

# $p$ -COMPACT GROUPS WITH ABELIAN WEYL GROUPS

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## §0. INTRODUCTION

In a recent series of papers, the authors have proposed a homotopy theoretic generalization of the notion of compact Lie groups to “ $p$ -compact” groups, and have shown that much of the discrete algebraic data for Lie groups, such as Weyl groups, can be developed in parallel for these “ $p$ -compact” groups. Since the Weyl group is a key ingredient in the classification of compact connected Lie groups, it’s natural to conjecture that it should play such a role for “ $p$ -compact ” groups also.

This paper explores the classification in the simplest case, where the Weyl group is generated by commuting reflections. The answer (appearing in 0.5A and 0.5B below) is that up to finite central quotients, the “ $p$ -compact” groups with abelian Weyl groups are products of the simply connected rank one examples. For odd primes, the rank one examples are just the “Sullivan” spheres, [25] and these have trivial centers, so the classification at odd primes reduces to earlier joint work of the authors with H. R. Miller, [8].

As is often the case for Lie groups, the prime 2 plays a special role in our analysis. In the work with Miller, [8], for odd primes  $p$ , cohomological methods based on Lannes’  $T$ -functor were used to prove strong uniqueness results for certain classifying spaces, characterizing them by the isomorphism type of the mod- $p$  cohomology algebra as an algebra over the Steenrod algebra. The key element of that proof was the construction of a function space analogue of the maximal torus. The function space calculation generalized the observation that in a connected compact Lie group  $G$ , the centralizer  $C_G(V)$  of the elements  $V$  of order  $p$  in the maximal torus  $T$ , for  $p$  an odd prime, is just the maximal torus  $T$ . But this may fail for  $p = 2$ —for example, in  $G = SU(2)$ ,  $V$  is the center of  $SU(2)$ , so its centralizer is all of  $SU(2)$ . A related failure is that even for a Lie group  $G$  such that  $H_*(G, \mathbb{Z})$  has no 2-torsion,  $H^*(BG, \mathbb{F}_2)$  may fail to all of  $H^*(BT, \mathbb{F}_2)^{W(G)}$ , in contrast to the odd prime case.

It turns out however, that this problem with  $SU(2)$  is essentially the entire extent of the failure. Our first result has a relatively simple proof in the case of Lie groups. A result similar to 0.1.2 appears in Broto-Henn, [5]:

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**Theorem 0.1.** *Let  $G$  be a compact connected Lie group.*

- (1) *If for an odd prime  $p$ , each element of order  $p$  in  $G$  is central, then  $G$  is a torus.*
- (2) *If each element of order 2 is central in  $G$ , then  $G$  is a product of a torus by products of  $SU(2)$  (quaternionic tori).*
- (3) *For an odd prime  $p$ , the mod  $p$  vector space  $V$  of elements of order  $p$  in a maximal torus  $T$  of  $G$  has  $C_G(V) = T$ .*
- (4) *For  $p = 2$  the mod 2 vector space  $V$  of elements of order 2 in a maximal torus  $T$ , the identity component of  $C_G(V)$  is a product of a torus by a quaternionic torus.*

*If  $H_*(G, \mathbb{Z})$  has no  $p$ -torsion, then*

$$H^*(BG, \mathbb{F}_p) \approx H^*(BC_G(V), \mathbb{F}_p)^{W_G(V)},$$

*where  $W_G(V) = N_G(V)/C_G(V)$ . For an odd prime  $p$ ,  $C_G(V) = T$  and  $W_G(V) = W(G)$ . For  $p = 2$  there is a short exact sequence*

$$1 \rightarrow E \rightarrow W(G) \rightarrow W_G(V) \rightarrow 1,$$

*where  $E$  is an elementary abelian 2-group.  $E$  can be identified with the Weyl group of  $C_G(V)$ .*

Our goal is to prove that with suitable analogues of the Lie structures used above, Theorem 0.1 holds for the class of connected  $p$ -compact groups. Recall from [13] that a  $p$ -compact group consists of a pair of spaces  $(BX, X)$  together with an homotopy equivalence  $g : \Omega BX \rightarrow X$  such that  $BX$  is  $F_p$ -complete in the sense of Bousfield-Kan, [3], that  $H^*(X, \mathbb{F}_p)$  is finite, and that the component group of  $X$  is a finite  $p$ -group. For such spaces [13] show that  $X$  has precise analogues of the usual homomorphisms, maximal torus, Weyl group, and centralizers. For example, the analogue of the centralizer is the loop space on the mapping space  $Map(BV, BX)_{Bf}$ . It is a main result of [13] and [10] that these centralizers inherit the  $p$ -compact group properties.

The centralizers of the elements of order  $p$  in the torus have special cohomological properties – in the  $p$ -torsion free homology case,  $H^*(BC_G(V), \mathbb{F}_p)$  is a sub-Hopf-algebra of  $H^*(BT, \mathbb{F}_p)$ , [8] and [27]. For odd primes, Hopf invariant one considerations force (even in the  $p$ -compact group case) the centralizer to be the torus. However, for  $p = 2$ , these considerations do not force in general the centralizer to be a torus. In fact, one original motivation for this work is the following characterization of these centralizers:

**Theorem 0.2.** *Let  $(BX, X)$  be a 2-compact group such that  $X$  is connected. Then  $BX$  is homotopy equivalent to the 2-completion of a product of  $BU(1)$ 's and  $BSU(2)$ 's under either of the following conditions:*

- (1)  *$H^*(BX, \mathbb{F}_2)$  is a finitely generated polynomial algebra with a Hopf algebra structure compatible with the action of  $\mathcal{A}_2$  or*
- (2) *each element of order 2 in  $X$  is central.*

In the proof of 0.2, we use one fast corollary of our work on maximal tori and Weyl groups in [13]:

**Proposition 0.3.** *If  $BX$  is a connected 2-complete space such that  $H^*(BX, \mathbf{F}_2)$  is a finitely generated polynomial algebra, and the cohomology is concentrated in dimensions divisible by 8, then  $BX$  is contractible.*

Proposition 0.3 also appears in [20] as an application of the [15] extension of mod 2 Hopf invariant one type results.

The second part of Theorem 0.1 is known for  $p$ -compact groups at an odd prime, [8]. The point at the the prime 2 is to identify the the homotopy type of the “centralizer” and to show that it accounts for the “inseparable” part of the normal extension given by [27].

**Corollary 0.4.** *Let  $(BX, X)$  be a 2-compact group such that  $X$  is connected and  $H^*(BX, \mathbf{F}_2)$  is concentrated in dimensions divisible by 2. Then there is a map*

$$f : BG = (BSU(2))^{r_1} \times (BU(1))^{r_2} \rightarrow BX$$

such that the homotopy fiber of  $f$  has finite mod 2 cohomology with a non-zero Euler characteristic. We also have

$$H^*(BX, \mathbf{F}_2) \approx H^*(BG, \mathbf{F}_2)^{W(X)},$$

but the action of  $W(X)$  need not be faithful on the mod 2 cohomology of  $BG$ . If the cohomology is concentrated in dimensions divisible by 4, then  $r_2 = 0$ , and  $X$  has a maximal rank subgroup isomorphic to a product of completed  $SU(2)$ 's. That is,  $X$  has a quaternionic maximal torus. In this case, the mod 2 reduction of  $W(X)$ ,  $W_X(V)$ , may be considered as a subgroup of  $\pi_0 \text{Aut}((\mathbf{F}_2)_\infty BSU(2)^r)$ . This group of components is isomorphic to the symmetric group  $\Sigma_r$  wreath product with the subgroup of squares in the 2-adic integer units. However, since  $W_X(V)$  is finite, it is conjugate to a subgroup of the  $\Sigma_r$  subgroup.

The existence of maximal tori and Weyl groups for  $p$ -compact groups yields us some primary data for classification. For example, for odd primes, the work of [8] easily classifies all connected  $p$ -compact groups with abelian Weyl groups, because these must be generated by reflections of order prime to  $p$ , and hence the order of the Weyl group is prime to  $p$ .

**Theorem 0.5A.** *Let  $(BX, X)$  be a  $p$ -compact group for an odd prime  $p$  such that  $X$  is connected. Then the following are equivalent:*

- (1)  $W(X)$  is abelian.
- (2)  $W(X)$  is a direct product of cyclic groups  $Z/rZ$ , where  $r$  is a divisor of  $(p - 1)$ .
- (3)  $BX$  is homotopy equivalent to a product of classifying spaces of “Sullivan” spheres,  $BS^{2r-1}$ .
- (4)  $H^*(BX, \mathbf{F}_p)$  is isomorphic over the Steenrod algebra to a tensor product of monogenic polynomial algebras, each closed under the Steenrod algebra action.

That is, the rank one connected  $p$ -compact groups for odd primes are the Sullivan spheres and these have trivial centers. However, for  $p = 2$ , the classification of connected 2-compact groups with abelian Weyl groups is more difficult. Somewhat surprisingly, we achieve the same classification as in the Lie case:

**Theorem 0.5B.** *Let  $(BX, X)$  be a 2–compact group such that  $X$  is connected. Then the following are equivalent*

- (1)  $W(X)$  is abelian.
- (2)  $W(X)$  is an elementary abelian 2–group.
- (3)  $H^*(BX, \mathbf{Z}_2) \otimes \mathbf{Q}$  is a finitely generated polynomial algebra on generators of dimensions 2 or 4.
- (4)  $H^*(X, \mathbf{F}_2)$  is isomorphic over the Steenrod algebra to the cohomology of a product of  $SU(2)$ ’s,  $SO(3)$ ’s, and  $U(1)$ .
- (5) there exists a Lie group  $G_X$  isomorphic to a product of  $U(1)$ ’s and  $SU(2)$ ’s with a central elementary abelian 2–subgroup  $E_X$ , such that

$$BX \approx B(G_X/E_X).$$

The cohomology rings of such  $BX$  need not be polynomial. The smallest non-polynomial example occurs as the quotient of a product of three  $SU(2)$ ’s divided by certain rank 2 subgroup of its center. In the cases that the cohomology is polynomial, the 2–complete homotopy type of the spaces  $BX$  is determined uniquely by the  $\mathcal{A}_2$ –structure on the mod 2 cohomology. The example of  $BSO(4)$  versus  $BSU(2) \times BSO(3)$  shows that the Steenrod algebra action as well as the abstract ring structure is needed. We show this uniqueness here only for the “semi-simple” case.

**Corollary 0.6.** *Let  $BY$  is a simply connected 2–complete space with  $H^*(BY, \mathbf{F}_2)$  a finitely generated polynomial algebra and  $\pi_1(Y)$  finite. Then the following are equivalent:*

- (1)  $W(Y)$  is abelian (and hence an elementary 2–group).
- (2) the indecomposables of  $H^*BY$  are concentrated in dimensions 2, 3, and 4.
- (3)  $BY$  is homotopy equivalent to the 2–completions of product of spaces  $BH(k)$ , where

$$H(k) \approx (SU(2)^k/\Delta(Z/2Z)).$$

equivalent to  $BY$ .

*Note.* Interesting examples of the groups  $H(k)$  of Corollary 0.6 arise in the analysis of centralizers in low rank 2–compact groups. For the Lie group  $G_2$ , the centralizer of any element of order 2 is  $SO(4)$  which is the product  $SU(2) \times SU(2)$  modulo the diagonal central  $Z/2Z$  subgroup. For the example  $DI(4)$  constructed in [12], the centralizer of any rank 2 elementary abelian subgroup is  $SU(2) \times SU(2) \times SU(2)$  modulo the diagonal central  $Z/2Z$  subgroup (up to completions).

Theorems 0.5X are suggested prototypes of a general classification of connected  $p$ –compact groups. The representation of the Weyl group on  $BT$  determines potential “simple” factors of  $X$  and then one works to decompose  $X$  (up to finite covers) as products of these simple factors. The methods in this paper are directly cohomologically based. It’s clear that this approach has its limits. Work in progress of the authors use the results of [13] and [14] in a more subtle way and we believe it will lead to a general theory of semi-simplicity for connected  $p$ –compact groups.

In the proof of Theorem 0.2, one ingredient needed beyond the cohomological tools of [8] is information about certain fibrations involving classifying spaces. We

can invoke the powerful results of [17] and its generalization in [14] here, although the particular results applied in this note are already proved in [12]:

**Theorem 0.8.** *Let  $G$  be a connected compact Lie group. Then the monoid of self equivalences of the 2-completion of  $BG$ ,  $Aut_{Id}(\mathbf{F}_2)_\infty BG$  which are homotopic to the identity is homotopy equivalent to the 2-completion of  $BC(G)$  and  $BAut_{Id}(\mathbf{F}_2)_\infty BG \approx B^2C(G)$ .*

This amounts to the assertion that the universal fibration

$$(\mathbf{F}_2)_\infty BG \rightarrow BMap_*((\mathbf{F}_2)_\infty BG, (\mathbf{F}_2)_\infty BG)_{Id} \rightarrow BMap((\mathbf{F}_2)_\infty BG, (\mathbf{F}_2)_\infty BG)_{Id}$$

is fiber homotopy equivalent to the 2-completion of the fibration

$$BG \rightarrow BG_{adjoint} \rightarrow B^2C(G).$$

**Corollary 0.9.** *Let  $G$  be a compact connected Lie group with a finite center and  $BG \rightarrow E \rightarrow B$  a fibration in which  $\pi_1(B, *)$  acts trivially on  $BG$ . Let  $p$  be a prime. Then the  $p$ -completion of the fibration is f.h.e. to the product fibration if  $H^2(B, C(G)) \otimes F_p$  is zero. In particular, if the center of  $G$  is trivial, the completed fibration is trivial.*

With suitable definitions of the center of  $p$ -compact group, both 0.8 and 0.9 are valid for  $p$ -compact groups also. These fibration theorems show that properties of centers and ideas of semisimplicity for Lie groups are strongly related. Moreover these concepts are mirrored in the homotopy theoretic properties of classifying spaces. Establishing such properties for  $p$ -compact groups is one goal for our future work.

*Method of proof.* : The method of proof of 0.2 is to pass to the 3-connected cover of  $BX$ . It's shown that this cover inherits the cohomological coproduct, and hence has polynomial generators in dimensions  $2^N$  for  $N > 1$ . We use the coproduct to show that the Weyl group is an elementary abelian 2-group and hence the rational cohomology of  $BX$  is generated by dimension 4 and 2. Hence the 3-connected cover of  $BX$  is a "fake" product of  $BSU(2)$ 's. Using homotopical analogues of the center, we construct a quotient space  $BY$  which is a "fake" product of  $B SO(3)$ 's. We then apply the diagram methods of [16] and [9] to inductively prove that  $BY$  is a product of 2-completed  $B SO(3)$ 's. Then we analyse the various fibrations that were used to reduce to the 3-connected case.

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*Notation..* All cohomology is with  $\mathbf{F}_2$  coefficients unless otherwise stated, and will be denoted by  $H^*X$ . The 2-adic integers are denoted by  $\mathbf{Z}_2$  and the usual rational numbers by  $\mathbf{Q}$ . The 2-completion used is the Bousfield-Kan completion and is denoted by  $(\mathbf{F}_2)_\infty X$ . The mod  $p$  Steenrod algebra is denoted as  $\mathcal{A}_p$ . The  $n$ -connected cover of the space  $X$  is denoted  $X \langle n \rangle$  and the  $n$ -th Postnikov approximation to  $X$  is denoted by  $X^{[n]}$ . The category of unstable  $\mathcal{A}_2$ -modules is  $\mathcal{U}$  and that of unstable algebras over  $\mathcal{A}_2$  is  $\mathcal{K}$ .

§1. CENTRAL QUOTIENTS OF  $U(1)^r \times SU(2)^s$ 

In this section we present a sketch of Theorem 0.1 for Lie groups (proofs are deferred to the end of the section):

**Lemma 1.1.** *Let  $G$  be a connected compact Lie group in which each element of order 2 is central. Then the Weyl group of  $G$  is an elementary abelian 2-group and the action of  $W(G)$  on the Lie algebra of the maximal torus is the product action.*

The next lemma is a special case of the fact that each connected compact Lie group has a finite cover is the product of a torus and a simply connected compact Lie group:

**Lemma 1.2.** *Let  $G$  be a connected compact Lie group in which the Weyl group is an elementary abelian 2-group. Then  $G$  has a finite covering which is isomorphic as Lie groups to a product of  $U(1)$ 's and  $SU(2)$ 's.*

**Lemma 1.3.** *Let  $G$  be the product of  $T = U(1)^r$  and  $H = SU(2)^s$ . Let  $C$  be a finite 2-group which is a subgroup of the center of  $G$ .*

- (1) *Every element of order 2 is central in  $G/C$  if and only if the projection of  $C$  into  $H$  is trivial.*
- (2)  *$\pi_1(G/C)$  is torsion-free if and only if the intersection of  $C$  with  $H$  is trivial.*

Before proving Theorem 0.1, we note a few homological properties of the quotients studied in this section. These properties will be useful for modeling the finite loop space analogues of Theorem 0.1:

**Proposition 1.4.** *Let  $\tilde{G}$  be the product of  $T = U(1)^r$  and  $H = SU(2)^s$ . Let  $G = \tilde{G}/C$ , where  $C$  is finite and central in  $\tilde{G}$ . Then*

- (1)  *$G$  is isomorphic to a quotient in which  $C$  is an elementary 2-group. In fact  $C$  can be replaced by  $C/(C^2)$ . Here  $C^2$  denotes the subgroup of  $C$  consisting of squares of elements of  $C$ .*
- (2)  *$G$  is homeomorphic as a topological space to a product of  $U(1)$ 's,  $SU(2)$ 's, and  $SO(3)$ 's. The number of  $SO(3)$  factors is the 2-rank of  $C \cap (1 \times H)$ .*
- (3) *If  $H^*BG$  is a finitely generated polynomial algebra, its indecomposables are concentrated in dimensions 2, 3, and 4.*
- (4)  *$H_*(G, \mathbb{Z})$  has torsion of order at most 2.*
- (5)  *$H^*(BG, \mathbb{Q})$  is a finitely generated polynomial algebra on generators of dimensions two and four.*

In the odd prime case, the Weyl group of  $G$  acts faithfully on the mod  $p$  cohomology of  $BT$  and hence its structure can be inferred entirely from the mod  $p$  cohomology of  $BG$ . At  $p = 2$  the action on cohomology is not always faithful, but we can at least recover the Weyl group up to an extension:

**Theorem 1.5.** *Let  $G$  be a compact connected Lie group. Let  $T$  be a maximal torus for  $G$ , and  $V$  the set of elements of order 2 in  $T$ . Then there is a short exact sequence of groups*

$$1 \rightarrow W(C_G(V)) \rightarrow W(G) \rightarrow N_G(V)/C_G(V) \rightarrow 1.$$

$W(C_G(V)) = (Z/2Z)^e$ . In terms of cohomological data,  $W_G(V) = N_G(V)/C_G(V)$  can be identified with the automorphism group of the fraction field extension corresponding to the ring extension

$$H^*BG/\ker(f^*) \rightarrow H^*BT.$$

In this setting,  $e$  is the inseparability degree of that extension. In cases that  $G$  has a unique (up to conjugacy) maximal elementary abelian 2-group  $U$ ,  $W_G(V)$  is just the subquotient

$$D_{W_G(U)}(V)/I_{W_G(U)}(V)$$

where  $D$  is the subgroup of  $W_G(U)$  that transports  $V$  into itself and  $I$  is the isotropy subgroup of  $V$ .

We remark that Theorem 1.5 has a valid translation to  $p$ -compact groups. From 1.5, we deduce another characterization of the Lie groups for which the Weyl group is an elementary 2-group:

**Lemma 1.6.** *Let  $G$  be a compact connected Lie group. Suppose that  $T$  is a maximal torus of  $G$  and  $V$  the subgroup of  $T$  of elements of order 2. Then the Weyl group of  $G$  is an elementary abelian 2-group if and only if each  $g$  in  $G$  that normalizes  $V$  also centralizes  $V$ .*

Again, 1.6 can be translated to  $p$ -compact groups.

*Proof of Theorem 0.1.* : By 1.1 and 1.2 there is a finite covering of  $G$  by a group  $\tilde{G}$  isomorphic to  $U(1)^r \times (SU(2))^s$ . Thus  $G = \tilde{G}/C$  for some finite central subgroup  $C$ . By 1.3, the projection of  $C$  to  $H = (SU(2))^s$  must be trivial. But in that case,  $\tilde{G}/C$  is abstractly isomorphic to  $\tilde{G}$ . For the cohomology statements, the odd prime case is in [8] ( Borel [2] only covers the prime to Weyl group order case). For  $p = 2$ , the centralizer is question is  $U(1)^s \times SU(2)^r$  and by [27] the extension of rings

$$H^*BG \rightarrow H^*BC_G(V)$$

is separable and Galois with Galois group  $W_G(V)$ . See Quillen,[22], also for this result up to  $F$ -isomorphism and localization.

*Proof of 1.1.* : Choose a maximal torus  $T$  for  $G$ . Let  $V$  be the group of elements of order 2 in  $T$ . If  $g \in N_G(T)$ , then conjugation by  $g$  takes  $V$  to  $V$  by the identity map. This implies that the mod 2 reduction of the action of  $W(G)$  on the integral lattice of  $L(T)$  is trivial. By 9.1,  $W(G)$  must be an elementary abelian 2-group. The action can be diagonalized over  $\mathbf{Q}$ .

*Proof of 1.2.* : Consider the adjoint form  $G_{adj}$  of  $G$ . It is a product of simple centerless compact Lie groups,  $G_i$ . The only Weyl group which is an elementary abelian 2-group and which is the Wey group of a simple Lie group is  $Z/2Z$ . Hence each factor of the adjoint form is a  $SO(3)$ , since the rank one simple groups are just  $SU(2)$  and  $SO(3)$ . But  $SU(2)$  has a non-trivial center so the only choice is  $SO(3)$ . General Lie theory, e.g. [26] gives that  $G$  has a finite cover by a product of a torus and a simply connected simple factors. Since the adjoint form is a product of  $SO(3)$ 's the corresponding terms in the finite cover must be  $SU(2)$ 's.

*Proof of 1.3.* : The center of  $G$  is isomorphic to the product  $T \times (Z/2Z)^{r_2}$ . Let  $(t, u)$  be an element in  $C$  such that  $u$  is not the identity. Denote by  $p$  the projection to  $G/C$ , and by  $p_{adj}$  the projection to  $G_{adj} = H_{adj}$ . Choose  $v$  in  $H$  such that  $v^2 = u$  and choose  $s$  in  $T$  such that  $s^2 = t$ . Then  $(p(s, v))^2 = p(s^2, v^2) = p(t, u) = Id$ , so  $p(s, v)$  has order 2 in  $G/C$ . Passing down to  $G_{adj}$ ,  $p_{adj}(s, v)$  is not the identity, because the kernel of  $H \rightarrow H_{adj}$  has no elements of order 4. But the center of  $H_{adj}$  is trivial by construction, so there must be  $h \in H$  such that  $p_{adj}(t, h)$  does not commute with  $p_{adj}(s, v)$ . That is, if the projection of  $C$  to  $H$  is non-trivial,  $G/C$  has elements of order 2 not in the center. Conversely, if the projection is trivial,  $G/C$  is abstractly isomorphic to  $G$ , since  $T/C$  is isomorphic to  $T$  for  $C$  finite.

If  $C$  contains a non-trivial element of  $H$ , then clearly  $\pi_1(G/C)$  has torsion. On the other hand, if  $\pi_1(G/C)$  has torsion, it's of exponent 2. Let  $\alpha$  be a class of order 2. Let  $G_\alpha$  be the covering group corresponding to  $\alpha$ .

*Proof of 1.4.* : (1) is clear. (3) and (4) follow directly from (2). For (2), the claim is that  $(T \times H)/C$  is homeomorphic to  $T \times (H/(C \cap (1 \times H)))$ . For a general compact connected Lie group  $G$ , let  $G'$  be the commutator subgroup. Then  $G'$  is connected and  $G/G'$  is a compact connected abelian Lie group, and hence a torus  $T$ . The map  $G \rightarrow T$  has a section, so  $G$  is homeomorphic (but not isomorphic) to  $G' \times T$ . In our example, for  $G = (T \times H)/E$ ,  $G'$  is isomorphic to  $H/(E \cap H)$ , since it is the image of the commutator subgroup in  $T \times H$  under the quotient map.

*Proof of 1.5.* From the inclusion  $V \rightarrow T$  are induced inclusions  $C_G(T) \rightarrow C_G(V)$  and  $N_G(T) \rightarrow N_G(V)$ . For a possibly disconnected compact Lie group  $H$ , define  $W(H) = N_H(T_H)/T_H$ . Then if  $H$  is maximal rank in  $G$ , there is an inclusion of Weyl groups  $W(H) \rightarrow W(G)$ . For such  $H$ , there is a natural surjection  $W(H) \rightarrow \pi_0(H)$ . In this case, we apply these facts to  $H = N_G(V)$  or  $C_G(V)$ .

$$W(C_G(V)) = (N_G(T) \cap C_G(V))/T.$$

So  $W(C_G(V))$  is kernel of  $W(G) \rightarrow W_G(V)$ . Also,  $W(N_G(V)) \rightarrow W(G)$  is an inclusion, but since  $W(N_G(T)) \rightarrow \pi_0(N_G(T)) = W(G)$  is onto, we have  $W(N_G(V)) = W(G)$ . Thus  $W(G) \rightarrow \pi_0(N_G(V))$  is onto.  $W_G(V)$  is a subgroup of  $Aut(V)$ , a finite group. The map  $N_G(V) \rightarrow W_G(V)$  therefore factors through  $\pi_0(N_G(V))$ . Putting these together, we have that  $W(G) \rightarrow W_G(V)$  is a surjection. Thus the short exact sequence

$$1 \rightarrow W(C_G(V)) \rightarrow W(G) \rightarrow W_G(V) \rightarrow 1$$

is established. The cohomological assertions follow from [22] together with [4].

*Proof of 1.6.* The condition is just that  $W_G(V)$  is the trivial group. From the identifications,  $W_G(V)$  is the quotient of  $W(G)$  by the elements that reduce to the identity mod 2. Hence  $W_G(V) = \{Id\}$  implies that  $W(G)$  is an elementary abelian 2–group. On the other hand, if  $W(G)$  is an elementary abelian 2–group, then  $G$  has been identified as a quotient of  $\tilde{G} = T \times H$ , where  $T$  is a torus and  $H$  is a product of  $SU(2)$ 's. On this covering group  $\tilde{G}$ ,  $W(G)$  acts trivially on  $V$ .

§2. FINITE LOOP SPACES WITH  $(\mathbf{Z}/2\mathbf{Z})^s$  AS WEYL GROUP

Recent work of the authors in [13] shows that for  $p$ -compact groups, there are a maximal torus and a Weyl group with most of the properties that the classical Lie group constructs possess. The definitions are modeled on those of [23], adapted to the  $p$ -compact groups. More precisely, let  $(BX, X)$  be a  $p$ -compact group such that  $X$  is connected. Then there exists a *maximal torus*, i.e. a map  $f : BT \rightarrow BX$  with many of the usual properties that  $BT \rightarrow BG$  has in the Lie group case. Here  $BT$  can mean an Eilenberg-MacLane space of type either  $K((\mathbf{Z}_p)^r, 2)$  or  $K((\mathbf{Z}/p^\infty\mathbf{Z})^r, 1)$ . The latter choice we denote as the *discrete* torus. The discrete form has the advantage that the elements of order a power of  $p$  can be visualized in the usual way as elements in a torus. In this environment,  $W(X)$  is just the component group of  $\text{Aut}(f : BT \rightarrow BX)$ , [23], [13].

**Proposition 2.1.** *Let  $(BX, X)$  be a 2-compact group such that  $BX$  is simply connected and  $\pi_*(BX) \otimes \mathbf{Q}$  has rank  $n$ . Then any one of the following conditions imply that  $W(X)$  is an elementary abelian 2-group contained in  $GL(n, \mathbf{Z}_2)$ :*

- (1) *The “rational” cohomology  $H^*(BX, \mathbf{Z}_2) \otimes \mathbf{Q}$  has a set of algebra generators over  $\mathbf{Q}_2$  concentrated in dimensions 2 and 4.*
- (2) *The mod 2 cohomology algebra  $H^*BX$  is a polynomial algebra and its indecomposables are concentrated in dimensions 2, 3, and 4.*
- (3) *The mod 2 cohomology algebra  $H^*BX$  has a coproduct consistent with the action of the mod 2 Steenrod algebra.*
- (4) *For each  $g : RP^\infty \rightarrow BX$ , the natural evaluation map*

$$e : \text{Map}(RP^\infty, BX)_g \rightarrow BX$$

*is a homotopy equivalence. That is, each element of order 2 in  $X$  is central in  $X$ . Consequently the subgroup of elements of order 2 in the maximal torus  $T$  is central in  $X$ .*

- (5) *If  $f : BT \rightarrow BX$  is the maximal torus, and  $i : BV \rightarrow BT$  the inclusion of the elements of order 2 in  $T$ , then for each  $w \in W(X)$ ,*

$$w \circ i = i.$$

*That is, the Weyl group action restricts to the trivial action on  $V$ .*

We need one key algebraic lemma. Although related to the congruence subgroup problem, the case needed has an elementary proof. The proof is supplied in appendix A.

**Lemma 2.2.** *Let  $W$  be a finite subgroup of  $GL(\mathbf{Z}_2, n)$ . Then the kernel of the mod 2 reduction map to  $GL(F_2, n)$  is a finite elementary abelian 2-subgroup of  $W$ .*

**Lemma 2.3.** *Let  $W$  in  $GL(N, \mathbf{C})$  be a reflection group with exactly  $n$  reflections, all of order 2. If the order of  $W$  is  $2^n$ , then  $W$  is an elementary 2-group. Note that it is not assumed that  $n = N$ .*

**Lemma 2.4.** *Let  $W$  in  $GL(N, \mathbb{C})$  be a reflection group with exactly  $n$  reflecting hyperplanes. If  $W$  is abelian, then  $W$  is isomorphic to a product of  $n$  cyclic reflection groups.*

Recall that  $f : Y \rightarrow Z$  is said to be central if the natural evaluation map  $eval : Map(Y, Z)_f \rightarrow Z$  is a homotopy equivalence. If this holds for  $Bf : BY \rightarrow BX$  we abbreviate by saying that  $Y$  is central in  $X$  or that  $f$  is central. We note some properties of this idea of centrality that are immediate in the Lie group case.

**Lemma 2.5.** *Let  $(BX, X)$  be a 2–compact group such that  $X$  is connected. Then*

- (1) *If  $f : BA \times BB \rightarrow BX$  has the property that  $f|_{* \times BB}$  and  $f|_{BA \times *}$  are each central, then  $f$  is central.*
- (2) *The subgroup  $V$  of order 2 in a maximal torus  $T$  for  $X$  are central in  $X$  if and only if for any  $Bg : B(Z/2Z) \rightarrow BX$ , we have  $Map(BZ/2Z, BX)_{Bg} \approx BX$ .*
- (3) *Suppose  $f : Y \rightarrow X$  is central in  $X$  and  $a$  and  $b$  in  $V$  are represented by  $Bi : BZ/2Z \rightarrow BV$  and  $Bj : BZ/2Z \rightarrow BV$  respectively. Then if  $Bf \circ Bi = Bf \circ Bj$  up to homotopy, then,  $a = b$  as elements of  $V$ .*

*Proof of 2.1.:*

Case (1): By [13], there is a Weyl group  $W(X)$  such that  $H^*(BX, \mathbf{Z}_2) \otimes \mathbf{Q} \approx (H^*(BT, \mathbf{Z}_2) \otimes \mathbf{Q})^{W(X)}$ .  $W(X)$  is a generalized reflection group generated by reflections of order 2. Classical formulas from [24] give the number of reflections in  $W(X)$  as

$$\sum (n_i - 2)/2$$

where  $\{n_i\}$  are the dimensions of the generators of the invariant polynomial algebra. The only roots of unity in  $\mathbf{Z}_2$  are plus and minus one, so each reflection has order 2. Hence in this case, the dimension 4 generators each contribute 1 to the sum, and the dimension 2 generators none. Hence the number of reflections is  $N$ , the number of dimension 4 generators. On the other hand,  $|W(X)| = \prod (n_i/2) = 2^N$ . By Lemma 2.3,  $W(X)$  is an elementary abelian 2–group.

Case(2): By Proposition 5.1 such a  $BX$  has its rational cohomology generated by dimensions 2 and 4, so case (1) applies.

Case (3) If  $H^*BX$  is a Hopf algebra, each component of the Lannes’  $T$ –functor  $T^V(H^*BX)$  is isomorphic to the component of the trivial map, [11], which is  $H^*BX$ , since  $H^*BX$  is noetherian. By [18] and [13] this implies that for any

$$f : BV \rightarrow BX$$

we have

$$H^*Map(BV, BX)_f \approx T_f^V(H^*BX) \approx H^*BX.$$

That is, condition (3) implies condition (4).

Case (4) : We show that condition (4) implies condition (5) Let  $Bf : BT \rightarrow BX$  be the maximal torus, and let  $Bi : BV \rightarrow BT$  represent the inclusion of the elements of order 2. Then  $V$  is central in  $X$  if and only if each  $Z/2Z$  is central in  $X$ . In one direction, in [13], it’s shown that for connected  $X$ ,  $h : BZ/2Z \rightarrow BX$  factors though  $Bf : BT \rightarrow BX$ . Case (4) by Lemma 2.4 implies that  $V$  is central in  $X$ . So it remains to show that  $V$  central in  $X$  implies that the Weyl group is

an elementary abelian 2–group. By the algebraic lemma (2.2), it’s enough to show that the Weyl group acts as the identity on  $V$  or  $BV$  up to homotopy. Suppose that there exists an element of  $W(X)$  which fails to act trivially on  $BV$  up to homotopy, i.e. as the identity on  $V$ . Then there exists an element  $a \in V$  and an element  $w \in W(X)$  [6 such  $w \circ a = b \neq a$ . But by the definition of  $W(X)$ , the inclusion into  $BX$  induces homotopic maps representing  $a$  and  $b$ . By 2.5.2,  $a = b$ , contradicting the claim that  $w \circ a \neq a$ .

Case (5) : Consider the discrete version of  $BT$ ,  $B(Z/2^\infty Z)^n$ .  $Aut(BT)$  acts on this. To get the action  $W(X)$ , use

$$0 \rightarrow \pi_2(BT) \rightarrow (\mathbf{Q} \otimes \pi_2(BT)) \rightarrow (Z/2^\infty Z)^n \rightarrow 0.$$

Then  $V$  can be identified with the image of  $((1/2)\pi_2(BT))$  under the projection. But this is also the mod 2 reduction of the action on  $\pi_2(BT)$ . By 2.2,  $W(X)$  is an elementary abelian 2–group.

*Proof of 2.3.* : Proof: Since  $W$  is finite, there exists a  $W$ –invariant hermitian inner product for  $V = \mathbb{C}^N$ . For each reflection  $w_i \in W$  choose a unit normal vector  $v_i$ . We need to know that  $w_i w_j = w_j w_i$  for all possible  $i$  and  $j$ . If  $n = 1$ , this is immediate. If  $n > 1$ , for each  $1 \leq k \leq n$ , define  $V_k$  to be the span of all the unit normal vectors except  $v_k$ . Now the action of  $W$  on  $V$  permutes the reflecting hyperplanes, and hence permutes the normal vectors, up to unit complex numbers. In particular, the action of  $w_k$  on  $V$  must take  $V_k$  into itself, because  $w_k(v_j) = a_{k,j} v_i$  where  $i \neq k$  unless  $k = j$ . By definition,  $w_k$  has only a single eigenvector for  $\lambda = -1$ , so its eigenvalues on  $V_k$  must be all 1’s. Hence  $w_k(v_j) = v_j$  if  $j \neq k$ . Hence  $w_k w_j = w_j w_k$  for all  $j$ . But  $k$  was arbitrary, so any two reflections commute. Thus  $W$  is an elementary abelian 2–group.

*Proof of 2.5.* : These appear in [14].

§3. REDUCTION TO THE CASE  $H^*BX \approx H^*BSU(2)^r$ 

Both 0.2 and 0.5B can be reduced to the  $H^*BSU(2)^r$  case by taking the 3–connected cover. In the polynomial algebra cases, we can do the calculations directly:

**Proposition 3.1.** *Let  $H^*BX$  be a finitely generated polynomial algebra with a coproduct over  $\mathcal{A}_2$ . Then the 3–connected cover  $BX \langle 3 \rangle$  has the same property.*

**Proposition 3.2.** *Let  $(BX, X)$  be a 2–compact group such that  $X$  is connected. Suppose that  $H^*BX$  is a polynomial algebra on generators of dimensions 2, 3, and 4. Then  $H^*BX \langle 3 \rangle \approx H^*BSU(2)^r$  for some  $r$  as algebras over  $\mathcal{A}_2$ .*

For the  $BX$  of 3.1, from section 2 it follows that the Rector-Weyl group for  $BX \langle 3 \rangle$  must be an elementary abelian 2–group and that the “rational” cohomology has generators only in dimension 4. Hence  $H^*BX \langle 3 \rangle$  is polynomial on dimension 4 generators.

**Proposition 3.3.** *For  $BX$  as in Theorem 0.2,  $H^*BX \langle 3 \rangle \approx H^*BSU(2)^r$  as algebras over the Steenrod algebra.*

If one wants to use only rational information and not explicit mod 2 cohomology information, the spirit of 3.2 and 3.3 is still valid. However, the proof requires results of Browder on finite  $H$ –spaces, [6]. We do not have a specific classifying space argument.

**Proposition 3.4.** *Let  $(BX, X)$  be a 2–compact group such that  $X$  is connected. Then  $(BX \langle 3 \rangle, \Omega BX \langle 3 \rangle)$  is a 2–compact group also.*

Using further results of Browder, we have

**Proposition 3.5.** *Let  $(BX, X)$  be a 2–compact group such that  $X$  is connected and  $\pi_*(BX) \otimes \mathbf{Q}$  is concentrated in dimensions 2 and 4. Then*

$$H^*BX \langle 3 \rangle \approx H^*BSU(2)^r$$

Thus all the classifying spaces that we’re seeking to characterize have 3–connected covers which are fake products of completed  $BSU(2)$ ’s.

§4. CONSTRUCTION OF THE ADJOINT FORM FOR  $BX \langle 3 \rangle$ 

In this section we construct the analogue of the adjoint form of  $SU(2)^r$ . Using the mod 2 cohomology we recognize the analogue of the center and show that the quotient is well defined.

**Proposition 4.1.** *Let  $(BX, X)$  be a 2–compact group such that  $BX$  is 3–connected and  $H^*BX \approx H^*BSU(2)^r$  as algebras over the Steenrod algebra. Then*

- (1) *There exists  $f : BV = (BZ/2Z)^r \rightarrow BX$  so that  $f^*$  is a monomorphism and  $eval : Map(BV, BX)_f \rightarrow BX$  is a homotopy equivalence and*
- (2) *The action map  $BV \times Map(BV, BX)_f \rightarrow Map(BV, BX)_f$  defines a space*

$$BY = (F_2)_\infty (EBV \times_{BV} Map(BV, BX)_f)$$

such that

$$H^*BY \approx H^*BSO(3)^r$$

as algebras in all dimensions and as modules over the Steenrod algebra though dimension three.

In the next section we'll show that property (2) above forces that

$$H^*BY \approx H^*BSO(3)^r$$

as algebras over the Steenrod algebra.

*Proof of 4.1.* : By Lannes [18], the existence of  $f$  is determined by the existence of a suitable  $f^*$ . By the isomorphism to  $H^*BSU(2)^r$ , such an algebraic map exists. By further work of Lannes,  $H^*Map(BV, BX)_f = T_f^V(H^*BX)$ , using the notation of [10]. Using either the fact that the cohomology is a Hopf algebra over the Steenrod algebra, [11] or by comparison to the classical case, this shows that the evaluation map induces a homotopy equivalence  $eval : Map(BV, BX)_f \rightarrow BX$ , since both spaces are 2-complete and the map in cohomology is an isomorphism. Now choose a model for  $BV$  such that it is an abelian topological group with the identity element as its basepoint. The action of  $BV$  on  $BV$  induces an action on  $Map(BV, BX)$  which restricts to an action on  $Map(BV, BX)_f$  since  $BV$  is connected. So this gives us a well defined way to define a  $BV$  action on space homotopy equivalent to  $BX$ , a *proxy* action in the language of [13]. Hence the Borel construction is well defined. The fibration sequence (up to homotopy)

$$\Omega BY \rightarrow BV \rightarrow BX$$

as far as algebras over the Steenrod algebras are concerned looks like the product of  $r$  copies of the fibration

$$SO(3) \rightarrow BZ/2Z \rightarrow BSU(2)$$

Therefore the Eilenberg-Moore Spectral Sequence for it collapses at  $E^2$ , and  $H^*\Omega BY \approx H^*SO(3)^r$  as algebras over the Steenrod algebra. Applying the Borel transgression, we see that  $H^*BY$  is a polynomial algebra on generators of dimensions 2 and 3, and that  $Sq^1$  is an isomorphism between dimensions 2 and 3. We can not however easily deduce the action of  $Sq^2$  on the dimension 3 generators. If only one or two dimension 3 generators are present, the expected action of  $Sq^2$  is almost immediate. However, the higher rank cases lead to an interesting linear algebra problem in characteristic 2, which is solved in section 5.

## §5. A DIAGONALIZATION CRITERIA

In this section we prove a useful criteria for showing that a matrix with entries in a field of characteristic  $p$  is diagonalizable over the field with  $q$  elements. The criteria is suggested by the information implied by the Steenrod algebra action in the case  $p = 2$  in analysing fake products of  $BSO(3)$ 's.

Let  $V_q$  be a vector space of dimension  $n$  over some finite field  $\mathbf{F}_q$  of characteristic  $p$ . Suppose that  $K$  is an algebraically closed field which contains  $\mathbf{F}_q$ . We form the vector space  $V$  over  $K$  by tensoring up with  $K$  over  $\mathbf{F}_q$ . This gives a preferred choice of  $\mathbf{F}_q$  structure for  $V$ . It specifies a Frobenius operator on the elements of  $V$ ,  $\phi : V \rightarrow V$  which is  $\mathbf{F}_q$  linear, by choosing a  $\mathbf{F}_q$  basis  $\{v_i\}$  for  $V_q$  and defining  $\phi(v) = \sum \alpha_i^q v_i$  if  $v = \sum \alpha_i v_i$  for  $\alpha_i \in K$ . This does not depend on the particular choice of  $\mathbf{F}_q$  basis for  $V_q$ . A subspace  $W$  of  $V$  is said to be defined over  $\mathbf{F}_q$  if  $\phi(W) \subset W$ . Notice that as a map of abelian groups,  $\phi$  is one to one and onto.

**Proposition 5.1.** *For  $M \in \text{End}_K(V)$ , with matrix  $\{m_{ij}\}$  with respect to the basis  $\{v_i\}$ , there exists  $B \in GL_n(\mathbf{F}_q)$  such that  $BMB^{-1}$  is diagonal with entries in  $K$  if and only if*

$$\phi(M) = (M)^q$$

where  $\phi(M)$  is the matrix with entries  $\{m_{ij}^q\}$

Notice that 5.1 is equivalent to showing that  $M$  is semi-simple, with eigenspaces defined over  $\mathbf{F}_q$ .

*Proof of 5.1.* : We must show that  $V$  decomposes as an  $M$ -module into eigenspaces which are defined over  $\mathbf{F}_q$ . Suppose that  $\lambda \in K$  is an eigenvalue for  $M$ , and denote the generalized eigenspace of  $K$ -vectors annihilated by some power of  $(M - \lambda)$  as  $V_\lambda$ . Now if  $(M - \lambda)^N v = 0$  then

$$\phi((M - \lambda)^N v) = (M^q - \lambda^q)^N \phi(v) = (M - \lambda)^{qN} \phi(v) = 0$$

so  $\phi(v) \in V_\lambda$ . That is,

$$\phi(V_\lambda) \subset V_\lambda.$$

Thus, without loss of generality, we can assume then that  $\lambda$  is the only eigenvalue for  $M$ . Suppose that  $V_\lambda$  is not spanned by eigenvectors. Then there exists  $N$  such that  $(M - \lambda)^{N+1} = 0$ , but  $(M - \lambda)^N v \neq 0$  for some  $v \in V$ . Since  $\phi$  is a monomorphism on  $V$ , for such a  $v$

$$0 \neq \phi((M - \lambda)^N v) = (M - \lambda)^{Nq} \phi(v).$$

This is impossible unless  $N = 0$ . That is, every vector in  $V_\lambda$  is an eigenvector.

In particular, there is a basis of  $V_\lambda$  of eigenvectors defined over  $\mathbf{F}_q$ .

**Corollary 5.2.** *Let  $R$  be the graded polynomial ring over  $\mathbf{F}_q$  in variables  $\{x_i\}$ , where  $\dim(x_i) = m$  for all  $i$ . Let  $M$  be an  $n \times n$  matrix with entries which are homogeneous linear forms in the  $\{x_i\}$ . If*

$$\phi(M) = M^q$$

then there exists an automorphism  $B$  of  $R$  such that  $BMB^{-1}$  is diagonal with entries in the  $\mathbf{F}_q$ -vector space spanned by the  $\{x_i\}$ .

*Proof of 5.2.* : Choose an algebraic closure  $K$  of  $\mathbf{F}_q$ , and let  $\{\alpha : x_i \rightarrow \alpha_i \in K\}$  be a point. Denote by  $M(\alpha)$  the matrix obtained by evaluating the entries of  $M$  at the point  $\alpha$ . Then for each such  $\alpha$ , by 5.1, there exists  $B(\alpha) \in GL_n(\mathbf{F}_q)$  such that

$$B_\alpha M(\alpha) B_\alpha^{-1}$$

is diagonal. Choose an  $\alpha$  such that its components are linearly independent over  $\mathbf{F}_q$ . Then consider

$$M' = B_\alpha M B_\alpha^{-1}.$$

Any off-diagonal term of  $M'$  is zero when evaluated at  $\alpha$ , by the choice of  $B_\alpha$ . Since  $B_\alpha$  has  $\mathbf{F}_q$  entries, and the terms of  $M$  are linear forms, the terms of  $M'(\alpha)$  are  $\mathbf{F}_q$ -linear combinations of the components of  $\alpha$ . By the choice of  $\alpha$ , since  $M'_{ij}(\alpha) = 0, i \neq j$ , each must be the trivial linear combination. That is  $M'_{ij} = 0 \in \mathbf{R}$  for  $i \neq j$ . That is, by choosing a sufficiently generic  $\alpha$ , we see that

$$B_\alpha M B_\alpha^{-1}$$

is diagonal. The diagonal entries are clearly in  $R$  and since  $R$  is a free commutative algebra, the action of  $B_\alpha$  on the vector space  $R^m$  extends to an automorphism of the algebra  $R$ .

§6. CLASSIFICATION OF UNSTABLE POLYNOMIAL  
ALGEBRAS WITH GENERATORS IN DIMENSIONS 2,3, AND 4

**Proposition 6.1.** *Let  $R^*$  be a finitely generated unstable algebra which is polynomial on generators of dimensions 2,3, or 4. Then*

- (1)  $Sq^1|R^3 = 0$ .
- (2)  $Sq^1 : R^2 \rightarrow R^3$  is a surjection.
- (3) There is a choice of algebra generators  $\{z_i\}$  generators of dimension 4 such that  $Sq^1 z_i = 0$  for all  $i$ .
- (4) The  $Sq^1$ -homology of  $R^*$  is a finitely generated polynomial algebra on generators in dimensions 2 and 4.

Thus far this has been formal work in unstable algebras over  $\mathcal{A}_2$ . For the topological results, we need information on the Bockstein spectral sequence. But if  $H^*BX = R^*$ , we see that the Bockstein spectral sequence collapses after taking the  $Sq^1$ -homology, since that answer is concentrated in even dimensions. In fact, it's polynomial with representatives for algebra generators  $\{x_i\}$  for  $i > N$ ,  $\{x_i^2\}$  for  $i \leq N$ , and the choice of generators  $\{z_k\}$  as constructed above with  $Sq^1 z_k = 0$ .

Thus in the topological case with  $R^* = H^*BX$ , we have

$$(H^*(BX, \mathbf{Z}_2)/torsion) \otimes \mathbf{F}_2 = \otimes_{i>N} \mathbf{F}_2[x_i] \otimes_{j \leq N} \mathbf{F}_2[x_j^2] \otimes_{k=1, \dots, l} \mathbf{F}_2[z_k]$$

and the rational cohomology is generated by classes of dimensions 2 and 4.

**Corollary 6.2.** *Let  $R^*$  be an unstable algebra over  $\mathcal{A}_2$  which is polynomial as an algebra and generated by elements in degrees 2 and 3. Suppose further that  $\dim_{\mathbf{F}_2} R^2 = N$  and  $Sq^1 : R^2 \rightarrow R^3$  is an isomorphism. Then  $R^* \approx \otimes_N H^*BSO(3)$  as algebras over  $\mathcal{A}_2$ . That is, there is a choice of generators  $\{x_i\}$  in dimension 2 so that setting  $y_i = Sq^1 x_i$  for each  $i$  yields  $Sq^2 y_i = x_i y_i$  for each  $i$ .*

In fact 6.2 is a special case of 6.3, for which more substantial calculations are required:

**Corollary 6.3.** *Let  $R^*$  be an unstable algebra over  $\mathcal{A}_2$  which is polynomial as an algebra and generated by elements in degrees 2, 3, and 4. Suppose further that  $\dim_{\mathbf{F}_2} R^2 = N$  and  $Sq^1 : R^2 \rightarrow R^3$  is an isomorphism. Then  $R^* \approx \otimes H^*BH(k)$ , for various  $k$ , where  $H(k)$  is the Lie group  $SU(2)^k/diag(Z/2)$ .  $H^*BH(k) = \mathbf{F}_2[x_2, y_3, z_1, \dots, z_{k-1}]$ , where  $\dim z_i = 4$ . The  $\mathcal{A}_2$  action is determined by*

- (1)  $Sq^1 x_2 = y_3$
- (2)  $Sq^1 y_3 = 0$
- (3)  $Sq^2 y_3 = x_2 y_3$
- (4)  $Sq^1 z_i = 0$
- (5)  $Sq^2 z_i = x_2 z_i$

In this labelling scheme,  $H_0 = SU(2)$ ,  $H_1 = SO(3)$ , and  $H_2 = SO(4)$ .

*Proof of 6.1.* : In the topological case with  $R^* = H^*BX$ , one can use the methods of [6] to analyse the Bockstein spectral sequence for  $X$  to obtain most of this information. However, we give a direct algebraic argument based on a careful analysis of the Steenrod algebra action in low dimensions. Choose an algebra basis

$\{x_i\}$  in dimension 2,  $\{y_j\}$  in dimension 3, and  $\{z_k\}$  in dimension 4. Let  $\vec{Y}$  denote the column vector with entries  $Y_j = y_j$ , for  $1 \leq j \leq N$ . For dimensional reasons

$$Sq^2 \vec{Y} = M(x) \vec{Y}$$

where  $M(x)$  is an  $N \times N$  matrix with entries from  $R^2$ . Notice that  $M(x)$  must have rank  $N$  as a map from  $R^3 \rightarrow R^5$ , since

$$Sq^1 Sq^2 \vec{Y} = \phi(\vec{Y})$$

where  $\phi(\vec{Y})$  is the vector with entries the square of the original entries of  $\vec{Y}$ . First we prove that  $Sq^1 \vec{Y} = 0$ . Consider  $Sq^1 Sq^2 \vec{Y} = Sq^3 \vec{Y} = \phi(\vec{Y}) = Sq^1(M(x)) \vec{Y} + M(x)(Sq^1 \vec{Y})$ . Each entry of  $Sq^1(M(x))$  is a linear form in the  $y$ -variables. Hence the remaining term in the equation,  $M(x)Sq^1(\vec{Y})$ , is on the one hand a function of only the  $y$ -variables, and on the other hand, a linear combination of terms of the form  $x_i y_j$ . Hence it is zero. Since  $M(x)$  has rank  $N$ , in fact  $Sq^1 \vec{Y} = 0$ .

Next we show that  $M(x)$  satisfies the equation

$$M(x)M(x) = M(\phi(x)).$$

Consider

$$Sq^2 Sq^2 \vec{Y} = Sq^3 Sq^1 \vec{Y} = 0$$

from above. Expanding, we obtain

$$Sq^2 Sq^2 \vec{Y} = Sq^2(M(x)\vec{Y}) = M(\phi(x))\vec{Y} + Sq^1(M)Sq^1 \vec{Y} + M(x)M(x)\vec{Y} = 0.$$

By 5.1 and 5.2 of the previous section, there is another choice of the  $y$ -generators so that with respect to this basis,  $M(x)$  is diagonal. That is, there exists a basis  $\{y_j\}$  for  $R^3$  and elements  $\{x_i\}$  of  $R^2$  so that

$$Sq^2 y_i = x_i y_i$$

for  $1 \leq i \leq N$ . Notice that since  $Sq^1 Sq^2 y_i = Sq^3 y_i = y_i^2 = (Sq^1 x_i) y_i$ , we have  $Sq^1 x_i = y_i$ . Extend the chosen  $\{x_i\}$  to a basis of  $R^2$  in such a way that  $Sq^1 x_i = 0$  if  $i > N$ . The final step is to show that there is a choice of the dimension 4 generators which are killed by  $Sq^1$ . Suppose  $z \in R^4$  is not a polynomial in the  $x$ -variables. Then if

$$Sq^1 z = \sum a_{ij} x_i y_j,$$

define

$$w = \sum_{i>N} a_{ij} x_i y_j.$$

Then

$$Sq^1 \left( \sum_{i>N} a_{ij} x_i x_j \right) = w.$$

Hence

$$Sq^1(z - w) = \sum_{i \leq N} a_{ij} x_i y_j.$$

But apply  $Sq^1$  again:

$$0 = \sum_{i \leq N} a_{ij} Sq^1(x_i) y_j = \sum_{i \leq N} a_{ij} y_i y_j$$

Hence  $a_{ij} = 0$  for all  $1 \leq i, j \leq N$ . So  $z - w$  represents the same indecomposable as  $z$  and has  $Sq^1$  zero.

*Proof 6.3.* This requires showing that the dimension four generators  $\{z_i\}$  can be chosen to have  $Sq^2 z_i = x z_i$  for some  $x \in R^2$ . Again we use the matrix notation. Let  $Z$  denote the column vector with  $i$ -th entry the generator  $z_i$ . For dimensiona reasons,

$$Sq^2 Z = A(x)Z + F(y) + G(x),$$

where  $A$  is a square matrix with entries linear forms in the  $x$ -variables, where  $F$  and  $G$  are column vectors with entries respectively quadratic and cubic homogeneous forms in the  $y$ - and  $x$ -variables. From 6.1, we can assume that  $Sq^1 Z = 0$ . Now apply the Adem relation  $Sq^2 Sq^2 Z = Sq^3 Sq^1 Z = 0$  to obtain

$$\begin{aligned} 0 &= Sq^2(A(x)Z) + Sq^2 F + Sq^2 G \\ &= A(x^2)Z + A(x)(A(x)Z + F + G) + Sq^2 F + Sq^2 G = \\ &= A(x^2)Z + A(x)^2 Z + A(x)F + A(x)G + Sq^2 F + Sq^2 G \end{aligned}$$

Hence as in 6.1,  $A(x) = A(x^2)$ , and we can change the choice of  $z$ -variables so that  $A(x)$  is diagonal. That is, let  $z$  be a typical generator. We can assume that

$$Sq^2 z = xz + f(y) + g(x).$$

Let  $Sq^{(0,1)}$  denote the Milnor primitive  $Sq^2 Sq^1 + Sq^1 Sq^2$ . Then

$$Sq^{(0,1)} Sq^2 z = Sq^2 Sq^3 z + Sq^1 Sq^2 Sq^2 z = Sq^5 z + Sq^4 Sq^1 z + Sq^1 Sq^3 Sq^1 z = 0,$$

using the Adem relations. Note that

- (1)  $Sq^{(0,1)} x_i = x_i y_i$
- (2)  $Sq^{(0,1)} y_i = y_i^2$

Hence

$$\begin{aligned} 0 &= (Sq^{(0,1)} x)z + x Sq^{0,1} z + Sq^{(0,1)} f + Sq^{(0,1)} g \\ &= (x Sq^1 x)z + x(Sq^1 Sq^2 z) + Sq^{(0,1)}(f + g) \\ &= (x Sq^1 x)z + x((Sq^1 x)z + Sq^1(f + g)) + Sq^{(0,1)}(f + g) \end{aligned}$$

That is,

$$(x Sq^1 x)z = x((Sq^1 x)z).$$

That is,  $x$  is either one of the canonical generators  $x_i$  or  $x = 0$ . We also deduce that

$$x(Sq^1(f + g)) + Sq^{(0,1)}(f + g) = 0$$

Now  $Sq^1 f = 0$ , so

$$xSq^1 g = Sq^{(0,1)}(f) + Sq^{(0,1)}(g).$$

But  $Sq^{(0,1)}(f)$  is purely a polynomial in the  $y$ -variables,  $Sq^{(0,1)}(f) = 0$ . If

$$f = \sum_{i \leq j} a_{ij} y_i y_j,$$

then

$$Sq^{(0,1)}(f) = \sum_{i \leq j} a_{ij} y_i y_j (y_i + y_j) = 0$$

Thus  $a_{ij} = 0$  if  $i < j$ . That is  $f$  is a square, in fact it's expressible uniquely in the form  $(Sq^1 u)^2 = f$ . If  $x = 0$ , then

$$z' = z + u^2$$

will be a suitable replacement for  $z$ .

If  $x \neq 0$ , we need an expression for  $g$  similar in spirit to that for  $f$ . Without loss of generality, we can assume that  $x = x_1$ . Write

$$g = \sum_{i \leq j \leq k} b_{ijk} x_i x_j x_k.$$

$$\begin{aligned} xSq^1 g &= x_1 \sum_{i \leq j \leq k} b_{ijk} (y_i x_j x_k + x_i y_j x_k + x_i x_j y_k) \\ &= Sq^{(0,1)}(g) = \sum_{i \leq j \leq k} b_{ijk} x_i x_j x_k (y_i + y_j + y_k) \end{aligned}$$

Comparing coefficients on the two sides, we see that

$$b_{123} = 0$$

unless  $i = 1, j = k$ . That is,  $g = xv^2$  for a two class  $v$ . Finally we must show that  $v = u$ , where  $f = (Sq^1 u)^2$ . But from

$$Sq^2 Sq^2 z = 0 = Sq^2(f + g) + x(f + g),$$

we have

$$\begin{aligned} Sq^2(xv^2 + (Sq^1 u)^2) &= x(xv^2 + (Sq^1 u)^2) \\ &= x^2 v^2 + x(Sq^1 v)^2 \end{aligned}$$

so since  $x \neq 0$ , we have

$$(Sq^1 u)^2 = (Sq^1 v)^2.$$

Taking square roots is unique in characteristic 2, and we had assumed that  $Sq^1$  was an isomorphism from dimension 2 to dimension 3, so we conclude that  $u = v$ . Thus

$$f + g = xu^2 + f = xu^2 + (Sq^1u)^2.$$

We can adjust our choice of  $z$  by

$$z' = z + u^2$$

and verify that  $Sq^2z' = xz'$ , and  $Sq^1z' = 0$ .

The mod 2 Weyl group for  $H(k)$ ,  $k > 0$ , can be identified with the subgroup of  $GL_{k+1}(\mathbf{F}_2)$  which pointwise fixes the span of the first  $k - 1$  basis vectors. We establish the identification of the unstable algebras of 6.3 with the tensor product of the cohomologies of  $BH(k)$ 's by choosing the canonical basis pairs in dimensions two and three,  $x_i, y_j$  with the property that  $y_i = Sq^1x_i$  and  $Sq^2y_i = x_iy_i$ . This uniquely defines these up to permutation. Next we pick the  $z_k$  as above. Each such  $z_k$  belongs to a particular  $x_i$ , in the sense that  $Sq^2z_k = z_kx_i$ . Several different  $z_k$ 's can use the same  $x_i$ . The decomposition is then by eigenspaces. A given  $x_i$  has an eigenspace of four dimensional classes  $\{z\}$  for which  $Sq^2z = x_iz$ . This contributes an  $H(k)$  factor, where  $k$  is the dimension over  $\mathbf{F}_2$  of that eigenspace. The  $SU(2)$  factors correspond to the zero eigenvalue.

§7. UNIQUENESS FOR THE ADJOINT AND SIMPLY CONNECTED FORMS

**Proposition 7.1.** *Let  $(BX, X)$  be a 2-compact group such that  $X$  is connected.*

- (1) *If  $H^*BX \approx H^*BSO(3)^r$  as algebras over  $\mathcal{A}_2$ , then  $BX$  is homotopy equivalent to  $(\mathbf{F}_2)_\infty(BSO(3)^r)$ .*
- (2) *If  $H^*BX \approx H^*BSU(2)^r$  as algebras over  $\mathcal{A}_2$ , then  $BX$  is homotopy equivalent to  $(\mathbf{F}_2)_\infty(BSU(2)^r)$ .*

The main ingredient is the [9] version of the [16] diagram description of  $BG$ . We quote from [9] and [13]:

**Definition 7.2.** *Let  $BX$  be a simply connected  $p$ -complete space such that  $H^*(\Omega BX, \mathbf{F}_p)$  is finite. Let  $A_{BX}$  be the category with objects  $\{(V, f)\}$  where  $f : BV \rightarrow BX$  such that  $V$  is a finite dimensional vector space over  $\mathbf{F}_p$  and  $H^*(BV, \mathbf{F}_p)$  is a finitely generated  $H^*(BX, \mathbf{F}_p)$  module via  $f^*$ . The morphisms are  $\psi : (V, f) \rightarrow (U, g)$  induced from an injection  $S_\psi : V \rightarrow U$  such that  $g \circ B(S_\psi) = f$  up to homotopy. We can also define a purely algebraic version. Let  $R^*$  be an unstable algebra over the Steenrod algebra which is noetherian.  $A_R$  has objects  $(V, f^*)$  corresponding to maps over the Steenrod algebra  $f^* : R^* \rightarrow H^*(BV, \mathbf{F}_p)$  which are finite. The morphisms are  $\psi : (V, f^*) \rightarrow (U, g^*)$  induced from an injection  $S_\psi : V \rightarrow U$  such that  $B(S_\psi)^* \circ g^* = f^*$ .*

If  $BX = BG$ , where  $G$  is a compact Lie group, then the category  $A_{BX}$  is equivalent to  $A_{H^*(BG, \mathbf{F}_p)}$ , by work of [22] and [18]. If  $H^*(\Omega BX, \mathbf{F}_p)$  is finite then Dwyer-Wilkerson in [9] and [13] generalize [16] to obtain a diagram description of  $BX$ :

**Theorem 7.3.** *Let  $BX$  be a simply connected  $p$ -complete space with  $H^*(\Omega BX, \mathbf{F}_p)$  finite. Define on the category  $A_{BX}$  a functor  $F$ , with  $F((V, f)) = \text{Map}(BV, BX)_f$ . To each morphism  $\psi : (V, f) \rightarrow (U, g)$ ,  $F$  assigns the induced map  $B(S_\psi)^* : \text{Map}(BU, BX)_g \rightarrow \text{Map}(BV, BX)_f$ . Then the  $p$ -completion of the homotopy direct limit of this functor over  $A_{BX}^{op}$  is homotopy equivalent to  $BX$ . Similarly, the functor  $\Psi : (V, f^*) \rightarrow T_f^V(H^*(BX, \mathbf{F}_p))$  has its inverse limit isomorphic over the Steenrod algebra to  $H^*(BX, \mathbf{F}_p)$  and the higher derived functors of  $\Psi$  are zero.*

The category  $A_Y$  can be defined more generally for a space  $Y$  with noetherian mod  $p$  cohomology. However, Theorem 7.3 even in the algebraic case is not true in general. One example is the wedge of two copies of  $RP^\infty$ . In this case the inverse limit is the cohomology of the disjoint union of a pair of  $RP^\infty$ 's. One hypothesis that suffices for the algebraic version is that  $R^*$  be a ring of invariants.

In our construction, we need a modified smaller version of the category  $A_{BX}$ . Let  $J$  be an ideal in  $H^*(BX, \mathbf{F}_p)$  which is closed under the Steenrod algebra action. Define the full subcategory  $A_{BX}(J)$  to include only those objects  $(V, f)$  such that  $J \in \ker(f^*)$ .

**Proposition 7.4.** *Suppose for  $BX$  in 7.2,*

$$H^*BX \approx R_1^* \otimes R_2^*$$

*as algebras over the Steenrod algebra, where  $R_1^*$  is a ring of invariants and  $R_2^*$  is noetherian. Take  $J$  to be the ideal generated by positive dimensional elements in*

$R_2 \otimes \mathbf{F}_p$ . Then

$$H^*(BX, \mathbf{F}_p) \approx \text{InvLim}_{A_{BX}(J)} T_f^V(H^*(BX, \mathbf{F}_p))$$

and the higher derived functors are zero. Topologically, we have that

$$BX \approx \text{hocolim}_{A_{BX}(J)} \text{Map}(BV, BX)_f$$

after  $p$ -completions.

Our application is to  $BY$ , where  $H^*BY \approx H^*BSO(3)^r$ . Choose  $R_1$  to be one tensor factor isomorphic over the Steenrod algebra to  $H^*BSO(3)$ , and  $R_2$  to be the remaining factors.

**Proposition 7.5.** *Suppose that  $H^*BY \approx H^*BSO(3)^r$  over the Steenrod algebra. Then there exists  $f : BY \rightarrow (\mathbf{F}_2)_\infty(BSO(3))$  such that  $f^*$  is an injection on mod 2 cohomology.*

*Proof of 7.1.* : For  $r = 1$ , this is the result of [7]. Suppose that 7.1 is known for  $r - 1$  factors. By 7.5, there is a fibration

$$\text{fib}(f) \rightarrow BY \rightarrow (\mathbf{F}_2)_\infty(BSO(3))$$

The Eilenberg-Moore spectral sequence collapses and shows that the mod 2 cohomology of the fiber is an example of the rank  $r - 1$  case. By induction  $\text{fib}(f)$  is h.e. to  $(\mathbf{F}_2)_\infty(BSO(3)^{r-1})$ . The classification of fibrations quoted in the introduction then shows that the fibration is f.h.e. to the trivial (product) fibration.

*Proof of 7.5.* : Assume that 7.1 is known for  $r - 1$  factors. Consider the diagram given by  $F : A_{BX}(J) \rightarrow TOP$ . There are up to isomorphism just two cohomology algebras occurring as  $H^*\text{Map}(BV, BX)_{Bf}$ :

- (1) for  $\dim V = 1$ ,  $H^*BSO(3)^{r-1} \otimes H^*BO(2)$  and
- (2) for  $\dim V = 2$ ,  $H^*BSO(3)^{r-1} \otimes H^*(RP^\infty)^2$ .

Using 7.5 and 7.1 for  $r - 1$  factors, we will show that only two homotopy types occur:

Denote by  $BX_1$  a space with cohomology of type (1) and by  $BX_2$  that of type (2). We first prove that

$$(1) \quad BX_1 \approx (\mathbf{F}_2)_\infty BSO(3)^{r-1} \times BO(2)$$

and that

$$(2) \quad BX_2 \approx (\mathbf{F}_2)_\infty BSO(3)^{r-1} \times (RP^\infty \times RP^\infty).$$

The 1-connected cover  $BX_1 \langle 1 \rangle$  has  $h : BX_1 \langle 1 \rangle \rightarrow K(\mathbf{Z}_2, 2)$  with  $H^*\text{fib}(h) \approx H^*BSO(3)^{r-1}$ . By the induction hypothesis,

$$\text{fib}(h) \approx (\mathbf{F}_2)_\infty BSO(3)^{r-1}.$$

By the fibration theorem, Theorem 0.8, the fibration

$$fib(h) \rightarrow BX_1 \langle 1 \rangle \rightarrow K(\mathbf{Z}_2, 2)$$

is f.h.e to the product fibration. Thus the fibration

$$BX_1 \langle 1 \rangle \rightarrow BX_1 \rightarrow RP^\infty$$

is classified by a map into  $BAut(BX_1 \langle 1 \rangle)$ . The fundamental group of this space is non-trivial, but the classifying map lifts to  $BAut_1(BX_1 \langle 1 \rangle)$ , which is  $K(\mathbf{Z}_2, 2)$ . There are two such maps, but the nontrivial one is the right choice. Thus  $BX_1$  is the desired product of completions of  $BSO(3)^{r-1}$  and  $BO(2)$ . A similar but easier analysis establishes  $BX_2$ .

In particular, the composite functor  $F^{[3]}$  into  $TOP$  that assigns to  $(V, f)$  the 3-rd Postnikov approximation to  $Map(BV, BX)_{Bf}$  takes, up to homotopy, only the values

$$(1) \quad K(Z/2Z, 2)^{r-1} \times (\mathbf{F}_2)_\infty BO(2)$$

and

$$(2) \quad K(Z/2Z, 2)^{r-1} \times BO(1)^2$$

Because of the choice of  $J$ , the mod 2 cohomology of the diagram is naturally isomorphic to that given by the functor  $\Psi : A_{H^*(BX)}(J) \rightarrow \mathcal{K}$ , given by Lannes'  $T$ -functor. But this latter diagram is naturally equivalent to  $T_f^V(H^*BX) \otimes T_0(R_2)$ , where  $T_0$  is the component of the trivial map. Hence by 7.3, the inverse limit is just  $R_1 \otimes R_2 = H^*BX$ , and this is the cohomology of the homotopy direct limit.

Now apply the 3-rd Postnikov approximation to the topological diagram. For the spaces of type (1), we obtain  $K(Z/2Z, 2)^{r-1} \times (\mathbf{F}_2)_\infty BO(2)$  and for the type (2) spaces,  $K(Z/2Z, 2)^{r-1} \times BO(1)^2$ . Hence the homotopy colimit has the cohomology of  $K(Z/2Z, 2)^{r-1} \times BSO(3)$ . Call this space  $M$  and choose a map  $h : M \rightarrow K(Z/2Z, 2)^{r-1}$  which induces an isomorphism onto the corresponding factor of the cohomology of  $M$ . Then there is a fibration

$$fib(h) \rightarrow M \rightarrow K(Z/2Z, 2)^{r-1}.$$

The Eilenberg-Moore spectral sequence collapses, and we have that  $H^*fib(h) \approx H^*BSO(3)$  as algebras over the Steenrod algebra. By the case for  $r = 1$ ,  $fib(h)$  is homotopy equivalent to  $(\mathbf{F}_2)_\infty BSO(3)$ . Thus Theorem 0.8 of the introduction applies and the fibration is f.h.e. to the product fibration. In particular, there is a map  $BX \rightarrow M \rightarrow (\mathbf{F}_2)_\infty BSO(3)$  which is non-trivial on mod 2 cohomology.

There is a variant of the above in which  $BX$  is replaced by a  $BY$  where  $BY$  is obtained by killing all but one of the non-zero elements in  $H^2$ . For the right choices, this gives a  $BY$  such that  $H^*BY$  is isomorphic to that of  $BSU(2)^{r-1} \times BSO(3)$ . We should point out that from the view of mod 2 cohomology, all maps  $BZ/2Z \rightarrow BSU(2)^r$  are central, and each can be permuted into any another by an automorphism of  $H^*BSU(2)^r$  over the Steenrod algebra. Topologically however, there can

be non-equivalent quotients at the classifying space level. For example, two possible quotients of  $SU(2) \times SU(2)$  by a central  $Z/2Z$  give  $SU(2) \times SO(3)$  and  $SO(4)$  respectively. These two have non-equivalent classifying spaces. This means that if one starts with a space with the mod 2 cohomology of a products of  $BSU(2)$ 's, mod 2 cohomology doesn't allow us to immediately choose a central  $Z/2Z$  so that the quotient  $BY/B(Z/2Z)$  has the cohomology of  $B SO(3) \times BSU(2)^{r-1}$ .

*Proof of 7.4.* : The main point is that if  $R_2$  is noetherian, then the component of the Lannes'  $T$ -functor corresponding to the trival map is naturally isomorphic to  $R_2$  itself, [10], and thus the functor  $T_f^V(R_1 \otimes R_2)$  is equivalent to  $T_f^V(R_1) \otimes R_2$ .

§8. HOMOTOPICAL UNIQUENESS

In [8], a large number of  $p$ -compact groups are shown to satisfy a rather strong uniqueness property: the  $\mathcal{A}_p$ -isomorphism type of  $H^*(BX, \mathbf{F}_p)$  determines the homotopy type of  $BX$ . [21] extends this type of result to many Lie groups. While it is plausible to ask for this type of uniqueness for the examples of this note also, given the possible complexity of the cohomology algebras, we choose to answer it only in the fundamental group is finite and the cohomology algebra is polynomial.

**Theorem 7.1.** *Let  $X$  and  $Y$  be semi-simple 2-compact groups with abelian Weyl groups and polynomial cohomology at the classifying space level. Then  $H^*BX \approx_{\mathcal{A}_2} H^*BY$  if and only if  $BX$  is homotopy equivalent to  $BY$ .*

A more reasonable general approach to uniqueness in terms of discrete algebraic data would be to suggest that the normalizer of the maximal torus contains sufficient information to determine  $BX$ . This is known to be true for  $BG$ , but under the proviso that one is testing against other  $BH$ .

**Theorem 7.2.** *Let  $X$  and  $Y$  be 2-compact groups with abelian Weyl groups. Let  $T_X$  and  $T_Y$  be discrete approximations to the maximal tori of  $X$  and  $Y$  respectively. Then  $N(T_X)$  is isomorphic to  $N(T_Y)$  if and only if  $BX$  is homotopy equivalent to  $BY$ .*

Finally, there is a background complication which was ignored in the earlier section. One is often computing quotients in different categories. The following comparison is proved in [14].

**Theorem 7.3.** *Let  $G$  be a compact Lie group such that the group of components is a  $p$ -group. Let  $A$  be a subgroup of the center of  $G$ , and  $i : A \rightarrow G$  the inclusion homomorphism. Suppose that  $(BX, X)$  is a  $p$ -compact group and  $g : BX \rightarrow F_{p^\infty}(BG)$  a homotopy equivalence. Then if  $f : BA \rightarrow BX$  is any map such that  $g \circ f \approx F_{p^\infty}(Bi)$ , then*

$$F_{p^\infty}B(G/A) \approx F_{p^\infty}(EBA \times_{BA} \text{Map}(BA, BX)_f).$$

§9. MOD  $p$  REDUCTIONS OF INVARIANTS

Let  $R$  be a commutative ring and  $J$  an ideal of  $R$ . If  $W$  is a finite subgroup of automorphisms of  $R$  such that  $WJ = J$ , there is an induced action of  $W$  on the quotient ring  $R/J$ . The long exact cohomology sequence for the coefficient sequence

$$0 \rightarrow J \rightarrow R \rightarrow R/J \rightarrow 0$$

shows that  $R^W \rightarrow (R/J)^W$  may fail to be a surjection. One topological case of interest has  $R = H^*(BT, \mathbf{Z}_p)$ ,  $J = pR$ , and  $W$  a finite subgroup of  $GL(n, \mathbf{Z}_p)$ . Somewhat surprisingly, the action of  $W$  is faithfully passed to the reduction, with a minor reservation for  $p = 2$ . The following has an elementary proof which the reader might like to reproduce:

**Lemma 9.1.** *Let  $A \in GL(n, \mathbf{Z}_p)$  be an element of finite order. If  $p > 2$ , then  $A$  is not in the kernel of the mod  $p$  reduction map  $GL(n, \mathbf{Z}_p) \rightarrow GL(n, \mathbf{F}_p)$ . If  $p = 2$ , and  $W$  is a finite subgroup of  $GL(n, \mathbf{Z}_2)$ , the intersection of  $W$  with the kernel of the reduction map is an elementary abelian 2–group.*

A corollary of the  $p$ -odd result is the observation that if  $(BX, X)$  is a connected  $p$ -compact group such that  $H^*(X, \mathbf{Z}_p)$  is torsion free, then in fact  $H^*(BT, \mathbf{F}_p)^{W(X)} \approx H^*(BX, \mathbf{F}_p)$ , [27], [8]. This fails in specific cases for  $p = 2$ . However, there is a common generalization

**Theorem 9.2.** *Let  $(BX, X)$  be a connected  $p$ -compact group such that  $H^*(X, \mathbf{Z}_p)$  is torsion free. Then*

$$H^*(BX, \mathbf{Z}_p) \approx H^*(BT_X, \mathbf{Z}_p)^{W(X)}$$

If  $p > 2$  this is true with  $\mathbf{F}_p$  coefficients. If  $p = 2$ ,

$$H^*(BX, \mathbf{F}_2) \approx H^*(BC_X({}_2T_X), \mathbf{F}_2)^{W(X)}$$

but the reduced action of  $W(X)$  is not effective if  $T_X \neq C_X({}_2T_X)$ . Here  ${}_2T_X$  denotes the elements of order 2 in a discrete maximal torus for  $X$ .

The proof of 9.2 uses two different ways of computing fraction field extension degrees. First, if  $W \rightarrow \text{Aut}(L)$  is a monomorphism of a finite group into the group of field automorphisms of a given field  $L$ , the Galois theory gives that the degree of the field extension  $L^W \rightarrow L$  is just the order of  $W$ ,  $|W|$ .

On the other hand, if  $R \rightarrow S$  is a finite extension of connected graded domains of finite type over a field  $K$ , then [1] provides a calculation of the degree of the corresponding fraction field extension:

**Lemma 9.3.** *For  $S$  and  $R$  as described above, the degree of the fraction field extension is*

$$\lim_{t \nearrow 1} (P(t, S)/P(t, R))$$

The easiest example of these two principles occurs for a connected compact Lie group  $G$ . Then a result of Borel identifies  $H^*(BG, \mathbf{Q})$  and  $H^*(BT, \mathbf{Q})^{W(G)}$ . So on the one hand the Galois theory gives the fraction field extension degree as  $|W(G)|$ . On the other hand, if  $(2n_1, \dots, 2n_r)$  are the degrees of the generators of the rational cohomology, then the Poincare series argument shows that  $|W(G)| = \prod (n_i)$ . There are of course many other paths to this conclusion. The advantage of the Poincare series argument is that it works in the possible absence of Galois extensions. We can generalize the Borel result somewhat:

**Lemma 9.4.** *Suppose  $(BX, X)$  is a connected  $p$ -compact group. Let  $f : BY \rightarrow BX$  be the inclusion of a maximal rank connected sub- $p$ -compact group. Then  $H^*(f, \mathbf{Z}_p) \otimes \mathbf{Q}$  has fraction field degree  $|W(X)|/|W(Y)|$ . If the  $\mathbf{Z}_p$ -adic cohomology of each is torsion free, the same result holds for the mod  $p$  fraction field degree.*

Proof of 9.2: Let  $BY = \text{Map}(B({}_2T), BX)_i = BC_X({}_2T)$ . Then both  $H^*(BT, F)$  and  $H^*(BY, F)$  are f.g. modules over  $H^*(BX, F)$ , for  $F = \mathbf{F}_p$  or  $F = \mathbf{Z}_p$ . Denote  $S = H^*(BT, \mathbf{Z}_p)$ ,  $R = H^*(BX, \mathbf{Z}_p)$ , and  $D = H^*(BC_X({}_2T), \mathbf{Z}_p)$ . If  $M$  is a graded

type free  $\mathbf{Z}_p$ -module of finite type, denote by  $P(t, M \otimes \mathbf{Q})$  the Poincare series for  $M \otimes \mathbf{Q}$ , and similarly for  $P(t, M \otimes \mathbf{F}_p)$ . By the torsion free hypothesis on  $X$ ,

$$P(t, R \otimes \mathbf{F}_p) = P(t, R \otimes \mathbf{Q}).$$

A similar statement is true for  $P(t, S \otimes \mathbf{F}_p)$ . Now the analogue of a theorem of Borel for connected  $p$ -compact groups, [13] asserts that

$$R \otimes \mathbf{Q} \approx (S \otimes \mathbf{Q})^{W(X)}$$

. Hence the degree of the fraction field extension corresponding to  $R \rightarrow S$  is by Galois theory exactly  $|W(X)|$ . Now  $R \subset S^{W(X)}$  by the definition of  $W(X)$ . But the fraction field degree of  $S^{W(X)} \rightarrow S$  is also  $|W(X)|$  so the degree of  $R \rightarrow S^{W(X)}$  is one. But  $S$  is integral over  $R$  and hence so is  $S^{W(X)}$ . Since  $R$  is integrally closed in its fraction field, we have  $R = S^{W(X)}$ .

We now perform a similar analysis for the mod  $p$  case. Again  $R \otimes \mathbf{F}_p$  is a finitely generated polynomial algebra, from the torsion free hypothesis on  $X$ .

Let  $E$  denote the kernel of the map  $W(G) \rightarrow \text{Aut}_p(T)$ . For odd  $p$ ,  $E$  is trivial and for  $p = 2$ ,  $E$  is an elementary abelian 2-group, by lemma 9.1. Consider first the odd prime case. Then we have the sequence of monomorphisms

$$R \otimes \mathbf{F}_p \rightarrow S^{W(X)} \otimes \mathbf{F}_p \rightarrow (S \otimes \mathbf{F}_p)^{W(X)} \rightarrow S \otimes \mathbf{F}_p$$

We wish to show that the leftmost two are isomorphisms. We do this by calculating degrees of fraction field extensions. From the characteristic zero calculation,  $R \otimes \mathbf{F}_p \rightarrow S \otimes \mathbf{F}_p$  has degree  $|W(X)|$ . By Galois theory,  $(S \otimes \mathbf{F}_p)^{W(X)} \rightarrow S \otimes \mathbf{F}_p$  has degree  $|W(G)/|E| = |W(G)|$ . Hence the degree of  $R \otimes \mathbf{F}_p \rightarrow (S \otimes \mathbf{F}_p)^{W(X)}$  is  $|W(G)|/|W(G)| = 1$ . But  $S \otimes \mathbf{F}_p$  is integral over  $R \otimes \mathbf{F}_p$ , so  $(S \otimes \mathbf{F}_p)^{W(X)}$  is also. Since  $R \otimes \mathbf{F}_p$  is integrally closed in its field of fractions, it must include  $(S \otimes \mathbf{F}_p)^{W(X)}$ . That is,  $R \otimes \mathbf{F}_p = H^*(BT, \mathbf{F}_p)^{W(X)}$ .

For the  $p = 2$  case, if  $|E| = 1$ , the above proof would suffice. If  $E \neq 1$ , then however,  $R \otimes \mathbf{F}_2 \neq H^*(BT, \mathbf{F}_2)^{W(X)}$ . For example, if  $X = SU(2)$ , this occurs.

In this case, we have already shown that  $C_X({}_2T)$  is a product of tori and  $SU(2)$ 's. Let  $r$  be the number of  $SU(2)$  factors. Then  $W(C_X({}_2T))$  is an elementary abelian 2-group of rank  $r$ . Hence by 9.3, the degree of the extension of fraction fields corresponding to  $H^*(BX, \mathbf{F}_2) \rightarrow H^*(BC_X({}_2T), \mathbf{F}_2)$  is  $|W(X)|/|W(C_X({}_2T))|$ . By [14], the Weyl group of  $C_X({}_2T)$  is also the subgroup of  $W(X)$  which pointwise fixes  ${}_2T$ . But this is just the  $E$  defined above.

Now  $W(X)/E$  acts on  ${}_2T$  and hence on  $H^*(BC_X({}_2T), \mathbf{F}_2)$ . We need to show that in fact

$$H^*(BX, \mathbf{F}_2) \approx H^*(BC_X({}_2T), \mathbf{F}_2)^{W(X)/E}$$

Consider

$$R \otimes \mathbf{F}_2 \rightarrow (D \otimes \mathbf{F}_2)^{W(X)/E} \rightarrow D \otimes \mathbf{F}_2$$

The composite degree is  $|W(X)|/|E|$ . The degree of the middle to rightmost is the order of  $W(X)/E$ , since this acts effectively on  $D \otimes \mathbf{F}_2$ . Hence by the integrality and integral closure arguments of above,

$$H^*(BX, \mathbf{F}_2) \approx H^*(BC_X({}_2T), \mathbf{F}_2)^{W(X)/E}.$$

## REFERENCES

- [[1]] J. F. Adams and C. W. Wilkerson, *Finite H-spaces and algebras over the Steenrod algebra*, Ann. of Math. **111** (1980), 95–143.
- [[2]] A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. **57** (1953), 115–207.
- [[3]] A.K. Bousfield and D.M. Kan, *Homotopy Limits, Completions and Localizations*, Lecture Notes in Mathematics vol. 304, Springer-Verlag, Berlin, 1972.
- [[4]] N. Bourbaki, *Commutative Algebra*.
- [[5]] C. Broto and H.-W. Henn, *Some remarks on central elementary abelian  $p$ -subgroups and cohomology of classifying spaces*, Quart. J. Math. Oxford **(2)44** (1993), 155–163.
- [[6]] W. Browder, *On Differential Hopf Algebras*, Transactions of the A.M.S. **107** (1963), 153–178.
- [[7]] W. G. Dwyer, H. R. Miller and C. W. Wilkerson, *The homotopic uniqueness of  $BS^3$* , Algebraic Topology, Barcelona 1986, Lecture Notes in Math. 1298, Springer, Berlin, 1987, pp. 90–105.
- [[8]] ———, *Homotopical uniqueness of classifying spaces*, Topology **31** (1992), 29–45.
- [[9]] W. G. Dwyer and C. W. Wilkerson, *A Cohomological Decomposition Theorem*, Topology **31** (1992), 433–443.
- [[10]] ———, *Smith theory and the functor  $T$* , Commentarii Math. Helv. (1991).
- [[11]] ———, *Spaces of null homotopic maps, ????*.
- [[12]] ———, *A New Finite H-space at the Prime 2*, Journal of the A.M.S **6** (1993), 33–63.
- [[13]] ———, *Homotopy fixed point methods for Lie groups and finite loop spaces*, Annals of Math. (to appear).
- [[14]] ———, *The center of a  $p$ -compact group* (1993), 1–44.
- [[15]] D. L. Gonçalves, *Mod 2 homotopy associative H-spaces*, Springer Lecture Notes in Mathematics **657** (1978), 198–216.
- [[16]] S. Jackowski and J. E. McClure, *A cohomology decomposition theorem*, Topology (1992), ???.
- [[17]] S. Jackowski, J. E. McClure and R. Oliver, *Homotopy classification of self-maps of  $BG$  via  $G$ -actions*, Annals of Math. **135** (1992), 183–226.
- [[18]] J. Lannes, *Sur la cohomologie modulo  $p$  des  $p$ -groupes Abéliens élémentaires*, Homotopy Theory, Proc. Durham Symp. 1985, edited by E. Rees and J.D.S. Jones, Cambridge Univ. Press, Cambridge, 1987.
- [[19]] ———, *Sur les espaces fonctionnels dont la source est le classifiant d'un  $p$ -groupe abélien élémentaire*, GET FINAL REFERENCE.
- [[20]] J. P. Lin, *Cup products and finite loop spaces*, Topology and its Applications **45** (1992), 73–84.
- [[21]] D. Notbohm, *Uniqueness of classifying spaces*, Goettingen preprint.
- [[22]] D. G. Quillen, *The spectrum of an equivariant cohomology ring: I*, Annals of Math. **94** (1971), 549–572.
- [[23]] D. L. Rector, *Subgroups of finite dimensional topological groups*, J. Pure and Applied Alg. **1** (1971), 253–273.
- [[24]] Shepherd and Todd, *Finite reflection groups*, Canadian J. Math. (1954).
- [[25]] D. Sullivan, *Geometric Topology, Part I: Localization, Periodicity and Galois Symmetry*, Lecture Notes, MIT, 1971.
- [[26]] T. tom Dieck, *Lie groups*, GET IT RIGHT.
- [[27]] C. W. Wilkerson, *Rings of invariants and inseparable forms of algebras over the Steenrod Algebra*, Proc. Camb. Phil. Soc. (to appear).

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