1. 15 points
Find a harmonic conjugate to \( u(x, y) = x^3y - xy^3 + 2x + 3y \) on \( C \). That is, find a harmonic function \( v(x, y) \) so that \( u + iv \) is complex analytic.

2. 20 points
Compute \( \oint_{\gamma} \frac{1}{z} \, dz \) where \( \gamma \) is each of the curves described below. Explain your work.

a) The straight line segments from 4 to 4 + i followed by the line segment from 4 + i to i.

b) (no parametrization is necessary to work this!)

3. (25 points)
Let \( C_r(0) \) denote the simple closed curve that is the positively oriented boundary of the disc of radius \( r \). Evaluate the following integrals. Explain.

a) \[
\oint_{C_1(0)} \frac{e^{3z}}{z - 3} \, dz
\]

b) \[
\oint_{C_1(0)} \frac{e^{-z}}{4z^2 + 1} \, dz
\]

c) \[
\oint_{C_1(0)} \frac{z^{11}}{(z - 1/2)^5} \, dz
\]

d) \[
\oint_{C_1(0)} \frac{1}{z^3} \, dz
\]
e)

\[ \oint_{C_2(0)} \frac{1}{(1 - z)(1 + z)} \, dz \]

Notice different curve!

4. (15 points)

Let \( C_R \) denote the semicircular curve given by \( z(t) = Re^{it} \) as \( t \) ranges from 0 to \( \pi \). Show that

\[ \oint_{C_R} \frac{e^{iz}}{(z^2 + 1)} \, dz \]

tends to zero as \( R \) tends to infinity.

BONUS (5 points) Use the above to find the value of

\[ \int_{\infty}^{\infty} \frac{\cos(x)}{(1 + x^2)} \, dx \]

Hint. Use the x-axis as half of curve and the semicircle from above as the other half.

5. (15 points)

Let \( f_1(z) = 1/z \), \( f_2(z) = z^2 \), and \( f_3(z) = -z \). Draw a sequence of three diagrams to illustrate the effect of the mapping \( f(z) = f_3(f_2(f_1(z))) \) on the domain \( \Omega \) below.

6. (10 points) Find all third roots of \( i \).