Groups and Symmetries
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After 25 years in Texas, I've lived in Hawaii, Switzerland, Canada, Israel, New York, Pennsylvania, Michigan, and Indiana. My research interests are algebraic topology, Lie groups and classifying spaces. My undergraduate teaching at Purdue has been concentrated in sophomore/junior level science-engineering mathematics. My hobbies include building computers and photography.
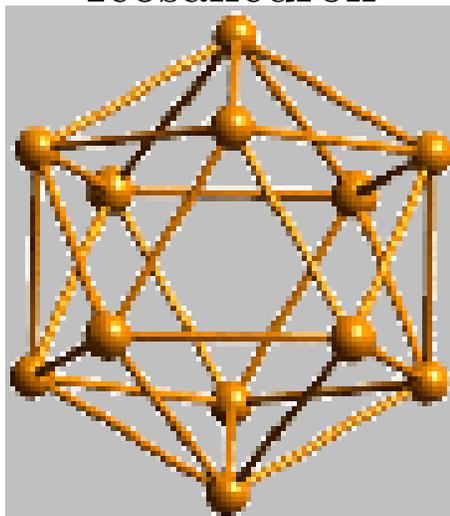
Home Page:

http://www.math.purdue.edu/˜ wilker

Groups and symmetries

The beginnings notions of groups go back to the classical Greeks and their realization of the possible symmetries of geometrical objects, for example the regular solids. The icosahedron below is an example of a regular solid:

Icosahedron

 Examples of Groups:
Symmetries of triangles, squares, circles.
(see overhead and toys)
Key Idea: Symmetries are reversible – you can always go back. Roughly, a group consists of operations on a set so that each operation can be

reversed and any two operations composed (multiplied).

Formal definition: *A group G is a set of elements $\{g\}$ together with a rule $G \times G \to G$ which is associative ($(ab)c = a(bc)$) with an identity element $\mathbf{e}$ such $\mathbf{e}g = g\mathbf{e} = g$ for all g in G, and such that each g has an inverse g^{-1} in G with $gg^{-1} = g^{-1}g = \mathbf{e}$ for all g in G.*

Examples:
Permutations: Given A,B,C, can write as ABC, ACB, CAB, etc. Go back to triangle example. Label vertices by A, B, C. Which permutations of A, B, C can we get by symmetries of the triangle? By rotation of the triangle, can get the "cyclic" permutations like CAB, etc. By a trick, flipping the triangle over, can get a non-cyclic permutation like CBA. (see live demo :)). So actually, the triangle has 6 symmetries, corresponding to the permutations of A,B,C . In general, can have permutations on N letters. This group is denoted by Σ_N or S_N and has order $N!$. Notice that $N!$

is a LARGE number if $N > 10$ or so.

Matrices and rotations: The 2×2 matrices with non-zero determinant. For example, rotation about $(0,0)$ by angle θ has matrix

$$\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

while the "flip" map has matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

More generally $SO(n)$ is the symmetries of n-space that preserve 0 and lengths and angles and orientation.

Symmetries of geometric figures: For example, for circle: rotation by any angle, plus the flip map. Notice that there are an infinite number of these. Sometimes called the infinite dihedral group $D_{2\infty}$. For the regular $n-$gon, the analog is the dihedral group of order $2n$, denoted by D_{2n}.

The subgroup of rotations is the cyclic group of order n, denoted C_n.

In the oral/overhead phase of the talk, I plan to talk briefly about some areas of math and science heavily influenced by groups. I will try to give computer demos of interesting (free) software for each area.

In outline form, here are the topics, together with some URL's for further investigation.

I. **Galois, Fermat, and Wiles** : Groups and equations
Goal: A tiny introduction to the use of groups in Number Theory and Fermat's Last Theorem.

E. Galois

Fermat's Last Theorem
Proved by A. Wiles, 1996.
If x, y, z are non-zero integers, show that $x^n + y^n = z^n$ has no solution if $n > 2$.
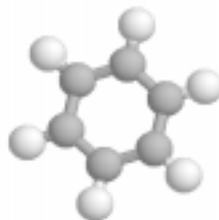
Software:
GP/PARI : ftp://megrez.math.u-bordeaux.fr/pub/
GAP : http://www-gap.dcs.st-and.ac.uk/~
gap
Powerful commercial programs: Maple, MatLab, Mathematica, Cayley, and Magma.

II. **Symmetry groups of Molecules**
Goal: Use group theory (representations) to simplify the quantum mechanics models for molecules

Benzene

Software:

If one examines the benzene molecule using for example the **RasMol** *visualizer, the molecule appears to be flat (planar) and essentially a hexagon. That is, we cam see a dihedral group of order 12 (rotations + flip), D_{12}. This information can be used to simplify the quantum mechanics model for benzene.*

This is just the tip of a very large iceberg. Molecular modeling is big business, both on the academic side and commercial applications.

RasMol: *(visualizer from PDB data)*
http://klaatu.oit.umass.edu/microbio/rasmol
Chemie : http://www.mdli.com/download/
Both Rasmol and Chemie can function as Netscape plugins.
Point Group Tutorial :
http://www.emory.edu/CHEMISTRY/pointgrp/
PDB Data:*(descriptions of 3-D structure of many molecules) Many places, e.g.*

http://www.sci.ouc.bc.ca/chem/molecule/molecule

III. Games: Rubik Cube and LightsOut

Goal: Use group theory to analyze and solve some popular games.

Rubik Cube
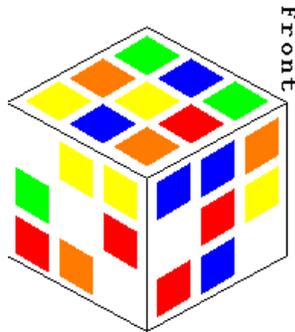
Rubik Software:
Solution in GAP: analyze the permutation, rewrite in terms of basic moves.

Rubik master web list:
http://www.best.com/s̃chubart/rc/resources.html

LightsOut Software:
Java, DOS and Win95 versions of both Cube and LightsOut on the Web, e.g. **http://www.download.com**

. I have written Maple and GAP solvers for Light-sOut.