DRAFT SUMMARY: COXETER ELEMENTS AND CENTERS

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Abstract. We show that the abstract group structure of the Weyl group often forces the structure of the center of the corresponding Lie group or p-compact group. For example, if G is a connected simple compact Lie group and $Center(W_G)$ is non-trivial, then $Center(G)$ is an elementary abelian 2-group. Generalizing to p-compact groups, if W_X is irreducible as a $\mathbf{Q}_p$ reflection group and $Center(W_X)$ is non-trivial, then the center C_X defined by Dwyer-Wilkerson is contractible if p is odd. The existence of a non-trivial center for W_X is detected by the rational cohomology type of X, so this principle is easy to apply.

If W has a trivial center, one can still bound C_X in terms W_X. The role of central elements in W_X must be replaced with that of elements analogous to the Coxeter elements of Lie Weyl groups. This gives sharp results – for any irreducible example X with Weyl group not that of a Lie group the center C_X is trivial. In the case of a Lie-like Weyl group, the possibilities for C_X are bounded by the analogous Lie examples. Similar triviality results and bounds hold for the fundamental group $\pi_1(X)$ and the k-invariant of the $BW \to BT \to BNT$ sequence.

§0. Introduction

If G is a compact Lie group, the group C_G of elements which commute with every element of G is a closed subgroup of G . It has two somewhat different characterizations:

(1) it is the maximal abelian central subgroup of G, or
(2) C_G is the fixed point set of the action of G on itself by conjugation.

As a compact abelian Lie group, C_G is the product of a torus and a finite abelian group. If G is semi-simple (e.g., $\pi_1(G)$ is finite), then C_G is finite. In cases where G is easily visualized as a matrix group, the center can be directly computed from the matrix description. In the other cases, one needs, for example as in Serre [23], information usually expressed in terms of the Lie algebra or representation theory of G.

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One feature of the Dwyer-Wilkerson program of studying Lie groups and H-spaces via classifying spaces has been to recast classical results so that the proofs need only minimal analytical content. The calculation of centers provides an interesting test of this view.

In [15], the authors provide an algorithm to calculate the center of a connected compact Lie group from information derivable from the normalizer of the torus – in most cases just from the Weyl group and its action on the maximal torus. That result is restated in section I, so we sketch a slightly weaker estimate here.

If G is connected, it has a maximal torus T and any two choices for T are conjugate. Moreover, each element of G is contained in at least one maximal torus and T is its own centralizer in G. It follows that C_G is exactly the intersection of all the maximal tori of G. However, we obviously prefer methods that use the action of W_G on a single maximal torus T_G. The first observation is that C_G is contained in the subgroup of T fixed by any element of $N_G(T)$ and hence by W_G. In Dwyer-Wilkerson [15] it was shown that this estimate is sharp up to 2-torsion.

Theorem I. *If G is a simply connected compact Lie group, then*

$$C_G = (T)^{W(G)},$$

the invariants of the $W(G)$-action on T. If G is not simply connected, the quotient group T^W/C_G is a (possibly trivial) elementary abelian 2-group.

One can calculate the $W(G)$-invariants as the intersection of the various $\{w_j\}$-invariants, where $\{w_j\}$ ranges over all elements from any generating set of $W(G)$. It's natural to try to minimize the problem to computing invariants with respect to only one element. Since any product of fewer than $n = rank(T)$ reflections must fix a nontrivial torus, the first interesting possibility is the product of exactly n reflections.

Coxeter [9] and [8] proved that if W is an irreducible real reflection group of rank n, then different choices of n generating reflections $\{s_j\}$ yield the same conjugacy class of elements $\{w_{Cox} = s_1 \dots s_n\}$. These Coxeter elements encode almost all the numerical invariants of the reflection group. In the 50's a connection between Coxeter elements and centers was observed, apparently on a case by case basis,

Theorem II. *If G is a simply connected connected compact Lie group, and w_{Cox} is a Coxeter element of $W(G)$, then*

$$C_G = T^{w_{Cox}}.$$

In this case $|C_G| = |det(I - w_{Cox})|$, computed on $\pi_1(T) \otimes \mathbf{Q}$.

A proof of II that avoids case by case analysis is given in section 2 of this paper. Theorems I and II appear to be "well-known" but with the possible exception of Bourbaki [2], don't seem to be present in the standard textbooks.

This exposition is intended as a rather leisurely stroll through some classical topics for readers who are interested in extensions to p-compact groups or seeing the standard results from a different point of view. It also provides the background for [12], which relates connectiveness of centralizers to homological torsion. This work could be also be viewed as a missing computational appendix to [15]. Finally, the sparcity of non-trivial centers helps to explain the effectiveness of centralizer diagrams in analysing classifying spaces.

The reader interested only in centers in the Lie case can restrict reading to the Section 1 and the appendix on Coxeter elements. A statement of results primarily for p-compact groups is given in Section 2, with proofs appearing in the later sections.

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§1. The Lie case: proofs of Theorems I and II

We begin with an easy observation:

Proposition 1.1. *If G is a connected compact Lie group with maximal torus T and Weyl group W_G, then*

1. $C_G \subset T^{W_G}$.
2. T^{W_G}/C_G *is a finite elementary abelian 2-group.*

Proof of 1.1. Represent $w \in W$ as $n \in N_G(T_G)$, the normalizer of T_G in G. If $t \in C_G$, then $ntn^{-1} = t$ and hence $t \in T_G^w$. This is true for all $w \in W$, so $t \in T_G^W$. For part (2), for each reflection s in W, the term T^s is up to index 2 the correct contribution to C_G, see 2.5 below. There is a similar characterization in the Lie case with the $\{U_\alpha\}$, see for example Bröcker-tom Dieck, [4].

Theorem 1.2 (Theorem I). *Let G be a connected simply connected compact Lie group, with maximal torus T_G and Weyl group W_G. Then*

$$C_G = (T_G)^{W_G}.$$

Theorem 1.3. *(Borel* [1]*, Steinberg* [28],[27]*) If G is a simply connected compact Lie group and $t \in T_G$ then the centralizer of t in G, $C_G(t)$, is connected.*

Proof of 1.2. Certainly $C_G \subset (T_G)^{W_G}$. Suppose that $t \in (T_G)^{W_G}$. Consider its centralizer $C_G(t)$. By 2.3, $C_G(t)$ is connected. T_G is a maximal torus for $C_G(t)$ and $W_{C_G(t)}$ is the subgroup of $W(G)$ that fixes t. By the choice of t, then, $W(C_G(t)) = W_G$. Since G is semi-simple, the dimension of G is determined by W_G and likewise for $C_G(t)$. So $dim(C_G(t)) = dim(G)$. Since G and $C_G(t)$ are closed connected manifolds, we must have $G = C_G(t)$. Hence $t \in C_G$. This is true for all $t \in (T_G)^{W_G}$, so $C_G = (T_G)^{W_G}$.

Coxeter elements are reviewed in Appendix A. Each Lie Weyl group has an unique conjugacy class of elements called Coxeter elements, $\{w_{Cox}\}$. These elements are products of n generating reflections for W.

Theorem 1.4 (Theorem II). *Let G be simply connected compact Lie group and w_{Cox} be a Coxeter element in W_G. Then $C_G = T^{w_{Cox}}$ and $|C_G| = |det(I - w_{Cox})|$. If G is not simply connected, then C_G is a quotient of $T^{w_{Cox}}$ but not necessarily all of it.*

The main change from the proof of Theorem I is the addition of a strong property of Coxeter elements – (see A.4) – if W is irreducible, then the Coxeter elements are not contained in any proper reflection subgroup of W.

Proof of 1.4. Since G is simply connected and compact, it is isomorphic to a direct product of simple simply connected compact Lie groups. Without loss of generality we can assume that G is simple. Let $t \in G$ be invariant under w_{Cox}. The centralizer $C_G(t)$ is connected and hence its Weyl group is a subreflection group of $W(G)$. Since t is fixed by w_{Cox}, certainly $w_{Cox} \in W(C_G(t))$. Hence by A.4, $W(C_G(t)) = W(G)$ since each is generated by reflections and contains w_{Cox}. In particular, $dim(W_G) = dim(C_G(t) = dim(G)$. Since $C_G(t)$ and G are connected compact manifolds, $C_G(t) = G$. That is, $T^{w_{Cox}} = C_G$.

The preceeding arguments identify the center C_G of a compact connected compact Lie group G as a a subgroup of the fixed points of the W_G action on the maximal torus T. Furthermore, in the simply connected case, the restriction to a Coxeter element gives the same invariants. We use this information to provide an explicit calculation of the center for the connected compact simple simply connected Lie groups.

One outstanding characteristic of the Coxeter element is that it fixes no nonzero vector in $\pi_1(T) \otimes \mathbf{Q}$.

Lemma. *Let $Cx \in GL(n, S)$, for S either $\mathcal{Z}$ or $\mathcal{Z}_p$, such that Cx has finite order and has no eigenvalues $= 1$. Let $L = S^n$ and $T = (L \otimes \mathbf{Q})$. Then*

$$|T^{Cx}| = |det(Cx - I)|$$

if $S = \mathcal{Z}$, and the p-th power factor of this for $S = \mathcal{Z}_p$.

The proof is in the appendix on Coxeter elements. The point is that in addition to the usual exponent bound provided by the order of the element Cx, one has the stronger bound $|det(Cx - I)|$ for the invariants. For example, for the Lie group E_8, the Coxeter element has order 60, but $T^{w_{Cox}}$ is the identity element.

The conjugacy class of Coxeter elements is usually quite large. Even larger still is the subset of elements of the Weyl group that share the property that they have no eigenvalues of $+1$. We dub these *generalized Coxeter* elements. Particularly useful examples of these are the non-trivial central elements in an irreducible Lie Weyl group. These elements have order 2 and act as the negative of the identity on $\pi_1(T_G)$. The above shows that C_G is then bounded in size by $(\mathcal{Z}/2\mathcal{Z})^\ell$, where ℓ is the rank of G. Furthermore an elementary cohomological argument shows that the exponent of the C_G is 2 in these cases. The Lie Weyl groups with this property include $\{B_n, C_n, D_{2n}, A_1, G_2, F_4, E_7, E_8\}$, but not $\{A_n, D_{2n+1}, or\ E_6\}$. Theorem $A.8$ uses work of Solomon to count the generalized Coxeter elements.

Direct calculation with Coxeter elements shows that for $\{G_2, F_4, E_8\}$

$$det(w_{Cox} - Id) = 1.$$

Hence any connected Lie group or p-compact group with these Weyl groups has trivial center.

On the other hand, such calculation reveals that $\{E_6, E_7\}$ have centers of order 3 and 2 respectively, and hence are determined up to isomorphism.

Likewise, the determinant method applied to Coxeter elements for $\{B_n, C_n\}$ give a bound of order 2 for the center of any connected compact Lie group or 2-compact group with these Weyl groups.

Only the cases $\{A_n, D_{2n}, D_{2n+1}\}$ require further calculations. For $\{D_n\}$ the determinant is 4 in all cases. If n is even, then the Weyl group $W(D_{2n})$ contains a diagonal central element of order 2. Hence the center of any form of D_{2n} is an elementary abelian 2-group. So the center of $Spin(4n)$ is $(\mathcal{Z}/2\mathcal{Z})^2$. For the case D_{2n+1}, more work is required to determine that the center of $Spin(2(2n+1))$ is cyclic of order 4.

For A_n, the determinant is $(n+1)$, and the Coxeter element is represented as the conjugacy class of the $(n+1)$-cycle in Σ_{n+1}. If one takes L to be the usual lattice for $SU(n+1)$ then one has the s.e.s.

$$0 \to L \to L' \to \mathcal{Z} \to 0$$

where L' has the permuation action of Σ_{n+1}, and $\mathcal{Z}$ the trivial action. produce a group of of order 2.

§2. Results for p-compact groups

The idea of a p-compact group was introduced in [13] to abstract the homotopical information available from the classifying space BG of a compact Lie group G. It focuses on the property that the space of pointed loops ΩBG is homotopy equivalent to the finite complex G. We weaken this to work one prime p at a time by considering pairs of p-complete spaces (BX, X) for which ΩBX is homotopy equivalent to X and $H^*(X, \mathbf{F}_p)$ is a finite set. The definition is motivated by the early work of D.L. Rector [rRector].

One major result [13] is that there are analogues of maximal tori and Weyl groups for p-compact groups, with the twist that the torus is a p-completion of the ordinary torus and the Weyl groups are defined over the p-adic numbers instead of the rational numbers. See Moller [20] or Notbohm [22] for overviews on classifying spaces and p-compact groups.

For (BX, X) a p-compact group, Dwyer-Wilkerson prove that $Map(BX, BX)_{id}$ is again a p-compact group and is calculable in algebraic terms. Namely, $\Omega Map(BX, BX)_{id} = C_X = \check{T}^{W_X}$ up to 2-torsion.

We use the work of [15] together with a generalization of Coxeter elements to complex reflection groups to show that many of the possibilities for centers and fundamental groups in the p-compact group case are zero.

Actually, the proof for the sporadic Lie Weyl groups of types (G_2, F_4, E_7, E_8) and the infinite families (B_n, C_n, and D_{2n}) is much simpler that one might expect from the size of these Weyl groups. The Weyl groups in these cases contain the element $\{-I\}$, as viewed in the representation on $\pi_1(T)$. This forces the center, fundamental group, and the ambient cohomology group for possible k-invariants to have exponent 2. The remaining Lie groups of types (A_n,D_{2n+1},E_6) have Weyl groups with trivial center.

The Coxeter element provides a convenient organising principle all Lie cases and the method is quite effective for the calculations and bounds for the center and fundamental group. On the other hand, the Coxeter element method does not provide an obvious good bound for the $H^3(W, L)$, home of the k-invariants for the extensions $T \to NT \to W$. To finish that work, currently one has to resort to explicit calculations involving comparisons to the symmetric group case. Similar principles apply in the p-compact group case – that is, the sporadic cases are easy because the Weyl group has a center of order prime to p, for p odd. The remaining infinite families of complex reflection groups are analogous to the $\{A_n\}$ or $\{D_{2n+1}\}$ cases.

Recall that for (BX, X) a p-compact group, Dwyer-Wilkerson prove that $BC_X = Map(BX, BX)_{id}$ is again a p-compact group and is calculable in algebraic terms as $\check{T}^{W_X}$ up to 2-torsion.

First, we give some consequences of our methods that do not depend on the classification of complex reflection groups:

Theorem III. *Let X be a CW complex such that $H^*(X, \mathbf{Q}) = \mathbf{Q}[x_1, \ldots, x_n]$ where $|x_i| = 2d_i$. If $H^*(\Omega X, \mathcal{F}_p)$ is finite and $r = gcd(\{d_i\})$ then $r \mid (p-1)$ for p odd and $r \mid 2$ for $p = 2$.*

Theorem III recovers a tough cohomological calculation by J.P. Lin showing that r cannot be 4 in the 2-torsion free case, [19].

As mentioned above, centers are related to spaces of self-maps, by work of [18] and [15]. Here is another conclusion couched in the rational homotopy type of X :

Theorem IV. *Let X be a CW complex such that $H^*(X, \mathbf{Q}) = \mathbf{Q}[x_1, \ldots, x_n]$ where $|x_i| = 2d_i$. Suppose that $H^*(\Omega X, \mathcal{F}_p)$ is finite and $r = gcd(\{d_i\})$. If $p = 2$ and $r = 2$, then*

$$Map((\mathbf{F}_2)_\infty X, (\mathbf{F}_2)_\infty X)_{id} \approx_{h.e.} (\mathbb{RP}^\infty)^k$$

for some $k \geq 0$. If $p > 2$ and $r \neq 1$, the mapping space is contractible. Here, the mapping space is that of unpointed continuous maps homotopic to the identity.

This method fails to yield new information in the case $r = 1$, such as $X = BSU(31)$. In cases where $k = 0$, such as for $W = W(E_8)$, the estimate provided by the center of W is not sharp. The best estimate can require the use of a Coxeter element of W instead of just C_W.

Classification of finite reflection groups over the complex numbers and p-adic numbers and analogues of the classical Coxeter elements provides stronger forms of IV:

Theorem V. *Let (BX, X) be a connected p-compact group with Weyl group W_X which is an irreducible p-adic reflection group. If the reflection representation of W is not definable over the rationals $\mathbf{Q}$, then the center $C_X = \Omega Map((\mathbf{F}_p)_\infty X, (\mathbf{F}_p)_\infty X)_{id}$ is contractible and the fundamental group $\pi_1(X)$ is trivial. If W_X is definable over $\mathbf{Q}$, then C_X is a subquotient of the center of the corresponding compact simply connected Lie group with the same Weyl group. In particular, if $p > 3$, then C_X is trivial unless $W_X \approx W(SU(pN)) = \Sigma_{pN}$ for some $N > 0$.*

Clark-Ewing [7] show for a reflection representation of a finite reflection group W the minimal field of definition is the extension of the rationals generated by the characters of the elements of W. Hence the restriction in Theorem V reduces to verifying that there is some element in W with corresponding matrix having trace not a rational number. For the irreducible complex reflection groups this information is available in [7].

We discuss briefly some of the algebra underpinning the above results. In the simply connected semi-simple Lie case, $T^{W_G} = H^0(W_G, T) = H^1(W_G, L)$. Here $L = \pi_1(T)$ is an integral lattice in the universal cover of T. For the p-compact group case, the theory is analogous, so $C_W(L) = H^1(W, \pi_1(T))$ serves as a good upper bound estimate. Now $L = \pi_1(T)$ is a p-adic integral lattice in $V = L \otimes \mathbf{Q}$. The algebraic content is summarized below:

Theorem VI. *Let L be a p-adic integral lattice in $V = \mathbf{Q}_p^n$ and W a finite subgroup of $Aut(L)$ generated by generalized reflections. Assume that the action of W on V is irreducible. Then*

(1) *If the reflection representation of W is not definable over $\mathbf{Q}$, then $C_W(L) = H^0(W, \check{T}) = H^1(W, L) = 0$.*
(2) *If W is equivalent to the Weyl group W_G of a simple simply connected Lie group G, then $C_W(L)$ is a subquotient of the classical G center. For odd primes, it is zero except for the cases $G = SU(pn)$ and for $p = 3$, also $G = E_6$.*

Similar results are obtainable for $H_0(W, L)$, a group which surjects onto the fundamental group in the Lie case, and for $H^3(W, L)$, home of the k-invariants which classify the $T \to NT \to W$ extension possibilities.

§3. The p-compact group case

For p-compact groups the usual torus must be replaced with a p-complete version for which $\pi_1(T)$ is a finitely generated free $\mathcal{Z}_p$ module, denoted by L again. There is a discrete model for T, $\check{T} = (L \otimes \mathbf{Q})/L$ which can be visualized as the points of p-th-power order in the standard torus. The p-completion of $\check{T}$ is homotopy equivalent to that of the standard torus.

Then the estimate of 2.1 holds for p-compact groups as well as an exact formula of [15]:

Definition 3.1. [15] *An element s of finite order in $Aut(V)$ is a (generalized) reflection if and only if s fixes a hyperplane of V. Suppose that X is a connected p-compact group with Weyl group W. If $s \in W$ is a reflection, with representative $x \in \check{N}T$. Then*

1. *the fixed point set $F(s)$ of s is the fixed point set of the action of x on $\check{T}$ by conjugation,*
2. *the singular hyperplane $H(s)$ of s is the maximal divisible subgroup of $F(s)$ (so that $H(s) \cong (\mathbf{Z}/p^\infty)^{r-1}$),*
3. *the singular coset $K(s)$ of s is the subset of $\check{T}$ given by elements of the form $x^{\mathrm{ord}(s)}$, as x runs through elements of $\check{\mathbf{N}}(T)$ which project to s in W, and*
4. *the singular set $\sigma(s)$ of s is the union $\sigma(s) = H(s) \cup K(s)$.*

Theorem 3.2. [15] *Let $\mathcal{X}$ be a connected p-compact group, T a maximal torus for $\mathcal{X}$, and $\check{T} \to T$ a discrete approximation for T. Define $C \subset \check{T}$ by*

$$C = \cap_s \sigma(s)$$

where the intersection is indexed by reflections $s \in W$. Then $C \to \mathcal{X}$ is a p-discrete center for $\mathcal{X}$.

Corollary 3.3. *Let (BX, X) be a connected compact Lie group or a connected p-compact group. If X is Lie or $p = 2$, $\overline{\check{T}^{W_X}}/C_X$ is a finite elementary abelian 2-group. Here the overline means the topological closure in T of the subgroup. If p is odd, C_X is the completion of $\check{T}^{W_X} = H^0(W_X, \check{T})$.*

Proof of 3.2. For p odd, the reflection s has order dividing $(p-1)$. Since $\check{T}$ is uniquely divisible for integers prime to p, $H(s) = Fix(s)$ in this case. For $p = 2$, $H(s)$ has index at most 2 in $Fix(s)$, so $Fix(s)/\sigma(s)$ is at most $\mathcal{Z}/2\mathcal{Z}$. Hence $(\cap_s \sigma(s))/\cap_s Fix(s)$ has exponent 1 or 2.

A deeper analysis of the singular set shows that if s does not act as the identity on the elements of order 2 in $\check{T}$, then $\sigma(s) = Fix(s)$. Hence if the irreducible factors of the universal cover of G do not contain one of the Dynkin types $\{B_n\}$ or $\{C_n\}$ then the 2.1 estimate is exact.

The Borel connectiveness theorem (2.3) used in the proof of 2.2 can also be applied to analyze the contributions of the singular sets of the reflections in 3.1. This leads to the same conclusion in the simply connected case. Unfortunately, the connectivity result is not yet known in the p-compact group context.

§4. Centers of Weyl Groups

Many classical Weyl groups have the property that the rings of rational invariant polynomials are concentrated in even weights (grading divisible by 4 if one uses topological conventions). This has implications for the structure of W and centrality questions.

Proposition 4.1. *Let K be a field of characteristic 0 and V a finite dimensional K-vector space. Suppose $W \subset \mathrm{Aut}(V)$ is a finite subgroup such that the representation is absolutely irreducible (doesn't split on extension of scalars). Then C_W is cyclic and represented on V as a scalar diagonal matrix θI, where θ is a root of unity in K.*

Taking $K = \mathbf{Q}$ yields that the center of the Weyl group of a simple Lie group has order at most 2 if the representation of W on $\pi_1(T) \otimes \mathbb{C}$ is irreducible. On the other hand, if $K = \mathbf{Q}_p$ with p odd, then the order of θ divides $(p-1)$ and thus is prime to p. For $K = \mathbf{Q}_2$, the only choice is $\theta = -1$.

Theorem 4.2. *Let k be a field and V a finite dimensional k-vector space. Denote by $S(V^{\#})$ the algebra of polynomial functions on V. Suppose that W is a finite irreducible subgroup of $Aut(V)$. If $S(V^{\#})^W$ is concentrated in weights divisible by r, then W contains a scalar diagonal matrix θI, where θ is a primitive r-th root of unity in k.*

Although 4.2 is purely algebraic, its applications to topology are farreaching. Here is an improvement to a result of Jim Lin, [19]:

Corollary 4.3. *If (BX, X) is a connected 2-compact group, then $H^*(BX, \mathcal{Z}_2) \otimes \mathbf{Q}$ cannot be concentrated in topological degrees divisible by 8. More generally, if (BX, X) is a connected p-compact group with degrees $\{d_1, \ldots, d_n\}$, then $gcd(d_1, \ldots, d_n)$ must divide $(p-1)$ for p odd, or 2 for $p = 2$.*

For example, S^7 has no structure as an associative H-space, (even at $p = 2$) because its classifying space would have rational cohomology concentrated in degrees divisible by 8.

4.3 follows from the observation that the field of 2-adics has no primitive 4-th roots of unity. Lin's original result was stated in terms of the mod 2 cohomology, but the Bockstein spectral sequence shows that 4.3 applies. A slight modification of 4.3 allows generators in dimension 2 to be excluded from the calculation of the *gcd*, so that, for example, the invariants of the upper triangular matrices over $\mathcal{F}_p$ with diagonal 1's are excluded from being the cohomology of a space, for p odd.

Corollary 4.4. *For R, k, and V as in 3.3, if $S(V^{\#})^W$ is concentrated in weights divisible by $r \neq 1$, then*

(1) *if $k = \mathbf{Q}$ or $k = \mathbf{Q}_2$, then $r = 2$ and $H^0(W, \check{T})$ is an elementary 2-group of rank less than or equal n.*

(2) *if* $k = \mathbf{Q}_p$ *for* $p > 2$, *then* r *is a divisor of* $(p-1)$ *and* $H^0(W, \check{T}) = 0$.

Actually, one can deduce even more information from the existence of a central element.

Theorem 4.5. *Suppose that* M *is a* $R[W]$ *module and* C_W *has an element* C *so that* $C|M = rId$, *for some unit* $r \in R$.

(1) *Then* $(r-1)H^j(W, M) = 0$ *for* $j \geq 0$.
(2) *If* $(r-1)$ *is a unit in* R, *then* $H^j(W, M) = 0$ *for* $j \geq 0$.

For example, the k-invariant of the extension $1 \to \check{T} \to N\check{T} \to W \to 1$ is an element of $H^2(W, \check{T}) = H^3(W, L)$. Hence we recover part of the results of Tits, [29],

Corollary 4.6. *Let* W *be a finite p-adic reflection group with degrees* $\{d_1, d_2, \ldots, d_n\}$ *and such that* $gcd(d_1, d_2, \ldots, d_n) = r > 1$. *Then for any lattice* L,

(1) *if* $p > 2$, $H^*(W, L) = 0$.
(2) *if* $p > 2$, *any extension*

$$1 \to \check{T} \to N \to W \to 1$$

is a semi-direct product.

(3) *if* $p = 2$, $2(H^*(W, L)) = 0$.
(4) *if* $p = 2$ *then* $H^2(W, L/2L) \to H^2(W, \check{T}) = H^3(W, L)$ *is a surjection.*

Part (4) implies that in these cases the NT extension is associated to one with with $L/2L$ as kernel. Tits shows this for all Lie situations. Notice that 4.6.4 applies to the Dwyer-Wilkerson exotic 2-compact group $DI(4)$ also.

Examples. The method of 4.6 applies to all forms of the classical B_n, C_n, D_{2n}, G_2, F_4, E_7, E_8, but not to A_n, D_{2n+1}, or E_6. It works for all the sporadic complex reflection groups not defined over the rationals as well as the Shephard-Todd family 3, the subset of $2a = \{G(m, r, n)\}$ with $gcd(m, mn/r) > 1$, and the dihedral groups $2b$ with degrees $(2, 2r)$. It does not apply to the remaining type $2a$'s or $2b$ groups. For the exceptions, one must be content with non-central generalized Coxeter elements.

Proof of 4.1. Extend scalars to use $V \otimes_K \bar{K}$, where $\bar{K}$ is the algebraic closure of K. By Schur's lemma, action by an element of the center is a W-map and hence a scalar unit. So C_W is embedded in the units of K, and hence, since it is finite, is cyclic. Let c be a generator. In $Aut(V \otimes \bar{K})$, $c = \lambda Id$, for some λ in $\bar{K}$. But then $trace(c) = dim(V)\lambda \in K$, not just in $\bar{K}$. Since $char(K) = 0$, $trace(c)/dim(V) = \lambda \in K$ also.

Proof of 4.2. This is similar to Humphreys, page 82 [17]. Without loss of generality, we can, by replacing k by a larger field, assume that k contains the r-th roots of unity. Let S be the fraction field of $S(V)$ and K that of $S(V^{\#})^W$. Then $K \to S$ is a Galois field extension with Galois group W.

Given an r-th root of unity $\eta \in k$, define a field automorphism $\phi : S \to S$ by $\phi(v) = \eta v$ for each $v \in V^{\#}$ and extend to $S(V^{\#})$ and the fraction field. Since $S(V^{\#})^W$ is concentrated in weights divisible by r, and $\phi x_N = (\eta^N)x = (\eta^r)N/rx = x$ if $r|N$, we have that $S(V^{\#})^W$ is fixed by ϕ. Since $K \to S$ is a Galois field extension, and hence normal, ϕ must be in the Galois group of S over K, W. Hence W contains the scalar diagonal matrix ηI and the value η is in the original field. Notice that ηI is a central element in W.

Proof of 4.3. This is immediate from the structure of the roots of unity in $\mathbf{Q}_p$ together with 4.2 .

Proof of 4.4. C_W has an element of order r that acts as a scalar on L.

Proof of 4.5. See for example, Hilton-Stammbach, Prop. X.X, or Dwyer, blank. Let C be central in W. Then action by C on L is a W-map, and hence induces a map $C_* : H^*(W, L) \to H^*(W, L)$. On the other hand, the induced map on BW by conjugation on W is homotopic to the identity. That is $C_* = Id$ for all $j \geq 0$.

Proof of 4.6. From the SES

$$0 \to L/pL \to \check{T} \to \check{T} \to\to 0$$

one has the short exact sequences $H^i(W, \check{T}) \to H^{i+1}(W, L/pL0 \to H^{i+1}(W, \check{T})$ for $i > 0$. Hence and $H^0(W, L) = 0$, it follows that $H^I(W, \check{T}) \to H^{i+1}(W, L)$ is an isomorphism for all $i \geq 0$. From $2H^i(W, L) = 0$, 4.5, and the SES

$$0 \to L \to L \to L/2L \to 0$$

, it follows that $H^i(W, L/2L) \to H^{i+1}(W, L)$ is onto for all $i \geq 0$. Thus $H^2(W, L/2L) \to H^3(W, L) \to H^2(W, \check{T})$ is onto, and the extensions pullback to extensions with $L/2L$ as fiber.

Or consider

$$0 \to L/2L \to \check{T} \to \check{T} \to 0$$

directly.

§5. Irreducible p-adic reflection groups

Recall the familiar classification of connected Dynkin diagrams associated to Lie groups and algebras. There are the infinite families A_n, B_n, C_n, and D_n, which are realized by the simply connected compact Lie groups $SU(n+1)$, $Spin(2n+1)$, $Sp(n)$, and $Spin(2n)$, respectively. In addition, there are the Weyl groups of the exceptional compact Lie groups G_2, F_4, E_6, E_7, and E_8.

The Shephard-Todd classification of irreducible finite complex reflection groups has a similar pattern – a few infinite families together with several sporadic examples, see Appendix B. For each group W in the Shephard-Todd list, the work of Clark-Ewing computes the possible p-adic fields $\mathbf{Q}_p$ over which the reflection representation can be defined. We focus on the infinite families in follows. are described in the text below.

The infinite Shephard-Todd family of type 1 coincides with the classical A_n family. The classical families $\{B_n\}$ and $\{C_n\}$ coincide as real reflection groups and are a subset of Shephard-Todd family $2a$, being the $\{G(2,1,n)\}$. Likewise, the classical $\{D_n\}$ are present in type $2a$ as the $\{G(2,2,n)\}$. The infinite families are topped off by the dihedral groups (type $2b$) and cyclic groups (type 3).

In our search for generalized Coxeter elements, it will be convenient to work with the broad divisions

(1) Weyl groups defined over $\mathbf{Q}$ (Weyl groups of Lie groups).
(2) sporadic with non-rational reflection representation
(3) infinite families with non-rational reflection representations.

Proposition 5.1. *Suppose that W is defined over $\mathbf{Q}$.*

(1) *If W has type A_1, B_n, C_n, D_{2n},G_2, F_4, E_7, or E_8, then W contains the scalar diagonal element $\{-I\}$ as a generalized Coxet er element.*
(2) *The type A_n has a Coxeter element of order $(n+1)$. No proper power is a generalized Coxeter element.*
(3) *The type D_n has a Coxeter element C of order $2(n-1)$ with $det(I-C) = 4$.*
(4) *The type $B_n = C_n$ has a Coxeter element C of order $2n$, with $|det(I-C)| = 2$.*
(5) *For the type G_2, the Coxeter element has order 6 and its 2-th power is a generalized Coxeter element of order 3. The $\{-I\}$ scalar diagonal is a generalized Coxeter element of order $2.|det(I-C)| = 1$.*
(6) *For the type F_4, the Coxeter element has order 12 and the 6-th and 4-th powers of the Coxeter elements are generalized Coxeter elements of order 2 and 3, respectively. $|det(I-C)| = 1$.*
(7) *For the type E_6 , the Coxeter element Cx has order 12. $(Cx)^4$ is a generalized Coxeter element of order 3. W does not contain any non-identity scalar diagonal matrices. $|det(I-C)| = 3$.*
(8) *For the type E_7, the Coxeter element has order 18 and its 9-th power is the generalized Coxeter element $\{-I\}$. $|det(I-C)| = 2$.*

(9) *For the type E_8, the Coxeter element has order* 30*. The* 15*-th power,* 10*-th power, and* 6*-th power are generalized Coxeter elements of order* 2*,* 3*, and* 5 *respectively.* $|det(I - C)| = 1$.

Notice that the abelian group structure is determined by the above information except for the cases $\{A_n\}$ and $\{D_{2n+1}\}$. We treat this question in an appendix, since its favor is somewhat different.

Proposition 5.2. *If W is a sporadic irreducible complex reflection group not defined over* $\mathbf{Q}$*, then W contains the central scalar diagonal element $\{-I\}$ as a generalized Coxeter element, or in the case of Type* 25*, a scalar diagonal matrix of order* 3*.*

This follows from glancing over the degrees of the invariants for these groups and applying 4.2.

Proposition 5.3. *Suppose that W is an irreducible complex reflection group in one of the Shephard-Todd families* 2*a,*2*b, or* 3*. Then*

(1) *If W is of type* 2*a, W is one of the $G(m, r, n)$ of Shephard-Todd. Here r must divide m, and in realizable cases, m divides $(p - 1)$, if $p > 2$, or $m = 1$ or* 2 *if $p = 2$. Given a basis $\{e_j\}$ of $\mathbb{C}^n$ and $\theta = e^{2\pi i/m}$, $G(m, r, n)$ is g enerated by the operations $e_i \to \theta^{\nu_i} e_{\sigma(i)}$, subject to the constraint that $\sum_i \nu_i = 0 \bmod r$. Here r divides m. There is a generalized Coxeter element C of order $2m$. If $gcd(mn/r, m) = d > 1$, then there is a central diagonal element of order d.*

(2) *If W is of type* 2*b and order $2m$, W is a dihedral group and there is generalized Coxeter element C of order m. It is not central in general, but if m is even, there is a central diagonal $-I$ matrix in W. In the realizable cases, m divides either $(p - 1)$ or $(p + 1)$ and hence if $p > 2$, m is prime to p.*

(3) *If W is of type 3 and order m, W is cyclic and there is a generalized Coxeter element C of order m. It is central. In the realizable cases, m divides $(p-1)$.*

The case of $DI(4)$. Dwyer-Wilkerson [14], showed the existence of a 2-compact group $DI(4)$ which realizes the complex reflection group of Shephard-Todd type 24, which is abstractly isomorphic to $\mathcal{Z}/2\mathcal{Z} \times GL(\mathcal{Z}/2\mathcal{Z}, 3)$. This group has a central $-I$, as well as generalized Coxeter elements of order 7 and 14. Thus any p-compact group X with this Weyl group would have C_X trivial. This example is the only non-Lie irreducible reflection group realizable over the 2-adics. It's reflections are of order 2, but the exponents $(3, 5, 13)$ fail to satisfy the "Poincare duality" present in those for irreducible real reflection groups.

Proof of 5.1. From the previous section, one knows that each W has a Coxeter element and that the eigenvalues are determined by the exponents or degrees of the fundamental invariants.

These degrees are known in all cases, so it is easy work to find certain generalized Coxeter elements, namely certain powers of the Coxeter element. Obviously C^k is a generalized Coxeter element for k a divisor of h if and only if $\eta^{-km_j} \neq 1$ for any $1 \leq j \leq \ell$. That is, if $k|h$ then C^k has a eigenvalue of $+1$ if and only if $km_j = 0$ mod (h) for some j. Typically, many of the $\{m_j\}$ are relatively prime to h, so not too many calculations are required.

We now analyze the cases:

- A_n : In the previous chapter it was remarked that the Coxeter element has order $h = (n+1)$ and that no proper power is a generalized Coxeter element.
- B_n,C_n : These yield isomorphic real reflection groups, although the Dynkin diagrams differ slightly. $\vec{m} = (1, 3, \ldots, 2n-1)$ and $h = 2n$. In this case, $C^n = -I$ is a generalized Coxeter element of order 2. If $2nm/k = q$ is odd, then q is in the list $\vec{m}$, and hence there are no generalized Coxeter elements of odd order constructible by this method. In this case, $det(I - w_{Cox}) = 2$.
- D_{2n+1} : In this case, $det(I - w_{Cox}) = 4$ and there are no generalized Coxter elements of order 2 that appear as powers of w_{Cox}. In the appendix, it is shown that the center is cyclic, so for the simply connected form it is $\mathcal{Z}/4\mathcal{Z}$. The case of the complex reflection group $G(2, 2, 2n+1)$ gives a generalized Coxeter element of order 4.
- D_{2n} : This contains a central $-I$ and $det(I - w_{Cox}) = 4$. Hence the center is $\mathcal{Z}/2\mathcal{Z} \times \mathcal{Z}/2\mathcal{Z}$.
- E_6 : In this case $det(I - w_{Cox}) = 3$. The Coxeter element has order 12 and its 4-th power is a generalized Coxeter element. Hence the center is $\mathcal{Z}/3\mathcal{Z}$ for the simply connected form.
- E_7 : $det(I - w_{Cox}) = 2$ and $-I$ is in W. Hence the simply connected form has center $\mathcal{Z}/2\mathcal{Z}$.
- E_8, F_4, G_2 : Each of these have $det(I - w_{Cox}) = 1$. It may be easier to see that each has a generalized Coxeter element of order 2, and also one of order 3. Hence the center is annihilated by 2 and 3 and is therefore zero.

Finally, the 2-torsion ambiguity is not a factor for many types:

Theorem 5.4. *If X is a connected simple 2-compact group with Weyl group W either A_n,D_n,G_2,F_4, E_6, E_7,E_8, or $DI(4)$ then $C_X = (\check{\breve{T}}_X)^W$.*

Proof of 5.4. In general, the induced map $W \to \mathrm{Aut}(L/2L$ has kernel an elementary abelian 2-group. In some cases, this kernel contains reflections, and in others, it does not. A key observation is that in formula 1.3, the singular set $\sigma(s)$ has the possibility of differing from $F(s)$ only if $s \in ker(W \to \mathrm{Aut}(L/2L))$. If s is not in this kernel, we are in the case analogous to $U(2)$, for which $H(s) = \sigma(s) = F(s)$, see section 8.7 of [11]. The set of reflections that reduce to the identity mod 2 is closed under conjugation and generate a normal elementary abelian subgroup of W. If all reflections in W are conjugate, then if W is not an elementary abelian 2-group, no reflection can be in the kernel. This happens for the exotic 2-compact group $DI(4)$ constructed in [14] and for any simple G of rank 2 or more with only single bonds in its Dynkin diagram, namely A_n, D_n, E_6, E_7, and E_8. The remaining cases of G_2 and F_4 likewise do not have large normal elementary abelian 2-subgroups. The Weyl groups for type $B_n = C_n$ do, and the full force of 2.3 is required to compute the center in these cases.

Proof of 5.3.

$2b$: The $2b$ examples contain the diagonal matrix (θ, θ^{-1}), for θ an m-th root of unity. If m is even, the $m/2$-th power of this matrix is $-I$.

$2a$: Notice that $G(m, m, n)$ is a subgroup of $G(m, r, n)$ for r a divisor of m, so it suffices to find a generalized Coxeter element for this group. In the the case $m = 2$, this is the Weyl group of D_n. Now if $h = gcd(m, n) > 1$, $G(m, m, n)$ contains a central element of order d. If $h = gcd(m, n) = 1$, then there exists $0 < b < m$ so that $bn = 1 \bmod m$. In this case, the diagonal matrix $M = (\theta^b, \ldots, \theta^b, \theta^{b-1})$ is in $G(m, m, n)$ and if $m > 2$ has no $+1$ eigenvalues. This leaves the case of $G(2, 2, 2n+1) = D_{2n+1}$. For this case, one can use the Lie Coxeter element, or observe that the diagonal matrix of $(2n-1)$ entries of -1 together with the two by two matrix of $[0, -1; 1, 0]$ has order 4.

§6. Calculations of fundamental groups

For compact connected Lie groups the centers and fundamental groups are closely related – the center of the simply connected form is the fundamental group of the adjoint form in the semi-simple case. For p-compact groups analogous relationships are expected but have been only partially established. In this note we only prove enough to demonstrate the triviality of the fundamental group of most irreducible connected p-compact groups.

Theorem 6.1. *If G is a connected compact Lie or p-compact group with Weyl group W and maximal torus T, the natural map $\pi_1(T) \to \pi_1(G)$ is a surjection and factors through the coinvariants $H_0(W, \pi_1(T))$.*

For G a connected compact Lie group, the map from 6.1, $H_0(W, \pi_1(T)) \to \pi_1(X)$ is always an isomorphism on odd torsion and is often a global isomorphism. We take this property as the definition of the algebraic version of the fundamental group:

Definition 6.2. *Let $p > 2$ and W a finite subgroup of $Aut(L)$ generated by reflections, where L is an integral p-adic lattice of V. The algebraic fundamental group $\pi_1(W, L)$ of the W action on L is the group of coinvariants $H_0(W, L)$.*

The relevance of the algebraic fundamental group is that it is an estimate for the actual fundamental group. For example,

Corollary 6.3. *If (BX, X) is a connected p-compact group with Weyl group W and maximal torus T, then if $\pi_1(W, \pi_1(T)) = 0$, then $\pi_1(X) = 0$. If $\pi_1(W, \pi_1(T))$ is p-torsion free, so is $\pi_1(X)$.*

For $p = 2$, one can enhance the definition of the algebraic fundamental group in a way similar to the formulas for the center in [15]. Note that the coinvariants can be computed as the quotient of L by the sublattice generated by the column spaces of $\{I - s_j\}$, where s_j is a reflection in W. Since $(I - s_j)$ has rank one rationally, each $I - s_j$ generates a rank one free module to the sublattice. For $p = 2$ or integrally, it may be the case that $I - s_j = 0 \bmod 2$, so that $(I - s_j) = 2A_j$. So there is the possibility of using the column space of A_j instead of that of $(I - s_j)$. One must use further information about NT to decided the choice. If s_j can be split back to an element of order 2 in NT, one takes the choice $(I - s_j)$. If not, use A_j. For example, for $SU(2)$ or $SO(3)$, there is one reflection $s_1 = -1$. $(1 - s_j) = 2$. For $SU(2)$, the reflection can only be lifted to an element of order 4, since the only element of order 2 in $SU(2)$ is its center. In $SO(3)$, the reflection lifts to an element of order 2. Hence for $SU(2)$, the sublattice is genenerated by 1, while for $SO(3)$, by 2. One can show that these modified coinvariants map onto the usual fundamental group.

Luckily, we can adapt the method of generalized Coxeter elements to estimate the algebraic fundamental group in most cases:

Proposition 6.4. *If (W, L, V) has a generalized Coxeter element of order N , then $N\pi_1(W, L) = 0$. In particular, if N is prime to p, then $\pi_1(W, L) = 0$.*

Corollary 6.5. *If W is any irreducible complex reflection group not defined over $\mathbf{Q}$, then $\pi_1(W, L) = 0$.*

Combining 6.2, 6.4, and results from the previous sections we have

Corollary 6.6. *Suppose $p > 2$ and (BX, X) is a connected p-compact group with Weyl group W and maximal torus T. Let $L = \pi_1(T)$ and suppose that $H^0(W, L) = 0$. Then the action of W on $L \otimes \mathbf{Q}$ gives a decomposition of W as direct product of irreducible p-adic reflection groups, $\{W_i\}$. If no factor W_i is isomorphic to the Weyl group of a $SU(pn)$ or for $p = 3$, E_6, then*

1. *$\pi_1(X) = 0$ and*
2. *$C_X = 0$.*
3. *BX is the product of p-compact groups $\{(BX_i, X_i)\}$ with Weyl groups $\{W_i\}$.*

Proof of 6.1. By the same argument as section XX from [11], each connected finite cover $\tilde{G} \to G$ pulls back to a connected cover $\tilde{T} \to T$. Thus each subgroup of $\pi_1(G)$ of finite index corresponds to at least one of $\pi(T)$. Thus on profinite completions, $\pi_1(T) \to \pi_1(X)$ is a surjection. In the p-compact case, we are finished. In the pure Lie case, $\pi_1(T)$ and $\pi_1(X)$ are finitely generated abelian groups, so the profinite surjection implies a surjection.

Proof of 6.2. Let C be a Coxeter element of order N prime to p. There is a natural surjection $H_0(< C >, L) \to H_0(W, L)$. Notice that $H_0(< C >, L) = L/(I - C)L = H^1(< C >, L)$ since $H^0(< C >, L) = 0$. Hence N annihilates $H^1(< C >, L)$ and hence also its quotient group $H_0(W, L) = \pi_1(W, L)$. $\pi_1(W, L)$ is a finite abelian p-group, so if N is prime to p, this implies that $\pi_1(W, L)$ is trivial.

Proof of 6.3. In the previous sections, it was demonstrated that each such group has a generalized Coxeter element of order prime to p, if the group is realized in $GL(n, \mathbf{Q}_p)$. Hence 5.3 follows from 5.2 .

Proof of 6.4. $\pi_1(W, L)$ is a quotient of the coinvariants $H_0(W, L)$, which itself is a quotient of $L/(1 - C)L = H_0(C, L) = H^1(C, L)$. This later groups is annihilated by $N = order(C)$. Hence so is $\pi_1(W, L)$.

Proof of 6.5. Such groups have a generalized Coxeter element of order prime to p.

Proof of 6.6. By the general structure stheorem of Dwyer-Wilkerson ,[16], X, since $H^0(W, L) = 0$ has a finite cover Y which is isomorphic to the product of p-compact groups $\{Y_j\}$, with each factor simply connected and simple. Given the constraints, these have trivial centers, so the theorem holds.

§7. The center's group structure in the A_n and D_{2n+1} cases

Lemma 7.1. *Let W be a finite group acting on a lattice L over $\mathcal{Z}$ or $\mathcal{Z}_p$. If $H^0(W, L) = 0$, then $H^1(W, L)$ is cyclic if and only if $H^0(W, L/pL)$ is zero or cyclic for all primes.*

Proposition 7.2. *Let G be the simply connected form of the group for*

$\{A_n\}$ *: for the lattice $L = \pi_1(T)$, $H^0(W, L/pL)$ is cyclic or zero for all primes p.*
$\{D_{2n+1}\}$ *: for the lattice $L = \pi_1(T)$ for $Spin(4n+2)$, $H^0(W, L/2L)$ is $\mathcal{Z}/2\mathcal{Z}$ and $H^0(W, L/pL) = 0$ for odd primes.*

To complete the analysis, one needs 7.2 for other lattices. Suppose that W acting on $L \otimes \mathbf{Q}$ is absolutely irreducible. Then on L/pL, there may still be proper W-subspaces. However, if L' is such that $L' \otimes \mathbf{Q} \approx L \otimes \mathbf{Q}$ as W-spaces, one knows that the composition factors of the action of W on L'/pL' and L/pL must agree (although the extensions involved may be different).

In the needed case, W is the symmetric group Σ_n acting on a rank $n-1$ lattice $L_{SU(n)}$, obtained as the submodule of the rank n module generated by a basis $\{e_j\}$ on which Σ_n acts by permuations. Then $L_{SU(n)}$ is spanned by the elements $\{e_i - e_j\}$ One could choose a basis $\{e_j - e_n\}$ if needed.

It is known that $L_{SU(n)}$ is absolutely irreducible. For p a prime, it is irreducible if p is prime to n. If p divides n, there is a trivial one dimensional submodule, denoted as F_1 here. There is a S.E.S. of W modules

$$0 \to F_1 \to L/pL \to B \to 0$$

and it is known that B is a simple W module. Hence the composition factors of L/pL are known for the $L_{SU(n)}$ case, and hence for any other lattice rationally equivalent to it. That is, there is an associated graded in all cases of either L/pL or $F_1 \oplus B$. Thus the invariants $H^0(W, L/pL)$ are either 0, or $\mathcal{Z}/p\mathcal{Z}$.

The mod 2 analysis for D_{2n+1} reduces to the action of W on which the normal elementary abelian 2-subgroup of W acts trivially, so the action factors through Σ_{2n+1} action on the $(2n+1)$-dimensional vector space $L/2L$ by permutation of basis elements. The above analysis thus applies.

The $\{D_{2n}\}$ case is unnecessary, since the center has exponent 2 and cardinality 4.

It's also an interesting exercise to work directly with the Coxeter element to deduce that its invariants are $\mathcal{Z}/n\mathcal{Z}$.

§8. The k-invariant in $H^3(W, L)$ and Tits' theorem

Tits [29] proved that for compact connected Lie groups G, the k-invariant classifying the extension

$$1 \to T_G \to N_G(T_G) \to W_G \to 1$$

has order 2, so that the extension can be consided to be induced from one of the form

$$1 \to L_G/2L_G \to N'_G \to W_G \to 1.$$

Tits called N'_G the *extended Coxeter group.*

We've already shown that for W Lie of type not E_6, A_n, or D_{2n+1}, this follows from the existence of a central element of order 2 in W. We collect here the remaining cases for such a *mano a mano* approach.

Proposition 8.1. *Let W be an irreducible p-adic reflection group and X be a connected p-compact group realizing W. Then $2H^3(W, L_X) = 0$.*

Proof of 8.1. After eliminating the cases covered by the existence in W of a nontrivial central element of order 2 or prime to p, the cases remaining to check are $\{W = G(m, r, n)\}$ for all p and the leftover Lie cases of A_n, D_{2n+1}, and E_6.

$\{A_n\}$ For the standard permutation lattice L_U, $H^3(\Sigma_n, L_U) = H^3(\Sigma_{n-1}, \mathcal{Z}) = \mathcal{Z}/2\mathcal{Z}$, for $n > 2$. For L_{SU}, we have

$$L_{SU} \to L_U \to L_U/L_{SU} \approx \mathcal{Z} \ .$$

The Σ_n-action on L_U/L_{SU} is trivial. Now we have a problem. Is $H^3(\Sigma_n, L_{SU})$ a $\mathcal{Z}/4ZZ$ or is the exponent 2.

$\{E_6\}$ This seems intractable by general techniques, so our argument here is special to E_6. W has a subgroup of index 27 isomorphic to $W(D_5) \times \{Id\}$. Hence the k-invariant is annihilated by 2, since this is true for $H^2(W(D_5, \mathcal{Z})$., as demonstrated below.

$\{G(m, r, n)\}$ if $p > 2$, these contain Σ_n as subgroup of index prime to p. The lattice on this subgroup is that permutation lattice for Σ_n, so $2H^3(W, L) = 2H^3(\Sigma_n, L) = 0$. If $p = 2$, then the only realizable values of (m, r, n) are $(2, 2, n)$ and $(2, 1, n)$. The $(2, 2, n)$ corresponds to the D_n case, and $(2, 1, n)$ is the degenerate case of the permutation action on n-letters, i.e. $W(U(n)$.

$\{D_{2n+1}\}$ $W(D_{2n}$ has index $2(2n + 1)$ in $W(D_{2n+1}$. For

Non-trivial k-invariants. . One can see the list in Curtis-Wiederhold-Williams, [10]. To add to these Lie examples, the exceptional 2-compact group $DI(4)$ of Dwyer-Wilkerson has a central element of order 2 in its Weyl group, so the k-invariant is annihilated by 2. However, comparing the Krull dimension of the mod 2 cohomology rings of the split extensions to that of $DI(4)$ shows that the k-invariant is not zero.

§Appendix A. Coxeter elements in real reflection groups

A *finite Coxeter group* W is a finite subgroup W of $GL(n,\mathbb{R})$ generated by reflections. One can assume that the matrices are orthogonal, but this is not always convenient or necessary. If the action of W is essential (i.e. $V^W = 0$), there is a choice of n distinct reflections $\{s_1, \ldots, s_n\}$ that generate W. The product $w_{Cox} = s_1 s_2 \ldots s_n$ is called a *Coxeter element*. These w_{Cox} have amazing properties.

Theorem A.1. *Suppose that W is an essential Coxeter group.*

1. *The choices of w_{Cox} determined by different sets of n generating reflections and different product orderings are unique up to conjugation within W.*
2. *w_{Cox} fixes no non-zero vector. That is, no eigenvalue for w_{Cox} is 1.*
3. *If the W-action is irreducible and $order(w_{Cox}) = h$, then the dimension of the eigenspace in $V \otimes \mathbb{C}$ of w_{Cox} for the eigenvalue $\eta = e^{2\pi i/h}$ is one.*
4. *If the W-action is irreducible, then the eigenvector v for η is not fixed by any reflection in W.*

There are other fascinating properties of w_{Cox}, including a relationship between the eigenvalues of w_{Cox} and the dimensions of the polynomial invariants of W. In the next section, we will use some numerology based on this connection.

Theorem A.2. *(see Chapter 3 of* [17]*) Let W be a finite real reflection group with fundamental invariants of degrees $\{d_1, \ldots, d_\ell\}$. Let $m_j = d_j - 1$.*

1. *$|W| = d_1 d_2 \ldots d_n$.*
2. *W contains exactly $N = \sum_j m_j$ reflections.*
3. *If W is the Weyl group of a Lie group G, then $dim(G) = \sum_i (2d_i - 1) = \sum_i (2m_j + 1)$.*
4. *If W is irreducible, the Coxeter elements $\{w_{Cox}\}$ have order $h = (2\sum_j m_j)/\ell = max_j(d_j)$.*
5. *If W is irreducible, then w_{Cox} has eigenvalues $\{\eta^{m_j}\}$, where $\eta = e^{2\pi i/h}$.*
6. *If W is irreducible, then $|det(1 - w_{Cox})| = \prod_i (1 - \eta^{m_i})$.*

In the literature, the $\{m_j\}$ are referred to as the *exponents* of W and the $\{d_j\}$ as the *degrees*. The *dimension* of W is defined, in analogy to the Lie case (3) above, to be $dim(W) = \sum_j (2d_j - 1) = \sum_j (2m_j + 1) = 2N + \ell$. If $W' \subset W$ is a subreflection group with $dim(W') = dim(W)$, then one must have $W' = W$, since the number of reflections is the same in each group, because $N = (dim(W) - \ell)/2$. In particular, the generators for W are contained in W'.

If W is not irreducible, but $V^W = 0$, then V is a direct sum of irreducible W-spaces $\{V_j\}$, and W splits as a product of subreflection groups $\{W_i\}$ which act irreducibly on V_i. In this case, the Coxeter element is the product of the Coxeter elements of the $\{W_j\}$. Parts of A.2 must be suitably modified to account for multiple factors.

Examples.

(1) For $A_n = \Sigma_{n+1}$, $\vec{m} = (1, 2, 3, \ldots, n)$, so $h = 2(\sum_i i)/n = (n+1)$. w_{Cox} can be taken as the standard $(n+1)$- cycle in Σ_{n+1} Notice that if k is a proper divisor of $(n+1)$ then ${w_{Cox}}^k$ has at least one eigenvalue equal to 1. The number of different Coxeter elements is $n(n-1)(n-2)\ldots 1$. The total number of w with $det(1-w) \neq 0$ is also $n(n-1)(n-2)\ldots 1$. Calculating $det|I - w_{Cox}| = (n+1)$ directly in this case is an interesting exercise.

(2) For $W(E_8)$, $\vec{m} = (1, 7, 11, 13, 17, 19, 23, 29)$, so $h = 30$, and $(w_{Cox})^k$ has no eigenvalue of $+1$ for k any proper divisor of h, since the $\{m_j\}$ are relatively prime to h in this case. There are 30 different conjugacy classes of elements w with $det(1-w) \neq 0$. $W(E_8)$ has $(2)(8)(12)(14)(18)(20)(23)(30)$ elements. One thirtieth are Coxeter elements and there are $(7)(11)(13)(17)(19)(23)(29)$ elements with $det(1-w) \neq 0$.

In general, W can have many elements with the property that $det(I - w) \neq 0$. The following theorem of Carter suggests that these arise as the Coxeter elements for maximal rank subreflection groups. Carter [6] provides the case for W defined over $\mathbf{Q}$. The extension to the real case involves checking the additional irreducible cases of H_3, H_4, and $I_2(m)$.

Proposition A.5. *Let W be a finite reflection subgroup of $GL(V)$, where V is an n-dimensional real vector space and $V^W = 0$.*

(1) *(Carter [6]) Any $w \in W$ can be written as a product of n or fewer reflections in W. The minimal number required is $n - dim(V^w)$.*

(2) *C has $det(1-C) \neq 0$ if and only if there exist a set of n reflections $\{s_1, \ldots, s_n\}$ in W such that $C = s_1 \ldots s_n$ and the root spaces $\{I - s_j\}$ span V.*

(3) *if $det(1-C) \neq 0$, then C is a Coxeter element for the subgroup W' generated by the $\{s_j\}$ from (2).*

This leads to the main fact necessary for the use of Coxeter elements in the computations:

Theorem A.4. *(R. Steinberg, e-mail) If W is irreducible, then the Coxeter elements are not contained in any proper reflection subgroup of W.*

Proof of A.5.

(1) : See the first section of Carter, [6].

(2) : The matrices $\{I - s_j\}$ have rank 1 , generated by some $\{p_j\}$ since the $\{s_j\}$ are reflections. Hence the $\{p_j\}$ are a basis for V. Notice that $s_j(p_j) = -p_j$. Now the argument is the same as that of Carter, Prop. 10.5.6, [5] for this property for the Coxeter element.

(3) : By part (1) C is the product of at most n reflections. If there are fewer than n, then the intersection of the fixed hyperplanes of the reflections is

positive dimensional and $dim(V^C) \neq 0$. On the other hand if C is the product of fewer than n reflections, or if the root vectors fail to span V, then the fixed point set of the product is at least as large as the intersection of the fixed point set of the reflections, which has non-zero dimension.

(4) : If w_{Cox} is contained in $W' \subset W$, then W' has no non-zero fixed points, so w_{Cox} is a product of n-reflections from W', $s_1 \dots s_n$ with roots vectors that span V. Hence W' is a product of irreducible W_i' acting irreducibly on V_i. The formulas of A.1 and A.2 hold for each W_i. From A.3 or A.3, w_{Cox} has an eigenvalue of $e^{2\pi i/h}$. This must occur as the eigenvalue for the restriction of w_{Cox} to one of the V_i. Hence $dim(W_i) = \ell_i(h) + \ell_i$ and

Proof of A.4. We thank Professor Steinberg for his hints on this question. From A.1.4, if w is a Coxeter element then the eigenspace for η is spanned by a vector v which is not fixed by any reflection in W. Suppose that $w \in W'$ for some proper reflection subgroup. Suppose that W' is a non-trivial product of reflection groups, $W = W_1 \times W_2$, acting on $V_1 \oplus V_2$. Then in the complexification, $v = v_1 \oplus v_2$ and $w(v) = w(v_1) \oplus w(v_2) = \eta v = \eta v_1 \oplus \eta v_2$. Thus both v_1 and v_2 are in the eigenspace for η. So one is zero, say v_2. But W_2 centralises V_1, so the reflections in W_2 fix v. Hence W_2 must be the trivial group. That is, W' is irreducible. We must now show that $W' = W$. By Carter, [6], w is the product of n reflections $\{s_\alpha\}$ in W' with rootspaces spanning V. Taking W'' to be the subreflection group generated by the $\{s_\alpha\}$, w is a Coxeter element for W''. If W'' is a nontrivial product, one reasons as before. So W'' must be irreducible and w is a Coxeter element for it. Hence $dim(W'') = (h+1)n = dim(W)$, so $W = W' = W''$.

Proof of A.3. Since G is simply connected and compact, it is isomorphic to a direct product of simple simply connected compact Lie groups. Without loss of generality we can assume that G is simple. Let $t \in G$ be invariant under w_{Cox}. The centralizer $C_G(t)$ is connected and its Weyl group is a subreflection group of $W(G)$. Since t is fixed by w_{Cox}, certainly $w_{Cox} \in W(C_G(t))$. Hence by A.3.5, $W(C_G(t)) = W(G)$. Hence $dim(W_G) = dim(C_G(t) = dim(G)$. Since $C_G(t)$ and G are connected compact manifolds, $C_G(t) = G$. That is, $T^{w_{Cox}} = C_G$.

Definition A.6. *Let k be $\mathbf{Q}$ or $\mathbf{Q}_p$ and R be $\mathcal{Z}$ or $\mathcal{Z}_p$. If W is a finite subgroup of $GL(n, R)$, an element $C \in W$ is a* **generalized Coxeter element** *if and only if $det(I - C) \neq 0$.*

The *regular* elements of Springer,[26] are generalizations of the property A.1.4. These regular elements are generalized Coxeter elements, but we don't require for our applications the detailed information present in [26].

Proposition A.7. *If w_{gCox} is a generalized Coxeter element in W, we have the following estimates (denoting the subgroup generated by w_{gCox} as C)*

(1) $H^0(W, \check{T}) \subset H^0(C, \check{T}) = H^1(C, L)$, *so* $|C| H^0(W, \check{T}) = 0$.

(2) $|H^0(C, \check{T})| = |det(I - w_{gCox})|$ *up to units in* R *and* $|H^0(W, \check{T})|$ *divides* $|det(I - w_{gCox})|$ *in* R.

Examples. In the Weyl group of E_8, the Coxeter element has order 30. Since the exponents $\vec{m}$ are each relatively prime to 30, the 15-th power and the 10-th power of the Coxeter element are generalized Coxeter elements of order 2 and 3, respectively. Thus, by A.5, $H^0(W, \check{T})$ is annihilated by 2 and 3. Hence it is zero. Or one can calculate that $|det(1 - C)| = 1$ for the Coxeter element.

There are three general methods of finding generalized Coxeter elements. The first is an abstract existence theorem:

Theorem A.8.

(1) *If* W *is a finite complex reflection group. Define* ρ_j *to be the number of elements in* W *with exactly* j *eigenvalues of* $+1$. *If* $\{d_1, \ldots, d_n\}$ *are the degrees of the basic invariants of* W, *then*

$$\prod_j (1 + (d_j - 1)x) = \sum_j \rho_j x^j.$$

[Solomon, [25]*]*

(2) *If* W *fixes no non-zero vector, then there are* $\prod_j (d_j - 1)$ *generalized Coxeter elements in* W.

Proof of A.8. Part (1) is a result of Shephard-Todd [24] and Solomon [25]. For part (2), take $j = n$. Then $\rho_n = (d_1 - 1)(d_2 - 1)\ldots(d_n - 1) = \prod_j (m_j)$. This is non-zero if and only if no d_j is $+1$. Since W has no fixed non-zero vectors, the invariants are trivial in degree one. That is, no d_j is $+1$, and thus $\rho_n \neq 0$ and there is a generalized Coxeter element in W.

The second general method is a relationship between the center of W and the degrees $\{d_j\}$. This will be developed in the next chapter.

Finally, according to $A.5$, in the real case, the generalized Coxeter elements arise of Coxeter elements of essential subreflection groups. In the case of Lie groups, such subgroups arise from the semi-simple maximal rank sugroups of G. Borel-deSiebenthal [3], catalogue these in the case that G is simple. For example, E_7 has a centralizer of type A_7, so the Weyl group of E_7 has a generalized Coxeter element of order 8 as well as its Coxeter element of order 18.

Proof of A.7. We assume that C has order k. The definition of a generalized Coxeter element implies that $H^0(C, V)$ is trivial. From the long exact sequence for the coefficients $0 \to L \to L \otimes \mathbf{Q} \to \check{T} \to 0$, it follows that $H^0(C, \check{T}) = H^1(C, L)$. Hence the groups have exponent at most k. Since C is cyclic of order k,

$$H^1(C, L) = ker(I + C + C^2 + \ldots C^{k-1})/image(I - C).$$

But $det(I-C) \neq 0$ and L is torsion free, so $ker(I-C) = 0$. Hence on V, $(I-C)$ is invertible. But $(I-C^k) = 0 = (I-C)(I+C+\dots C^{k-1})$, so $(I+C+C^2+\dots C^{k-1}) = 0$ on V and L. Hence $ker(I+C+C^2+\dots C^{k-1}) = L$, and

$$H^1(C,L) = L/(I-C)L.$$

This quotient group by the theory of elementary divisors has order $|det(I-C)|$ in the $\mathcal{Z}$ case and the p-th part of this in the p-adic case.

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