3t^2}{3y^2 - 4},\ y(1) = 0$



(c) $y' = y^2/0.85,\ y(0) = 1$



(d) $y' = 10y - 11e^{-t},\ y(0) = 1$



9. Consider the initial value problem $y' = xy + y^2,\ y(3) = -1$.

(a) Is the solution increasing or decreasing near $x = 3$?

(b) Is the solution concave upward or downward near $x = 3$?

(c) Are the Euler tangent line approximations of the solution near $x = 3$ greater than or less than the solution?

10. Find the first four nonzero terms of the Taylor series about $c = 1$ of the solution of the initial value problem $y' = x^2y,\ y(1) = 2$.

MA 266 SPR 00 REVIEW 3 PRACTICE QUESTION ANSWERS

1. (a) homogeneous, exact (b) homogeneous (c) none of these types (d) linear, exact (e) separable, exact

2. $y = \dfrac{1 - e^{2x}}{1 + e^{2x}}$

3. $y = \dfrac{x^2}{4} + \dfrac{C}{x^2}$

4. $x\dfrac{dv}{dx} + v = \dfrac{1+v}{1-v}$

5. $\dfrac{1}{2}\left(\dfrac{y}{x}\right)^2 = \ln|x| + C$

6. $x^2y + x + y^2 = 1$

7. $y_2 = 1.75$

8. (a) The initial value problem does not have a unique solution near x_0. The functions $y(t) = 0$ and $y(t) = t^3$ are both solutions.

(b) The approximation extends beyond where the solution is valid . The solution approaches a point where $y'(x)$ becomes unbounded.

(c) The approximation extends beyond where the solution is valid. The solution approaches a vertical asymptote.

(d) The solution is unstable. Solutions that have slightly different initial values diverge from the desired solution.

9. (a) $(x, y) \approx (3, -1)$ implies $y' = xy + y^2 \approx -2 < 0$, so y is decreasing.

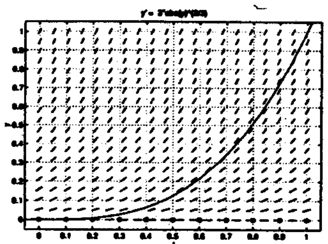
(b) $(x, y) \approx (3, -1)$ implies $y'' = xy' + y + 2yy' \approx -3 < 0$, so y is concave downward.

(c) The Euler approximations of the solution near $x = 3$ are greater than the solution.

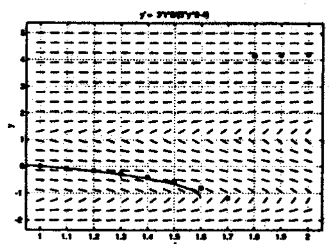
10. $y = 2 + 2(x-1) + 3(x-1)^2 + 3(x-1)^3 + \cdots$

8. Use the given direction fields and the graph of an Euler tangent line approximation of a solution of an initial value problem to explain why the approximation is not a good approximation of the solution.

(a) $y' = 3y^{2/3},\ y(0) = 0$



(b) $y' = \dfrac{3t^2}{3y^2 - 4},\ y(1) = 0$



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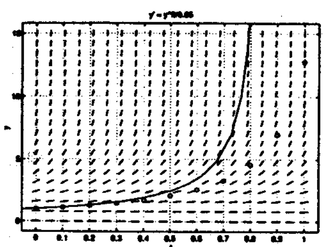

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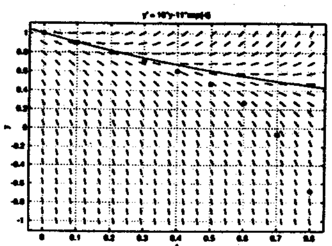

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MA 266 SPR 00 REVIEW 3 PRACTICE QUESTION ANSWERS

1. (a) homogeneous, exact (b) homogeneous (c) none of these types (d) linear, exact (e) separable, exact

2. $y = \dfrac{1 - e^{2x}}{1 + e^{2x}}$

3. $y = \dfrac{x^2}{4} + \dfrac{C}{x^2}$

4. $x\dfrac{dv}{dx} + v = \dfrac{1+v}{1-v}$

5. $\dfrac{1}{2}\left(\dfrac{y}{x}\right)^2 = \ln|x| + C$

6. $x^2y + x + y^2 = 1$

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8. (a) The initial value problem does not have a unique solution near x_0. The functions $y(t) = 0$ and $y(t) = t^3$ are both solutions.

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(b) $(x, y) \approx (3, -1)$ implies $y'' = xy' + y + 2yy' \approx -3 < 0$, so y is concave downward.

(c) The Euler approximations of the solution near $x = 3$ are greater than the solution.

10. $y = 2 + 2(x-1) + 3(x-1)^2 + 3(x-1)^3 + \cdots$

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1. (a) $L[y] = y'' - 3y' + 2y$. Evaluate $L[e^t]$, $L[e^{2t}]$, $L[e^{-t}]$.

(b) $L[y] = y'' - 4y' + 4y$. Evaluate $L[e^{2t}]$, $L[te^{2t}]$, $L[t^2e^{2t}]$.

(c) $L[y] = y'' - 4y' + 5y$. Evaluate $L[e^{2t}\cos t]$, $L[e^{2t}\sin t]$, $L[\sin t]$.

2. Suppose that y_0 is a solution of $t^2y'' + ty' + y = t^2$ and $L[y] = t^2y'' + ty' + y$. Evaluate $L[y_0 + t^2 - 2t + 1]$.

3. Find the largest open interval for which the initial value problem $y'' + \frac{1}{t}y' + \frac{1}{t-2}y = \frac{1}{t-3}$, $y(1) = 3$, $y'(1) = 2$, has a solution.

4. (a) Show that $y_1 = t$ and $y_2 = t^{-1}$ are solutions of the differential equation $t^2y'' + ty' - y = 0$.
(b) Evaluate the Wronskian $W(t, t^{-1})(t)$.
(c) Find the solution of the initial value problem $t^2y'' + ty' - y = 0$, $y(1) = 2$, $y'(1) = 4$.

In Problems 5–7 find the general solution of the homogeneous differential equations in (a) and use the method of undetermined coefficients to find the form of a particular solution of the nonhomogeneous equations in (b) and (c).

5. (a) $y'' - 5y' + 6y = 0$

(b) $y'' - 5y' + 6y = t^2$

(c) $y'' - 5y' + 6y = e^{2t} + \cos(3t)$

6. (a) $y'' - 6y' + 9y = 0$

(b) $y'' - 6y' + 9y = te^{3t}$

(c) $y'' - 6y' + 9y = e^t + \cos(3t)$

7. (a) $y'' - 2y' + 10y = 0$

(b) $y'' - 2y' + 10y = e^t + \cos(3t)$

(c) $y'' - 2y' + 10y = e^t \cdot \cos(3t)$

8. Find the general solution of the differential equation $y'' - y' = 4t$.

9. The differential equation $t^2y'' + ty' - y = 0$ has solution $y_1(t) = t$.

(a) Use the method of reduction of order to find a differential equation satisfied by v, where $y(t) = tv(t)$ is a solution of $t^2y'' + ty' - y = 0$.

(b) Solve the differential equation in (a) to find a solution of $t^2y'' + ty' - y = 0$ that is not a constant multiple of y_1.

(c) Find the general solution of the differential equation $t^2y'' + ty' - y = 0$.

MA 266 SPR 00 REVIEW 4 PRACTICE QUESTION ANSWERS

1. (a) $L[e^t]=0,\ L[e^{2t}]=0,\ L[e^{-t}]=6e^{-t}$

   (b) $L[e^{2t}]=0,\ L[te^{2t}]=0,\ L[t^2e^{2t}]=6e^{2t}$

   (c) $L[e^{2t}\cos t]=0,\ L[e^{2t}\sin t]=0,\ L[\sin t]=2\sin t-4\cos t$

2. $L[y_0+t^2-2t+1]=6t^2-4t+1$

3. $0<t<2$

4. (b) $W(t,t^{-1})(t)=-2t^{-1}$ (c)$y=3t-t^{-1}$

5. (a) $y=C_1e^{2t}+C_2e^{3t}$

   (b) $y=At^2+Bt+C$

   (c) $y=Ate^{2t}+B\cos(3t)+C\sin(3t)$

6. (a) $y=C_1e^{3t}+C_2te^{3t}$

   (b) $y=t^2(At+B)e^{3t}$

   (c) $y=Ae^t+B\cos(3t)+C\sin(3t)$

7. (a) $y=(C_1\cos(3t)+C_2\sin(3t))e^t$

   (b) $y=Ae^t+B\cos(3t)+C\sin(3t)$

   (c) $y=t(A\cos(3t)+B\sin(3t))e^t$

8. $y=C_1+C_2e^t-2t^2-4t$

9. (a) $t^3v''+3t^2v'=0$

   (b) $y=t^{-1}$ or $y=a_1t^{-1}+a_2t,\ a_1\neq 0$

   (c) y=$C_1t+C_2t^{-1}$

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VARIATION OF PARAMETERS

If y_1 and y_2 are solutions of $y''+py'+qy=0$ with $W(y_1,y_2)\neq 0$, we can find a particular solution of $y''+py'+qy=g$ of the form $y(t)=u_1(t)y_1(t)+u_2(t)y_2(t)$. The simplifying assumption (1) $u_1'y_1+u_2'y_2=0$ gives $y'=u_1y_1'+u_2y_2'$. Then substitution of y,y', and y'' into $y''+py'+qy=g$ and simplifying gives the differential equation (2) $u_1'y_1'+u_2'y_2'=g$. Equations (1) and (2) can be solved for u_1' and u_2'. This gives the following:

THEOREM: If p,q, and g are continuous on an open interval I, y_1 and y_2 are solutions of the homogeneous differential equation $y''+p(t)y'+q(t)y=0$, and $W(y_1,y_2)\neq 0$, then the nonhomogeneous differential equation
$y''+p(t)y'+q(t)y=g(t)$ has particular solution
$$Y(t)=-y_1(t)\int\frac{y_2(t)g(t)}{W(y_1,y_2)(t)}\,dt+y_2(t)\int\frac{y_1(t)g(t)}{W(y_1,y_2)(t)}\,dt$$
and general solution $y=c_1y_1(t)+c_2y_2(t)+Y(t)$.

HIGHER ORDER LINEAR EQUATIONS

The theory is similar to that for second order linear equations. This includes the interval in which solutions exist, the form of general solutions of homogeneous and nonhomogeneous equations with constant coefficients, and the method of undetermined coefficients.

APPLICATIONS

Spring–mass system, $mu''+\gamma y'+ky=F_0\cos(\omega t)$:

The mass of an unforced system $mu''+\gamma y'+ky=0$ does not oscillate if the system is either critically damped, $\gamma=2\sqrt{km}$, or overdamped, $\gamma>2\sqrt{km}$. Otherwise, the system oscillates.

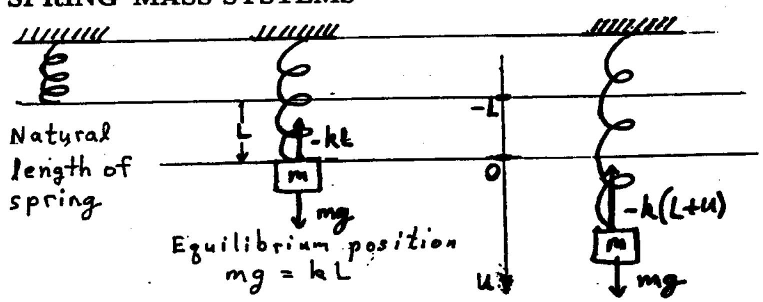
A forced, undamped system becomes unbounded as $t\to\infty$ if and only if $\omega=\sqrt{k/m}$.
A damped system is always bounded.

Know how to find steady–state solutions.

Know how to interpret initial conditions and graphs of solutions.

Know how to use the formulas $R\cos\delta=A,\ R\sin\delta=B,\ R=\sqrt{A^2+B^2}$, $A\cos(\omega_0t)+B\sin(\omega_0t)=R\cos(\omega_0t-\delta)$.

SPRING–MASS SYSTEMS



mass, $m = \dfrac{\text{Weight}}{g}$, $\quad g = 32\ \text{ft/sec}^2 = 9.8\ \text{m/sec}^2 = 980\ \text{cm/sec}^2$,
gravitational force = weight = mg,
spring constant, $k = \dfrac{\text{Force}}{\text{Displacement}}$,
$u(t)$ = displacement from equilibrium position, where $mg = kL$,
spring force = $-k(L + u(t))$,
damping constant, $\gamma = \dfrac{\text{Force}}{\text{Speed}}$,
damping force = $-\gamma u'(t)$,
applied external force = $F(t)$,

$$\text{(mass)(acceleration)} = \text{(sum of all external forces)},$$

$$mu''(t) = mg - k(L + u(t)) - \gamma u'(t) + F(t),$$

$$\boxed{mu''(t) + \gamma u'(t) + ku(t) = F(t),\ u(0) = u_0,\ u'(0) = u_0',}$$

System	Length	Mass	Time	Force
English	feet	slugs	seconds	pounds
mks	meters	kilograms	seconds	newtons
cgs	centimeters	grams	seconds	dynes

$$\boxed{A\cos(\omega_0 t) + B\sin(\omega_0 t) = R\cos(\omega_0 t - \delta),}$$

where

$$R\cos(\omega_0 t - \delta) = R(\cos(\omega_0 t)\cos\delta + \sin(\omega_0 t)\sin\delta),$$

so $R\cos\delta = A,\ R\sin\delta = B,\ R = \sqrt{A^2 + B^2},\ \tan\delta = B/A.$

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In Problems 1–3 find the general solution of the homogeneous differential equations in (a) and use the method of undetermined coefficients to find the form of a particular solution of the nonhomogeneous equation in (b).

1. (a) $y''' - y' = 0$

(b) $y''' - y' = t + e^t$

2. (a) $y''' - y'' - y' + y = 0$

(b) $y''' - y'' - y' + y = e^t + \cos t$

3. (a) $y''' - y = 0$

(b) $y''' - y = te^{-t/2}\cos(\sqrt{3}t/2)$

4. Find the general solution of the differential equation $y''' + y' = t^2$.

5. Find the solution of the initial value problem
$y''' - 2y'' + y' = 0,\ y(0) = 2,\ y'(0) = 0,\ y''(0) = 1.$

6. (a) Find the general solution of the differential equation
$y'' + 5y' + 6y = 26\cos(2t)$.

(b) Find the steady–state solution of the differential equation
$y'' + 5y' + 6y = 26\cos(2t)$.

7. Use the formulas $R\cos\delta = A,\ R\sin\delta = B,\ R = \sqrt{A^2 + B^2}$,
$A\cos(\omega_0 t) + B\sin(\omega_0 t) = R\cos(\omega_0 t - \delta)$ to find R and δ such that
$-3\cos(2t) + 4\sin(2t) = R\cos(2t - \delta)$.

8. For what nonnegative values of m will the the solution of the initial value problem $mu'' + 4u = 8\cos(4t)$, $u(0) = 4$, $u'(0) = 0$, become unbounded as $t \to \infty$?.

9. For what nonnegative values of γ will the the solution of the initial value problem $u'' + \gamma u' + 4u = 0$, $u(0) = 4$, $u'(0) = 0$, oscillate?

10. A mass that weighs 4 pounds stretches a spring 0.25 feet. The mass is acted upon by an external force of $2\cos t$ pounds and moves in a medium that imparts a viscous force of 6 pounds when the speed of the mass is 3 feet/sec. At time $t = 0$ the mass is 0.5 feet below the equilibrium position of the system and the mass is moving upward at 5 feet/sec. Set up an initial value problem that describes the motion of the mass. You do not need to solve the initial value problem.

11. The differential equation $t^2y'' - ty' = 0$ has solutions $y_1 = 1$ and $y_2 = t^2$. Use the method of variation of parameters to find a solution of $t^2y'' - ty' = 4t^4$.

12. The differential equation $t^2y'' - 2ty' + 2y = 0$ has solution $y_1 = t$. Find the general solution of $t^2y'' - 2ty' + 2y = 2t^2$.

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1. (a) $y = C_1 + C_2e^{-t} + C_3e^t$

(b) $y = t(At + B) + Cte^t$

2. (a) $y = C_1e^t + C_2te^t + C_3e^{-t}$

(b) $y = At^2e^t + B\cos t + C\sin t$

3. (a) $y = C_1e^t + C_2e^{-t/2}\cos(\sqrt{3}t/2) + C_3e^{-t/2}\sin(\sqrt{3}t/2)$

(b) $y = te^{-t/2}((At + B)\cos(\sqrt{3}t/2) + (Ct + D)\sin(\sqrt{3}t/2))$

4. $y = C_1 + C_2\cos t + C_3\sin t + \frac{1}{3}t^3 - 2t$

5. $y = 3 \div e^t + te^t$

6. (a) $y = C_1e^{-2t} + C_2e^{-3t} + \frac{1}{2}\cos(2t) + \frac{5}{2}\sin(2t)$

(b) $y(\text{steady–state}) = \frac{1}{2}\cos(2t) + \frac{5}{2}\sin(2t)$

7. $R = 5$, $\delta = \tan^{-1}(-4/3) + \pi \approx 2.214$.

8. $m = 1/4$

9. $0 \le \gamma < 4$

10. $\frac{1}{8}u'' + 2u' + 16u = 2\cos t$, $u(0) = \frac{1}{2}$, $u'(0) = -5$

11. $y = u_1 + t^2u_2 = -\frac{1}{2}t^4 + t^2 \cdot t^2 = \frac{1}{2}t^4$

12. $y = C_1t + C_2t^2 + 2t^2\ln t$