MA453 HW#4. 1

§5.1. #8, 20, 22, 25, 26.

#8. $\left(x^2+\frac{3}{x}\right)^{15}=\sum_{k=0}^{15}\binom{15}{k}(x^2)^k\left(\frac{3}{x}\right)^{15-k}=\sum_{k=0}^{15}\binom{15}{k}3^{15-k}x^{2k-(15-k)}$

$=\sum_{k=0}^{15}\binom{15}{k}3^{15-k}x^{3k-15}$, $3k-15=18 \Rightarrow k=11$

$\therefore$ coefficient of x is $\binom{15}{11}3^{15-11}=\binom{15}{4}3^4=110565$

$\therefore$ ans : $\underline{110565}$

#20. coeff. of the middle term of $(1+x)^{2n}=\binom{2n}{n}$ $\cdots$ (a)

$\therefore$ the sum of the coeff. of the two middle terms of $(1+x)^{2n-1}=\binom{2n-1}{n-1}+\binom{2n-1}{n}$. $\cdots$ (b)

$$\binom{2n-1}{n-1}+\binom{2n-1}{n}=\frac{(2n-1)!}{(n-1)!(n)!}+\frac{(2n-1)!}{n!(n-1)!}=\frac{2(2n-1)!}{n!(n-1)!}\times\frac{n}{n}$$

$$=\frac{2n\cdot(2n-1)!}{n!\,n(n-1)!}=\frac{(2n)!}{n!\,n!}=\binom{2n}{n}$$

$\therefore$ (a) = (b). ▨

#22. k consecutive positive integers : $n, (n+1), (n+2), \cdots, (n+k-1)$.

Show : $n(n+1)(n+2)\cdots(n+k-1)=k!N$, N is an integer.

that is, $\frac{n(n+1)(n+2)\cdots(n+k-1)}{k!}$ is an integer.

$$\because \frac{n(n+1)(n+2)\cdots(n+k-1)}{k!}=\frac{[n(n+1)(n+2)\cdots(n+k-1)][(n-1)(n-2)\cdots 1]}{k!\,[(n-1)(n-2)\cdots 1]}$$

$$=\frac{(n+k-1)!}{k!\,(n-1)!}=\binom{n+k-1}{k}$$: integer.

▨

MA453 HW#4

§5.1 8, 20, 22, 25, 26

#25. $0 = (1-1)^n = [(-1)+1]^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k (1)^{n-k}$

$$= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n}$$

#26. $2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$ $\cdots$ (a)

by #25, $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0$ $\cdots$ (b)

(a) − (b) $\Rightarrow$ $2\left[\binom{n}{1} + \binom{n}{3} + \cdots\right] = 2^n$

$\Rightarrow$ $\binom{n}{1} + \binom{n}{3} \qquad = 2^{n-1}$

§5.2 #10, 14

#10. $n^{13} \equiv n \pmod{13}$ $\forall n$ by thm 5.4 ($\because$ 13 is a prime)

$\therefore n^{13} - n = 13k$ k: integer. & $2730 = 13 \cdot 210$

$\therefore 2730 \mid n^{13} - n$.

#14. 2, $2^2 = 4$, $2^3 = 8$, $2^4 = 16 \equiv 6 \pmod{10}$ $2^5 \equiv 2 \pmod{10}$

$\therefore 2^{4k+1} \equiv 2 \pmod{10}$

$2^{4k+2} \equiv 4$ "

$2^{4k+3} \equiv 8$ "

$2^{4k} \equiv 6$ "

$\therefore 2^{400} = 2^{4 \cdot 100} \equiv 6 \pmod{10}$

$\therefore$ unit digit of 2^{400} is 6