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HIGHER CONJUGATION COHOMOLOGY IN COMMUTATIVE HOPF ALGEBRAS

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ABSTRACT. We study the action of the symmetric group Σ_n on a tensor product of $n-1$ copies of a commutative Hopf algebra A, defined by the second author [8]. We show that for 'nice' Hopf algebras, the cohomology algebra $H^*(\Sigma_n; A^{\otimes n-1})$ is independent of the coproduct if $n.(n-2)!$ is invertible in the ground ring.

Let A be a graded, connected, unital, counital, associative, coassociative Hopf algebra. In section 8 of [6] it was shown how A has a 'conjugation' or 'antipode' χ satisfying the equality

$$\mu \circ (1 \otimes \chi) \circ \Delta = \eta \circ \epsilon,$$

where μ and Δ are the product and coproduct and η and ϵ are the unit and counit/augmentation. In particular, $\chi(1) = 1$ and, for x of positive degree,

$$\chi(x) = -x + \sum_i x_i' \chi(x_i'')$$

where $\Delta(x) = x \otimes 1 + 1 \otimes x + \sum_i x_i' \otimes x_i''$. If A is commutative then $\chi^2 = 1$ and so gives an action of Σ_2 on A.

The second author extended this in [8] by providing, for each $n \geq 2$, an action of the symmetric group Σ_n on $A^{\otimes n-1}$ when A is commutative. If σ_i denotes the transposition $i \leftrightarrow i+1$, then Σ_n is generated by $\sigma_1, \cdots, \sigma_{n-1}$, and the action on $A^{\otimes n-1}$ is given by:

$$\sigma_1 = [(\mu \otimes 1) \circ (\chi \otimes \Delta)] \otimes 1^{\otimes n-3}$$

$$\sigma_i = 1^{\otimes i-2} \otimes [(1 \otimes (\mu \circ (\mu \otimes 1)) \otimes 1) \circ (\Delta \otimes \chi \otimes \Delta)] \otimes 1^{\otimes n-i-2} \text{ if } 1 < i < n-1$$

$$\sigma_{n-1} = 1^{\otimes n-3} \otimes [(1 \otimes \mu) \circ (\Delta \otimes \chi)]$$

Note that each σ_i acts multiplicatively, as does χ in the case $n = 2$

So for all $n \geq 2$, we have a multiplicative action of Σ_n on $A^{\otimes n-1}$. While it may seem unusual to have Σ_n acting on an $n-1$-fold product, these actions do in fact arise quite naturally in homotopy theory as will be explained in later. Motivated by this connection, we wished to calculate the cohomology of Σ_n with coefficients in $A^{\otimes n-1}$. However, in [3] we saw how complicated this calculation could be when we attempted it for $n = 2$ and for A an object familiar to algebraic topologists: the mod 2 dual Steenrod algebra.

Since these Σ_n actions explicitly involve χ and Δ, and the former involves the latter, it would be desirable if we could make the coproduct Δ as simple as possible. The following theorem, which is our main result, gives conditions under which we can do this without changing the cohomology ring that we wish to calculate.

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Theorem 1. *Let A be a commutative graded Hopf algebra whose underlying algebra is a tensor product of monogenic Hopf algebras and whose coproduct is coassociative. Let $\tilde{A}$ be this tensor product, considered as a Hopf algebra. So the underlying algebra of $\tilde{A}$ is the same as that of A but the coproduct is much simpler in that all the generators are primitive. If n and $(n-2)!$ are invertible in the ground ring, R, then there is an isomorphism of algebras*

$$\tilde{A}^{\otimes n-1} \longrightarrow A^{\otimes n-1}$$

which commutes with the Σ_n action thus inducing an isomorphism of cohomology rings

$$H^*(\Sigma_n; \tilde{A}^{\otimes n-1}) \approx H^*(\Sigma_n; A^{\otimes n-1}).$$

So, in particular, if A is a graded, connected, commutative Hopf algebra, of finite type over a perfect field then by the Borel-Hopf theorem ([6] Theorem 7.11), A satisfies the hypotheses of Theorem 1. Our Theorem is more general than this in that we can work over a ring instead of a field, and we do not need A to be of finite type so, for example, $A = \mathbb{Z}/8[x_1, x_2, \cdots]$, where $|x_i| = 2$ for all i, satisfies the hypotheses of Theorem 1, but not of the Borel-Hopf theorem.

The hypothesis that $n.(n-2)!$ be invertible is rather curious and we will discuss this further at the end of Section 2.

Of course, if we make the stronger assumption that $n!$ is invertible in R, then the cohomology ring is zero in positive degrees, but even in this case the theorem is of significance since it greatly simplifies the calculation of $H^0(\Sigma_n; A^{\otimes n-1}) = (A^{\otimes n-1})^{\Sigma_n} = \{x \in A^{\otimes n-1} \mid \sigma x = x \text{ for all } \sigma \in \Sigma_n\}$. In particular, if R is a field and A is a polynomial algebra, then the theorem puts this calculation into the realm of classical invariant theory. This is because Σ_n acts multiplicatively on $\tilde{A}^{\otimes n-1}$ and preserves the R-subspace $Q \subset \tilde{A}^{\otimes n-1}$ spanned by the generators. So we have Σ_n acting on a vector space Q, and the action on $\tilde{A}^{\otimes n-1}$ is just the action on the polynomial algebra $R[Q]$ $(= \tilde{A}^{\otimes n-1})$ induced from this Σ_n action on Q.

In fact one does not necessarily need A to be polynomial. For example, Molien's theorem can be re-worked as follows:

Theorem 2.

1) *Suppose R is a field of characteristic coprime to $n!$. Let A be generated by $x_1, x_2, \cdots$ of degrees $d_1, d_2, \cdots$ and heights $h_1, h_2, \cdots$ (chosen from $\mathbb{N} \cup \infty$). Then the Poincaré series, $p(t)$, for $(A^{\otimes n-1})^{\Sigma_n}$ is given by*

$$p(t) = \frac{1}{n!} \sum_{g \in \Sigma_n} \prod_i \frac{\det(1 - gt^{d_i h_i})}{\det(1 - gt^{d_i})},$$

where, in the i-th factor on the right, the determinants are those of the given operator acting on the space $\langle x_i \otimes 1^{\otimes n-2}, \cdots, 1^{\otimes n-2} \otimes x_i \rangle$.

2) *If R is a field whose characteristic is not coprime to $n!$ then this Poincaré series bounds that for $(A^{\otimes n-1})^{\Sigma_n}$ coefficient-wise.*

By way of example, if $n = 2$ and A is a polynomial algebra on generators in degrees $d_1, d_2, \cdots$ then the first part of Theorem 2 leads to the following explicit formula

Corollary 3. *If R is a field of characteristic different from 2, the Poincaré series for A^{Σ_2} is*

$$p(t) = \frac{\prod_i(1+t^{d_i}) + \prod_i(1-t^{d_i})}{2\prod_i(1-t^{2d_i})}.$$

In fact, in the case $n = 2$ Theorem 1 leads to the following explicit 'model' for the conjugation invariants.

Theorem 4. *Suppose A is a Hopf algebra which, as an algebra, is a tensor product of monogenic Hopf algebras and suppose that 2 is invertible in the ground ring. Let A^{Σ_2} denote the subring of conjugation invariants, and let A_E denote the subalgebra of A spanned by the monomials whose exponents sum to an even number. Then there is an isomorphism of algebras*

$$A^{\Sigma_2} \approx A_E$$

This is a simple consequence of Theorem 1 and the fact that $\chi(x) = -x$ if x is primitive.

We note that if $p > 2$ then Theorem 4 along with Theorem 2 completely solves the 'conjugation invariants' problem for the mod p dual Steenrod algebra, in marked contrast to the partial solution [3] available when $p = 2$.

Why should Σ_n act on $A^{\otimes n-1}$?

Let E be a ring spectrum and let $A = \pi_*(E \wedge E)$, the set of "co-operations" in the cohomology theory associated to E. (For example, if E is the $\mathbb{F}_p$ Eilenberg-Mac Lane spectrum then A is the mod p dual Steenrod algebra.) If E is sufficiently nice then A will be a commutative Hopf algebra (more generally it will be a Hopf *algebroid*). The conjugation map, χ, on A is precisely the map induced on $\pi_*(E\wedge E)$ by switching the factors in the smash product $E \wedge E$ ([1] Lecture 3). Analogously, we can take the homotopy of a smash product of n copies of E, $\pi_*(E \wedge \cdots \wedge E)$, and there will be a natural action of Σ_n induced by permuting the factors. But $\pi_*(E^{\wedge n}) = E_*(E^{\wedge n-1})$, the cohomology of an $n-1$-fold product of copies of E and, for suitable E, this is isomorphic to $A^{\otimes n-1}$. Thus there is an action of Σ_n on $A^{\otimes n-1}$ and it is shown in [8] that this action satisfies the formulae given at the start of this paper.

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