MATH 303 Second Exam on 7, 8.1, 10.1-2
Wilkerson 9:30AM Section
November 19, 1998

No books, notes, calculators. Show all work for credit on these pages. Ask the instructor if you need help understanding what’s needed for a complete answer. (90 minutes).

Scoring Grid:

1. (20 points) For the system
\[
\begin{align*}
\vec{x}''(t) &= A\vec{x}(t) \\
\frac{dx_1}{dt} &= 3x_1 + x_2 \\
\frac{dx_2}{dt} &= x_1 + 3x_2
\end{align*}
\]

(a) find the eigenvalues of the matrix \( A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \).

(b) find the eigenvectors for the matrix \( A \).

(c) use these eigenvectors and eigenvalues to give a general solution to the system.

(d) Find a solution for which \( \vec{x}(0) = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \).
2. (15 points) Suppose that $A$ is a $2 \times 2$ matrix of real numbers and that $A$ has complex eigenvalues $r = 2 \pm i$, and the vector
\[
\begin{bmatrix}
i \\
1
\end{bmatrix}
\]
is a complex eigenvector associated to $r = 2 + i$.

(a) (5 points) Write $e^{(2+i)t}$ in terms of real exponentials, sines and cosines.

(b) (10 points) Give a pair of linearly independent REAL solutions to the linear system $\ddot{\mathbf{x}}(t) = A\mathbf{x}(t)$, where $A$ is as described.
3. (15 points)

(a) (10 points) Suppose that \( A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) has eigenvalues \( \pm 1 \) and eigenvectors \( v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) and \( v_{-1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \). Then

\[
X(t) = \begin{bmatrix} e^t & e^{-t} \\ e^t & e^{-t} \end{bmatrix}
\]

is a fundamental solution matrix for \( \mathbf{x}'(t) = A \mathbf{x}(t) \). If \( \mathbf{x}'(t) = A \mathbf{x}(t) + \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix} \) has a solution of the form \( \mathbf{x}(t) = X(t) \mathbf{u}(t) \), find \( \mathbf{u}(t) \) and give a particular solutions using the variation of parameters method. Don’t carry out integrations needed to fully solve.

(b) (5 points) Let \( B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \). Find the eigenvalues and two linearly independent eigenvectors for \( B \).
4. (10 points) Use the simple *Euler Method* with a step size of \( h = \frac{1}{2} \) to find an approximate value of \( y(2) \) where \( y(x) \) is the solution to the initial value problem,

\[
\frac{dy}{dx} = f(x, y) = x + y
\]

with \( y(0) = \frac{1}{2} \). Hint: make a small table of values for \( x, y, \) and \( f(x, y) \).

5. (10 points) Define the function \( f(x) \) on the interval \( [0, \pi] \) which is +1 for \( 0 \leq x, \pi/2 \) and 0 for \( \pi/2 \leq x \leq \pi \). Give a graph of \( f(x) \) and compute the Fourier sin series for this \( f(x) \).
6. (15 points) Consider the ordinary differential equation

\[ X''(x) + \lambda X(x) = 0 \]

on the interval \( 0 \leq x \leq \pi \), subject to the boundary conditions

\[
\begin{align*}
X(0) &= 0 \\
\frac{dX}{dx}(\pi) &= 0.
\end{align*}
\]

For what positive values of the constant \( \lambda \) do there exist solutions \( X(x) \) to this problem which are not identically zero?
7. (15 points) Consider the Heat Equation problem 

\[ u_{xx} = u_t \]

on the region \(0 < x < \pi\) and \(0 < t\) with \(u(0, t) = 0, u(\pi, t) = 0\) and \(u(x, 0) = f(x)\).

(a) (8 points) Write the solution assuming that \(f(x)\) has Fourier sin series

\[ \sum_{n=1}^{\infty} b_n \sin (nx). \]

(b) (7 points) Assume that \(f(x) = \sin(x) + 2\sin(3x)\). Write out the solution in this case. Estimate \(u(\pi/2, 0)\) and \(u(\pi/2, 1)\). (you don’t need exact numerical answer). Estimate the value of \(u(\pi/2, t)\) for very large \(t\).