Math 266 Review 4a, Spring 2001
Complex numbers and the complex exponential function

The complex numbers are needed to provide roots for certain polynomials, such as $x^2 = -1$. The roots of this particular polynomial are called $\pm i$. The convention is that the complex number $a + bi$ is graphed as the point (a, b) in the plane. We can also think of the complex numbers in terms of polar coordinates, so that $a + bi = r(cos(\theta) + isin(\theta))$, where $r = \sqrt{(a^2 + b^2)}$ and $\theta = arctangent(b/a)$.

Example: $1 + i = \sqrt{2}(cos(\pi/4) + isin(\pi/4)) = \sqrt{2}e^{i\pi/4}$
Arithmetic with complex numbers for addition and multiplication is not so bad.
Example: $(1+2i)+(3-3i) = ((1+3)+(2-3)i) = 4-i$ and $(2+i)(3-5i) = (6 - 10i + 3i - 5(i^2)) = 6 - 7i - 5(-1) = 6 + 5 - 7i = 11 - 7i$.

For division, it's convenient to have the idea of *complex conjugation*: – the conjugate of $z = (a+bi)$ is $\bar{z} = (a-bi)$. Here "a" is often called the part of $z = a+bi$, and "b" is the imaginary part of $(a+bi)$. NOT: the imaginary part of z is a real number.

The *absolute value* or *magnitude* of $z = a + ib$ is $\sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$.

Then division $(c + di)/(a + bi) = \bar{z}(c + di)/(z\bar{z}) = (a - bi)(c + di)/(a^2 + b^2)$.

Example: $(2+3i)/(1-i) = (1+i)(2+3i)/(1+i)(1-i) = (2-3+5i)/(1+1)$.
Euler's Formula:

$$e^{(a+bi)t} = e^{at}(cos(bt) + isin(bt)$$

In particular

$$e^{i\omega t} = cos(\omega t) + isin(\omega t)$$

Conversely, $cos(\omega t) = (e^{i\omega t} + e^{-i\omega t})/2$
and $sin(\omega t) = (e^{i\omega t} - e^{-i\omega t})/2i$.

Examples: Use of Euler's formula to find roots and powers.
Integral powers are easy: if $z = 5e^{2\pi i/5}$, then $z^3 = 5^3e^{6\pi i/5}$.
To solve $r^N = 1$, note that $1 = e^{02\pi i} = e^{2M\pi i}$ for any integer M. Hence the

numbers $r_k = e^{2\pi ki/N}$ are all solutions of $r^N = 1$, for any integer k. It suffices to take $0 \leq k \leq N-1$ in order to get N distinct roots of unity.
Example: The third roots of unity: $\{1, e^{2\pi i/3}, e^{4\pi i/3}\} = \{1, (-1 \pm \sqrt{(-3)})/2\}$.

Complex numbers and Euler's formula allow one to write the solutions to linear constant coefficient DE as either complex exponentials or as products of real exponentials with sines and cosines.

Fundamental Theorem of Algebra: The polynomial $p(x) = x^n + a_1 x^{n-1} \ldots + a_0 = 0$ has n complex roots (counting multiplicities). If the coefficients are real, then the roots are either real, or appear in complex conjugate pairs. If n is odd, at least one of the roots is real.

Example: $r^3 - 1 = 0$. Roots are $r = 1, (-1 \pm i\sqrt{3})/2$

Theorem: If $P(D) = D^n + a_1 D^{n-1} \ldots a_0$ is a linear differential operator with constant REAL coefficients, then if $z(t)$ is a solution to $P(D)z = 0$, then so is the complex conjugate $\bar{z}(t)$ and each of the real and imaginary parts of $z(t)$.

Example. $y'' + 4y = 0$. One solution is e^{2it}. The conjugate is e^{-2it}, the real part is $cos(2t)$, the imaginary part is $sin(2t)$. Each of these latter three is a solution also.

Example. $y'' + y = 0, y(0) = 1, y'(0) = 0$.
Method 1: One fundamental solution set is $\{e^{it}, e^{-it}\}$. Set $y = Ae^{it} + Be^{-it}$. Then $y' = iAe^{it} - Be^{-it}$. Then $y(0) = A + B = 0$ and $y'(0) = i(A - B) = 0$. Hence $A = B$ Thus $A = B = 1/2$, and $y(t) = (e^{it} + e^{-t})/2 = cos(t)$.
Method 2: Another fundamental solution set is $\{cos(t), sin(t)\}$. Set $y(t) = c_1 cos(t) + c_2 sin(t)$, and $y' = -c_1 sin(t) + c_2 cos(t)$. So $y(0) = 1 = c_1(1) + c_2(0)$ and $y'(0) = -c_1(0) + c_2(1) = 0$. That is, $c_2 = 0$ and $c_1 = 1$. That is, $y(t) = cos(t)$.