MATH 303 - 12/16/97 FINAL EXAM - Alternate WILKERSON SECTION	Fall 97	Name: ____________ ID.NO: ____________ ____________

PLEASE, BE NEAT AND SHOW ALL YOUR WORK; CIRCLE YOUR ANSWER. NO NOTES, BOOKS, CALCULATORS, TAPE PLAYERS, or COMPUTERS.

Problem Number	Possible Points	Points Earned
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
SUBTOTAL	80	
I	20	
II	20	
III	20	
IV	20	
V	20	
VI	20	
SUBTOTAL	120	
TOTAL	200	

These multiple choice and short answer questions are worth 10 points each. Very little partial credit will be given. Supporting work will be considered in marginal cases. Circle on these sheets the correct answers. On questions with several correct answers, your score will be computed on the basis of the number of your correct responses minus the the number of your incorrect responses.

1. The only solution to

$$\mathbf{x}' = \frac{\mathbf{dx}}{dt} = \mathbf{Ax}$$

with $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ is

A. $5e^{2t}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 2e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

B. $5e^{-2t}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 2e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

C. $e^{2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^{t}\begin{bmatrix} 0 \\ 2 \end{bmatrix}$

D. $5e^{2t}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 2e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

E. $e^{t}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

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2. Which of the following pairs of vector functions are a basis for the space of all solutions to the system of differential equations (circle all that work).

$$\mathbf{x}' = \frac{d\mathbf{x}}{dt} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}\mathbf{x}$$

A. $e^{2it}\begin{bmatrix} i \\ 1 \end{bmatrix}$ and $e^{-2it}\begin{bmatrix} -i \\ 1 \end{bmatrix}$

B. $\begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$ and $\begin{bmatrix} -\cos(2t) \\ \sin(2t) \end{bmatrix}$

C. $e^{2it}\begin{bmatrix} -i \\ 1 \end{bmatrix}$ and $e^{-2it}\begin{bmatrix} i \\ 1 \end{bmatrix}$

D. $e^{2t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

E. $\begin{bmatrix} -\sin(2t) \\ \cos(2t) \end{bmatrix}$ and $\begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix}$

3. Suppose that all the solutions of a 2×2 homogeneous linear system of differential equations $\vec{x}\,' = A\vec{x}$ tend to zero as $t \to \infty$. Then which of the following are possible eigenvalues for$A$:

A. 1 and -1

B. i and $-i$

C. $1+i$ and $1-i$

D. $-1+i$ and $-1-i$

E. -2 and -7

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4. The general solution of the linear system of differential equations

$$\begin{aligned} x_1' &= x_1 + 2x_2 \\ x_2' &= 4x_1 + 3x_2 \end{aligned}$$

is equal to

A. $c_1 \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} e^{5t} \\ 2e^{5t} \end{bmatrix}$

B. $c_1 \begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} e^{5t} \\ -2e^{5t} \end{bmatrix}$

C. $c_1 \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} e^{5t} \\ -2e^{5t} \end{bmatrix}$

D. $c_1 \begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} e^{5t} \\ 2e^{5t} \end{bmatrix}$

E. None of the above.

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5. The function $x_2(t)$ determined by the initial value problem

$$\begin{aligned} x_1' &= x_1 \\ x_2' &= x_1 + x_2 \end{aligned}$$

with initial conditions $x_1(0) = 1$ and $x_2(0) = 1$ is given by

A. $x_2 = e^t + te^t$

B. $x_2 = e^t$

C. $x_2 = e^{-t} + te^t$

D. $x_2 = cosh(t)$

E. $x_2 = 1 + t$

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6. The ODE

$$y'' + xy' + y = 0$$

has an ordinary point at 0. That is, there is a power series solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Which of the following best expresses the solution with $y(0) = 2, \quad y'(0) = 1$:

A. $a_0 = 2, \quad a_1 = 1$ and $a_{n+2} = -(n+1)a_n/(n-1)$

B. $a_0 = 2, \quad a_1 = 1$ and $a_{n+1} = (n+3)a_n/((n+1)(n+2))$

C. $a_0 = 2, \quad a_1 = 0$ and $a_{n+2} = -a_n/(n+2)$

D. $a_0 = 2, \quad a_1 = 1$ and $a_{n+2} = -a_n/(n+2)$

E. $a_0 = 2, \quad a_1 = 0$ and $a_{n+2} = -(n+1)a_n/(n-1)$

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7. The function $f(x) = 0$ for $-1 \leq x < 0$ and $f(x) = 2$ for $0 \leq x < 1$ has a Fourier expansion

$$f(x) \approx a_0/2 + \sum_{n=1}^{\infty}(a_n cos(n\pi x) + b_n sin(n\pi x))$$

A. Give the general formula for the coefficients a_n and b_n for a function $g(x)$ defined on the interval $[-L, L]$.

B. Compute the coeeficients for the the function $f(x)$ defined above.

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8. Which of the following functions are **even** functions of x? (That is, $f(-x) = f(x)$)

A. $sin(x)sin(3x)$

B. $sin(2x)$

C. $sin(x)cos(x)$

D. $x\, sin(x)$

E. $x^2|x|$

No notes, books, or calculators allowed. Show all work in the exam pages and hand the entire examination back at the end of the two hours.

These discussion problems are worth 20 points each. Show all work. If you have questions about what is required, ask the instructor.

I. Real Solutions to systems

A (2×2) matrix of real numbers $\mathbb{A}$ has complex eigenvalues $r = -2 \pm 3i$, and the vector

$$\begin{pmatrix} 5+7i \\ 1 \end{pmatrix}$$

is a complex eigenvector associated to $r = -2 + 3i$.

a) Find a *real valued* general solution to the linear system

$$\mathbf{x}' = \mathbb{A}\mathbf{x}$$

.

b) Fit the constants in this general solution to the initial value

$$\mathbf{x}(0) = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

II. Power Series Solutions to ODE

a) For the Euler type equation

$$x^2y'' + 4xy' + 2y = 0$$

give the general solution. Fit the constants to $y(1) = 1, y'(1) = 0$.

b) A power series solution to an ODE is given by $y = a_0 + a_3x^3 + a_6x^6 + a_9x^9 + \cdots$ where the coefficients are defined recursively by

$$a_{n+3} = \frac{n^2 + n}{27n^2 + 17}a_n \qquad n = 0, 3, 6, \cdots$$

Determine the radius of convergence of this power series. *(Do not try to compute the coefficients.)*

III. Laplace and Inverse Laplace Transforms

Hints: $\mathcal{L}(e^{ct}f(t)) = F(s-c)$ and $\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}F(s)$.

a) Find the inverse Laplace transform of $F(s) = 1/((s-1)(s+1))$

b) Find the inverse Laplace transform of $G(s) = 1/(s^2+4s+2)$

c) Find the Laplace transform of $e^{-2t}sin(3t)$.

d) Find the Laplace transform of $f(t)$, where $f(t) = t$ for $0 \le t \le 1$ and $f(t) = 1$ for $t \ge 1$. Hint: use the step functions $u_c(t)$ to write out $f(x)$.

IV. Boundary Values and Second Order Equations. Consider the ordinary differential equation

$$X''(x) + cX(x) = 0$$

on the interval $0 \leq x \leq \pi$.

a) Suppose that $c = 0$. What is the general solution for $X(x)$?

b) Suppose that $c < 0$, so $c = -\lambda^2$. What is the general solution for $X(x)$?

c) Suppose that $c > 0$, so $c = +\lambda^2$. What is the general solution for $X(x)$?

d) Suppose that $c > 0$, so $c = +\lambda^2$, and that in addition X is subject to the boundary conditions

$$X'(0) = 0, X(\pi) = 0.$$

Note that the condition is on $X'(0)$ **not** $X(0)$**.** Find all values of the constant c such that there exist solutions $X(x)$ to this problem which are not identically zero.

V. Using Laplace Transforms to solve ODE initial values problems.

Find the Laplace Transform $Y(s)$ of the solution to the problem,

$$y'' + 2y' + 1y = \delta(t-2), \qquad y(0) = 1,\ y'(0) = 5.$$

Here $\delta(t)$ is the Dirac delta "function". If you don't know the Laplace transform of $\delta(t)$, try the step function $u_2(t)$. **FIND ONLY $Y(s)$. DO NOT TRY TO FIND $y(t)$.**

VI. Separation of Variables

The two dimensional potential problem (stable heat equation) satisfies the PDE

$$u_{xx} + u_{yy} = 0.$$

Suppose that the domain is the square $\{0 \le x \le 1, \quad 0 \le y \le 1\}$.

a) Look for solutions of the form

$$u(x, y) = X(x)Y(y)$$

and separate variables to find the ODE's satisfied by X and Y.

b) Suppose that the square is kept at temperature 0 on the top and bottom sides, and temperature 0 on the left side. Sketch the square with this informtion indicated. Write this boundary information as conditions on the solutions X and Y.

c) Suppose that in addition, the right side is kept at temperature 0. Estimate the temperature at the center of the square, $(x, y) = (0.5, 0.5)$.