NAME ______________________ Id. No, ______________________

Final Examination, Math 525 Spring 2003, Wilkerson Section.

May 6, 2003, 3:20PM-5:20PM REC 112

No notes, books, calculators, tape players, earphones. Show all work. Use back of pages for scratch.

Problem	Score	Problem	Score
I.(25)		VI.(15)	
II.(20)		VII.(30)	
III.(25)		VIII.(25)	
IV.(20)		IX.(25)	
V.(15)			
Total(200)			

Recall that the Moebius transformation $z \to \frac{az+b}{cz+d}$ with $ad - bx \neq 0$ can be specified with the choice of three points $\{z_1, z_2, z_3\}$ that are taken to the points $\{0, 1, \infty\}$ by $f(z) = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$. Here $z_1 \to 0,\ z_3 \to \infty$ and z_2 is chosen so that $z_2 \to 1$.

1. Given the Moebius transformation

$$G(z) = 3\frac{(z+2)}{(z+1)}$$

sketch the image of the following sets: a) $|z| \leq 1$

b) The lines $y = x$ and $y = -1$

c) What are the preimages (inverse images $= G^{-1}$) of $\{0,\ 1,\ \infty\}$?

2. a) Find a Moebius transformation that maps the region described as the disc of radius 2 centered at $z = 2$ with the interior of the disc of radius 1 centered at $z = 1$ deleted to the vertical strip , $2 < x < 4$, $\quad -\infty < y < \infty$. Draw a picture also.

b) Use this to define a harmonic function $h(x, y)$ that is 5 on the points with $|z - 2| = 2$ and $h(x, y) = 3$ if $|z - 1| = 1$, and $z \neq 0$.

Here are some powerful theorems from the course. You may refer to them below.

Riemann Mapping Theorem If D_1 and D_2 are open domains in C which are not all of C, then there exists $f : D_1 \to D_2$ which is analytic, one to one, and onto.

Louiville's Theorem If $f : C \to C$ is analytic and bounded, then f is constant.

Maximum Modulus Principle If f is analytic on D, then if $max(|f|)$ is attained on D, then f is a constant.

Roche's theorem If f and h are analytic inside and on the simple closed contour Γ, and if the strict inequality $|h(z)| < |f(z)|$ holds at each point on Γ, then f and $f + h$ must have the same total number of zeroes (counting multiplicities) inside Γ.

3. a) Let $f : C \to C$ be analytic such that $Re(f) > 0$. Show that f is a constant. Hint: Show that $|f|$ is bounded on C and then use Louiville's theorem.

b) Show that the polynomial $z^5 + 2z + 1$ has five roots inside the disk of radius 2.

4. For each of the following functions, find the singular points (including ∞) and classify as to pole of order m, removable singularity, or essential singularity. Give the residue at each singularity. If the point is a removable singularity, give the limit at the point.

a)

$$\frac{tan(z)}{(z)}$$

b)

$$\frac{(z^3 - 27)}{((z^2 - 4z + 4)(z - 5))}$$

c) Give the Laurent series of $e^{sin(\frac{1}{z^2})}$ ANd analyse the singularities.

5. Set up the calculation of

$$P.V. \int_0^\infty \frac{1}{(x^2+1)}\, dx$$

. Label all paths neatly, and detail which path integrals go to zero and which do no. Then carry out the calculation.

6.

For $u(x, y) = e^{2x}((x^2 - y^2)cos(2y) - (2xy)sin(2y)$ find the harmonic conjugate v to u. That is, find a harmonic function $v(x, y)$ so that $u + iv$ is complex analytic.

7.

Let $C_r(0)$ denote the simple closed curve that is the positively oriented boundary of the disc of radius r . Evaluate the following integrals. Explain.

a)

$$\oint_{C_1(0)} \frac{e^{3z}}{z-\frac{1}{2}}\, dz$$

b)

$$\oint_{C_10(0)} \frac{1}{(z-1)(z-2)(z-3)(z-4)}\, dz$$

c)

$$\oint_{C_1(0)} \frac{cos(2z)}{(z-\pi/6)^8}\, dz$$

d)

$$\oint_{C_1(0)} \frac{1-cos(z)}{z^5}\, dz$$

8. Compute

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^4+1)}\, dx$$

9.

a) Describe a 1-1 onto analytic map from the horizontal strip $-\infty < x < \infty$, $0 < y < 1$ to the sector between the positive real axis and the line $x = y$ in the first quadrant.

b) Use this to define a function on the sector which is harmonic and equal to 5 on the $x - axis$, $x > 0$, and 10 on the line $x = y,\ x > 0$.