MA453.

Solution.

Exercises 1.1

p7 #2. Since $(5-\sqrt{3})^2 = 28-10\sqrt{3}$ and $(2-\sqrt{3})^2 = 7-4\sqrt{3}$,

$$\sqrt{28-10\sqrt{3}} - \sqrt{7-4\sqrt{3}} = \sqrt{(5-\sqrt{3})^2} - \sqrt{(2-\sqrt{3})^2}$$

$$= |5-\sqrt{3}| - |2-\sqrt{3}| = 5-\sqrt{3}-2+\sqrt{3} = 3$$

ans: 3.

Exercises 2.1

p16 #13. $$\frac{\sqrt{3}+5i}{2-\sqrt{3}i} \times \frac{2+\sqrt{3}i}{2+\sqrt{3}i} = \frac{2\sqrt{3}-5\sqrt{3}+3i+10i}{4+7} = -\frac{3\sqrt{3}}{7} + \frac{13}{7}i$$

ans: $-\frac{3\sqrt{3}}{7} + \frac{13}{7}i$

Exercises 2.2

p23. #7. $\sqrt[3]{|1+i|} = \sqrt[3]{\sqrt{1^2+1^2}} = \sqrt[6]{2} \approx 1.1225$

$\arg(\sqrt[3]{1+i}) = \frac{1}{3}\arg(1+i) = \frac{1}{3}\arctan 1 = \frac{1}{3}\frac{\pi}{4} = \frac{\pi}{12} = 15^\circ$

Set $w = \cos 15^\circ + i\sin 15^\circ = .966 + .259i$

& $\zeta = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i = -.5 + .866i$

$\zeta^2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i = -.5 - .866i$

$\therefore \sqrt[3]{1+i} \approx 1.1225\{w, w\zeta, w\zeta^2\}$

$\approx 1.1225\{.966+.259i, -.707+.707i, -.259-.966i\}$

$= \{1.08+.29i, -.79+.79i, -.29-1.08i\}$

ans: $\{1.08+.29i, -.79+.79i, -.29-1.08i\}$

p23. #14. $z = \dfrac{3 \pm \sqrt{9-(9+2i)}}{1} = 3 \pm \sqrt{-2i} \underset{(*)}{=} 3 \pm (-1+i) = \begin{cases} 2+i \\ 4-i \end{cases}$

$$\left(\begin{aligned} (*) &: \sqrt{|-2i|} = \sqrt{\sqrt{4}} = \sqrt{2} \\ &\arg(\sqrt{-2i}) = \tfrac{1}{2}\arg(-2i) = \tfrac{1}{2}\cdot\tfrac{3\pi}{2} = \tfrac{3\pi}{4} \\ &\therefore \sqrt{-2i} = \sqrt{2}\left(\cos\tfrac{3\pi}{4} + i\sin\tfrac{3\pi}{4}\right) = \sqrt{2}\left(-\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}i\right) = -1+i \end{aligned} \right)$$

$\therefore$ ans : $2+i$, $4-i$

#19. $(1+\omega^2)^{16} = (-\omega)^{16} = \omega^{16} = \omega^{3\cdot 5+1} = (\omega^3)^5\cdot\omega \underset{\omega^3=1}{=} \omega.$

$$\left(\begin{aligned} &1+\omega+\omega^2 = 0 \\ &\Rightarrow 1+\omega^2 = -\omega \end{aligned} \right)$$

HW#2.

§2.5. #4. Let $\zeta = \cos 2\pi/7 + i \sin 2\pi/7$.

Then $\sqrt[7]{1} = \{\zeta, \zeta^2, \zeta^3, \zeta^4, \zeta^5, \zeta^6, 1\}$ with orderes

$\{7, 7, 7, 7, 7, 7, 1\}$

§3.1 #2. $x^3 + 12x - 12 = 0$. $p = 12$, $q = -12$.

$$z^3 = \frac{-q \pm \sqrt{q^2 + 4p^3/27}}{2} = \frac{12 \pm \sqrt{400}}{2} = 16 \text{ or } -4$$

Choose $z^3 = 16$. Then $z_1 = 16^{1/3} = 2\sqrt[3]{2}$, $z_2 = 2\sqrt[3]{2}\omega$, $z_3 = 2\sqrt[3]{2}\omega^2$

$$x_1 = z_1 - \frac{p}{3z_1} = 2\sqrt[3]{2} - \frac{12}{3 \cdot 2\sqrt[3]{2}} = 2\sqrt[3]{2} - \frac{2}{\sqrt[3]{2}} = 2\sqrt[3]{2} - \sqrt[3]{4} = 4^{2/3} - 4^{1/3}$$

$$x_2 = z_2 - \frac{p}{3z_2} = 2\sqrt[3]{2}\omega - \frac{12}{3 \cdot 2\sqrt[3]{2}\omega} = 2\sqrt[3]{2}\omega - \frac{2}{\sqrt[3]{2}\omega} = 2\sqrt[3]{2}\omega - \sqrt[3]{4}/\omega$$
$$= 4^{2/3}\omega - 4^{1/3}\omega^2$$

$$x_3 = z_3 - \frac{p}{3z_3} = 2\sqrt[3]{2}\omega^2 - \frac{12}{3 \cdot 2\sqrt[3]{2}\omega^2} = 2\sqrt[3]{2}\omega^2 - \frac{2}{\sqrt[3]{2}\omega^2} = 2\sqrt[3]{2}\omega^2 - \sqrt[3]{4}/\omega^2$$
$$= 4^{2/3}\omega^2 - 4^{1/3}\omega.$$

#13. $y^3 + py + q = 0$.

Let x_1, x_2, x_3 be the solutions of $y^3 + py + q = 0$.

Then $x_1 + x_2 + x_3 = 0$. $x_1x_2 + x_2x_3 + x_3x_1 = p$. $x_1x_2x_3 = -q$.

We know that $(x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2$

$(x_1 + x_2 + x_3)^2 = x_1^2 + x_2^2 + x_3^2 + 2(x_1x_2 + x_2x_3 + x_3x_1)$

$x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3 = (x_1 + x_2 + x_3)(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_3x_1)$

$$\therefore [(x_1-x_2)(x_2-x_3)(x_3-x_1)]^2 = (x_1-x_2)^2(x_2-x_3)^2(x_3-x_1)^2$$

$$= [(x_1+x_2)^2 \quad 4x_1x_2][(x_2+x_3)^2 \quad 4x_2x_3][(x_3+x_1)^2 \quad 4x_1x_3]$$

$x_1+x_2+x_3=0.$
$(x_1x_2x_3=-q$

$$= [x_3^2+\frac{4q}{x_3}][x_1^2+\frac{4q}{x_1}][x_2^2+\frac{4q}{x_2}]$$

$$= \frac{x_3^3+4q}{x_3}\cdot\frac{x_1^3+4q}{x_1}\cdot\frac{x_2^3+4q}{x_2}$$

$x_1x_2x_3=-q$

$$= \frac{(4q+x_1^3)(4q+x_2^3)(4q+x_3^3)}{-q}$$

$x_1^3+px_1+q=0.$
$\Rightarrow x_1^3=-px_1-q$

$$= \frac{(4q-px_1-q)(4q-px_2-q)(4q-px_3-q)}{-q}$$

$$= \frac{(3q-px_1)(3q-px_2)(3q-px_3)}{-q}$$

$(x-a)(x-b)(x-c)$
$=x^3-(a+b+c)x^2$
$+(ab+bc+ca)x-abc$

$$= \frac{(3q)^3-p(x_1+x_2+x_3)\cdot(3q)^2+p^2(x_1x_2+x_2x_3+x_3x_1)(3q)-p^3x_1x_2x_3}{-q}$$

$$= \frac{27q^3+p^3\cdot 3q+p^3\cdot q}{-q} = -27q^2-4p^3$$

$\therefore$ discriminant of the eq. $y^3+py+q=0$ is $-4p^3-27q^2$

§3.3. #2. $f(2)=-12$, $f(3)=2$.

$\therefore$ begin with $x_1=2$. Since $f'(x)=6x^2-10x+1$, $f'(2)=5$.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{-12}{5} = 4.4$$

similarly, $x_3 = 3.4710$

$x_4 = 3.0337$

$x_5 = 2.9233$

$x_6 = 2.9164$

$x_7 = 2.9164$ $\quad\therefore x \approx 2.9164.$

§ 4.1. #4 $x=0 \Rightarrow 0+0+0+1 \equiv 0$ is impossible

$x=1 \Rightarrow 1+1+1+1=4 \equiv 0$ in $\mathbb{Z}_2$ $\therefore x=1$

§4.1 #26.

$x = a_0 + a_1 \cdot 10 + 10^2 a_2 + \cdots + 10^n a_n.$ $0 \le a_i \le 9,$ $a_n \ne 0.$

Show: $x \equiv 0 \pmod 9 \iff a_0 + a_1 + \cdots + a_n \equiv 0 \pmod 9$

Since $10^m - 1 = (10-1)(10^{m-1} + 10^{m-2} + \cdots + 1)$

$= 9(10^{m-1} + 10^{m-2} + \cdots + 1),$ $\forall m \ge 2$

$10^m - 1 \equiv 0 \pmod 9$ $\forall m \ge 2$

$10^m \equiv 1 \pmod 9$ $\forall m \ge 2.$

$\therefore$ $a_0 + 10 a_1 + 10^2 a_2 + \cdots + 10^n a_n$

$\underset{9}{\equiv} a_0 + a_1 + a_2 + \cdots + a_n \underset{9}{\equiv} 0$ $\therefore a_0 + a_1 + \cdots + a_n \equiv 0 \pmod 9$

$\because 10^m \equiv 1 \pmod 9$ $\because x \equiv 0 \pmod 9$

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Solution.

§4.2 #34. $3x^3-3x^2+17x-4=0$.

rational root: a/b. $(a,b)=1$. $a|4$, $b|3$. (by exercise 33).

$\therefore$ $a|4 \Rightarrow a=\pm1, \pm2, \pm4$.

$b|3 \Rightarrow b=\pm1, \pm3$.

i) $a=\pm1, b=\pm1 \rightarrow x=\pm1 \Rightarrow 3(\pm1)-3\cdot1+17(\pm1)-4 = 13, -27 \neq 0$.

$\therefore$ ±1 are not solutions

ii) $a=\pm1, b=\pm3 \rightarrow x=\pm\frac{1}{3} \Rightarrow 3\cdot(\pm\frac{1}{27})-3\cdot\frac{1}{9}+17(\pm\frac{1}{3})-4 \neq 0$.

iii) $a=\pm2, b=\pm1 \rightarrow x=\pm2$

iv) $a=\pm2, b=\pm3 \rightarrow x=\pm\frac{2}{3}$

v) $a=\pm4, b=\pm1 \rightarrow x=\pm4$

vi) $a=\pm4, b=\pm3 \rightarrow x=\pm\frac{4}{3}$

similarly, we can show they are not solutions.

$\therefore$ there are no rational solutions for $3x^3-3x^2+17x-4=0$.

§4.2 #8. $67=1\cdot66+1$

$\Rightarrow 1=67\cdot1+(-1)\cdot66$

$-1\equiv66$: multi. inverse of 66 in $\mathbb{Z}_{67}$

§4.4 #6. $\forall a+b\sqrt{-5},\ c+d\sqrt{-5} \in \mathbb{Z}[\sqrt{-5}]$,

$(a+b\sqrt{-5}) \pm (c+d\sqrt{-5}) = (a\pm c)+(b\pm d)\sqrt{-5} \in \mathbb{Z}[\sqrt{-5}] \quad \because a+c,\ b+d \in \mathbb{Z}$.

$(a+b\sqrt{-5})(c+d\sqrt{-5}) = (ac-5db)+(ad+bc)\sqrt{-5} \in \mathbb{Z}[\sqrt{-5}]$.

$\because ac-5bd,\ ad+bc \in \mathbb{Z}$.

#7. (a) $N(z)=1$ i.e. $a^2+5b^2=1 \Rightarrow$ i) $a=0 \rightarrow 5b^2=1 \quad \nexists b\in\mathbb{Z} \ni: 5b^2=1$.

ii) $b=0 \rightarrow a^2=1 \quad \therefore a=\pm1. \quad \therefore z=\pm1$.

iii) $b\neq0 \rightarrow b^2\geq1 \Rightarrow 5b^2\geq5. \quad \therefore \nexists a,b \ni: a^2+5b^2=1$.

trivially, if $z=\pm1$, then $N(z)=(\pm1)^2+0^2=1$.

(b) $\forall z=a+b\sqrt{-5}\in\mathbb{Z}\sqrt{-5}, \quad N(z)=a^2+5b^2\neq3$.

$\because$ i) $b=0 \rightarrow a^2\neq3$.

ii) $b\neq0 \rightarrow 5b^2\geq5 \quad \therefore \nexists a,b \ni: a^2+5b^2=3$.

(c). $N(z)=a^2+5b^2=9$.

i) $b=0 \Rightarrow a^2=9 \quad a=\pm3. \qquad \therefore z=\pm3$.

ii) $b^2=1. \Rightarrow a^2=4 \quad a=\pm2 \qquad \therefore z=\pm2+\sqrt{-5},\ \pm2-\sqrt{-5}$.

$\therefore z=\pm3,\ \pm2+\sqrt{-5},\ \pm2-\sqrt{-5}$ //

(d) assume $z=a+b\sqrt{-5},\ \omega=c+d\sqrt{-5} \Rightarrow z\omega=(ac-5bd)+(ad+bc)\sqrt{-5}$.

$N(z\omega)=(ac-5bd)^2+5(ad+bc)^2=a^2c^2-10acbd+25b^2d^2+5a^2d^2+10adbc+5b^2c^2$

$N(z)N(\omega)=(a^2+5b^2)(c^2+5d^2)=a^2c^2+5a^2d^2 \quad b^2c^2 \quad b^2d^2$ //

$\therefore N(z)N(\omega)=N(z\omega)$.