First Midterm, Wilkerson section of Math 303, October 7, 1998

No notes, books, or calculators allowed. Show all work on these pages. For full credit, adequate work must be shown. Table of transforms is on last page.

1. For the differential equation $y'' + 4y' + 4y = 0$, let $y = \sum_{0}^{\infty} a_n x^n$ be the series solution at $x = 0$.

A) Find the recurrence relation for $a_{n+2}$ in terms of $a_{n+1}$ and $a_n$. [10]

B) Find the terms $a_2, a_3$ (in terms of $a_0$ and $a_1$). [10]
C) If the solution $y(t)$ has $y(0) = 0$ and $y'(0) = 1$, solve for $a_0, a_1, a_2, a_3$. Can you give a closed formula for $y(t)$. [5]

2. Solve the Euler equation: $x^2y'' - 2xy' + 2y = 0$ [10]
and give a solution satisfying $y(1) = 1, y'(1) = 0$. [5]

3. For the differential equations $x^2y'' - 4xy' + (x + 6)y = 0$.
   A. Show that this D.E. has a regular singular point at $x = 0$. [5]

   B. Write the indicial equation and solve for its roots. [5]
C. If \( y_1 \) is the solution corresponding to the larger root \( r_1 \) of the indicial equation in part (A), then the second solution \( y_2 \) corresponding to the smaller root \( r_2 \) is of the form: (CIRCLE a, b, c, d, or e):

(Show all reasoning for your conclusions.)

\[ a) \ a_0 + a_1 x + \ldots + a_n x^n + \ldots \]

\[ b) \ x^{r_2} \sum_{0}^{\infty} b_n x^n \]

\[ c) \ x^{r_2}(1 + \sum_{1}^{\infty} b_n x^n) + ay_1 \ln(x) \]

\[ d) \ x^{r_2} \sum_{1}^{\infty} b_n x^n + y_1 \ln(x) \]

\[ e) \ None \ of \ these \ answers. \]

4. Solve for \( Y(s) \), the Laplace transform of the solution \( y(t) \) of the differential equation

\[ y'' + 4y' + 4y = 4e^{-2t} + u_2(t) \]

with initial conditions \( y(0) = 1, y'(0) = -2 \). Here \( u_2(t) = u(t - 2) \) is the step function which is 0 for \( t < 2 \) and 1 for \( t \geq 2 \).
5. Using step functions, write the following function $f$ as a single formula. Give a rough graph of $f$. Find the Laplace transform for $f$. 

$$f(x) = \begin{cases} 
        e^t & \text{if } 0 \leq t < \ln(2) \\
        2 & \text{if } \ln(2) \leq t < 2 \\
        4-t & \text{if } 2 \leq t < 4 \\
        0 & \text{if } 4 \leq t 
\end{cases}$$
6A. Find the inverse Laplace transforms of $F(s) = \frac{2s + 7}{(s - 8)(s - 5)}$.

6B. Find the inverse Laplace transforms $F(s) = \frac{2s + 7}{(s - 8)(s - 5)}e^{-3s}$.

6C. Find the inverse Laplace transforms $F(s) = \frac{2s + 1}{s^2 - 6s + 10}$.

6D. For $f = e^{at}$ and $g = e^{bt}$ compute the convolution product function $f \ast g$.

What is the Laplace transform of $(f \ast g)(t)$?
<table>
<thead>
<tr>
<th>Function</th>
<th>Laplace Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1/s$</td>
</tr>
<tr>
<td>$e^{at} f(t)$</td>
<td>$F(s-a)$</td>
</tr>
<tr>
<td>$\sin(at)$</td>
<td>$a/(s^2 + a^2)$</td>
</tr>
<tr>
<td>$\cos(at)$</td>
<td>$s/(s^2 + a^2)$</td>
</tr>
<tr>
<td>$t^n$</td>
<td>$n!/s^{n+1}$</td>
</tr>
<tr>
<td>$u(t-a)f(t-a)$</td>
<td>$F(s)e^{-as}$</td>
</tr>
<tr>
<td>$\delta(t-a)$</td>
<td>$e^{-as}$</td>
</tr>
<tr>
<td>$f'(t)$</td>
<td>$sF(s) - f(0)$</td>
</tr>
<tr>
<td>$f''(t)$</td>
<td>$s^2F(s) - sf(0) - f'(0)$</td>
</tr>
<tr>
<td>$f * g$</td>
<td>$L[f]L[g]$</td>
</tr>
</tbody>
</table>

Here $f * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau$. 