Homotopy fixed point methods for Lie groups and finite loop spaces

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§1. Introduction

A *loop space* X is by definition a triple (X, BX, e) in which X is a space, BX is a connected pointed space, and $e\colon X \to \Omega BX$ is a homotopy equivalence from X to the space ΩBX of based loops in BX. We will say that a loop space X is *finite* if the integral homology $H_*(X, \mathbf{Z})$ is finitely generated as a graded abelian group, i.e., if X appears at least homologically to be a finite complex. In this paper we prove the following theorem.

1.1 Theorem. *If X is a finite loop space, then for any prime number p the cohomology algebra $H^*(BX, \mathbf{F}_p)$ is finitely generated (as an algebra).*

Any compact Lie group G is a finite loop space: G is a loop space because it is homotopy equivalent to ΩBG (where BG is the usual classifying space for principal G-bundles) and G is finite because it is a compact smooth manifold. It was proved in [**22**] that for a compact Lie group G the algebra $H^*(BG, \mathbf{F}_p)$ is finitely generated. Theorem 1.1 answers an old question of J. Moore by extending this result to finite loop spaces. In fact, we obtain a new homotopy theoretic proof of the result in [**22**].

The method behind the proof of Theorem 1.1 is really more interesting than the theorem itself. We immediately (§2) reduce 1.1 to an analogous theorem about other loop spaces called *p-compact groups*, and then we study p-compact groups in great detail. By definition a p-compact group is purely homotopy theoretic in nature: it is essentially just a finite loop space with all of its structure concentrated densely at the single prime p. Our experience shows, however, that a p-compact group possesses much of the rich internal structure of a compact Lie group. It seems to us that p-compact groups are remarkable objects, and one of the main goals of the paper is to introduce them and study their properties.

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We begin the paper by showing that the elementary geometric apparatus of Lie group theory can be re-interpreted so as to apply to a p-compact group X: this includes the notion of a subgroup of X, centralizer of a subgroup of X, quotient of X by a central subgroup, etc. The homotopy theoretic definitions of some of these concepts are given in §3, and we spend the next few sections showing that the definitions are plausible ones. Many unusual questions come up (what is an element of order p in X? are there any? is the centralizer in X of an element of order p a subgroup of X? is an element of order p contained in its centralizer?) and new techniques have to be devised to answer them. However, when the dust has settled and the machinery is in place, we are able to use a straightforward inductive scheme (cf. 1.2) to produce a maximal torus T in a p-compact group X, show that the normalizer $\mathcal{N}(T)$ of T is an extension of T by a finite pseudoreflection group W (the Weyl group of X), and prove that the homogeneous space $X/\mathcal{N}_p(T)$ has Euler characteristic prime to p. (Here $\mathcal{N}_p(T)$ is the inverse image in $\mathcal{N}(T)$ of a p-Sylow subgroup of W.) It is not hard to see directly that $H^*(B\mathcal{N}_p(T), \mathbf{F}_p)$ is a finitely generated algebra, and finite generation for $H^*(BX, \mathbf{F}_p)$ follows from a standard lemma that uses the Becker-Gottlieb transfer of the fibration $X/\mathcal{N}_p(T) \to B\mathcal{N}_p(T) \to BX$.

The philosophy behind this paper is due to Rector [**20**]; in drawing a serious analogy between the homotopy theory of loop spaces and the geometry of Lie groups we are following his lead. From a practical point of view we are heavily dependent on the results of Lannes [**15**]; in combination with our previous work [**9**], these results allow us to study homotopy fixed point sets of finite p-group actions with the same Smith theory techniques which are used in the classical case to study fixed point sets (§2). Some of the homotopy fixed point sets which arise are described below (1.3).

1.2 A sketch of the main argument. Our study of maximal tori in p-compact groups is directly parallel to an unorthodox geometric method for studying maximal tori in a compact Lie group. We will describe this Lie theoretic approach and indicate where in the paper the corresponding homotopy theoretic points come up. Let G be a compact Lie group. A *toral subgroup* T of G is a subgroup isomorphic to a product $\mathrm{SO}(2)^n$ of circle groups (6.3); T is called a *maximal torus* (8.9) if it is equal to the identity component of its centralizer (3.4) in G. If G is a discrete group, then the identity subgroup is a maximal torus for G. To construct such a maximal torus if G is not discrete, we first find a nontrivial element of order p in the identity component of G (5.4), and then show that the resulting homomorphism $f_1\colon \mathbf{Z}/p \to G$ extends to a homomorphism

$f_\infty\colon \mathbf{Z}/p^\infty \to G$ (5.5). The closure A of the image of f_∞ (6.6, 6.7) is a nontrivial connected abelian subgroup of G (connected because $\mathbf{Z}/p^\infty$ has no finite quotients) and so is a toral subgroup of G. Let H be the centralizer in G of A and let T/A be a maximal torus in the quotient group H/A; such a T/A can be assumed to exist by induction, since H/A is a group of smaller dimension (6.13) than G. Let T be the inverse image of T/A in H. It is not hard to see that T is a maximal torus for H (proof of 8.13) and consequently (8.14) a maximal torus for G (observe in this last connection that T contains A (8.2, 8.11) and so the centralizer of T in G is equal to the centralizer of T inside the centralizer of A in G (8.4), i.e., equal to the centralizer of T in H). Now let $\mathcal{N}(T) \subset G$ be the normalizer of T in G (9.8); since T is a maximal torus, $\mathcal{N}(T)/T$ is a finite (discrete) group (9.5). By inspection the fixed point set of the left translation action of T on the homogeneous space G/T is exactly the set $\mathcal{N}(T)/T$, and so by elementary transformation group theory the Euler characteristic of G/T is equal to the cardinality of NT/T (9.5). We are using here the fact that under reasonable hypotheses the Euler characteristic (4.3) of the fixed point set of a torus action on a compact space Y is equal to the Euler characteristic of Y (4.7). It follows from the multiplicativity of the Euler characteristic in finite coverings (4.14) that the Euler characteristic of G/NT is 1. A similar argument shows that any other maximal torus T' of G is conjugate to T (9.4) (the action of T' on G/T must have a fixed point (8.11) because the Euler characteristic of G/T is nonzero) and that if G is connected then T is actually equal to its own centralizer in G (9.1). It is now possible to prove that $H^*(BG, \mathbf{F}_p)$ is a finitely generated algebra by using transfer (9.13) in the fibration $G/\mathcal{N}(T) \to B\mathcal{N}(T) \to BG$ (2.4) and appealing to the arguments of §12 for the fact that $H^*(B\mathcal{N}(T), \mathbf{F}_p)$ is finitely generated.

1.3 Homotopy fixed point sets. A fundamental part of our technique involves interpreting group theoretic constructions as fixed point sets, identifying the corresponding homotopy fixed point sets, and then using the homotopy objects as substitutes for the geometric ones. Here are some examples.

(1) The centralizer in G of the image of a homomorphism $f\colon K \to G$ is the fixed point set of the action of K on G via f by conjugation. The corresponding homotopy fixed point set is the space of sections over BK of the fibration with fibre G associated to this conjugation action of K on G. Up to homotopy this fibration is the pullback over Bf of the free loop space fibration over BG [**11**]; its space of sections is the loop space on the component of the space of maps $BK \to BG$ containing Bf (3.4, proof of 5.8).

(2) The space of elements in G of order p (i.e., the subspace of G given by $\{g|\,g^p=1\}$) is the fixed point set of a certain action of $\mathbf{Z}/p$ on G^{p-1} (5.12); for instance, if $p=2$ this is the fixed point set of the map $G\to G$ given by $g\mapsto g^{-1}$. The corresponding homotopy fixed point set is the space of basepoint preserving maps $B\mathbf{Z}/p\to BG$ (proof of 5.4). See 5.8 for a much more elaborate example like this.

(3) The Weyl group of G is the fixed point set of the left translation action of T on G/T. The corresponding homotopy fixed point set is the topological monoid of maps $BT\to BT$ over BG (9.5).

1.4 Organization of the paper. Section 2 gives the definition of a p-compact group and reduces Theorem 1.1 to a corresponding result about p-compact groups. Section 3 tabulates (most of) the dictionary we will use to translate from the geometry of Lie groups to the homotopy theory of p-compact groups. Section 4 contains crucial technical theorems about the cohomology of homotopy fixed point sets; these theorems may be of independent interest. Section 5 studies representations of a finite p-group in a p-compact group, and §6 extends this to a study of representations of p-compact toral groups (these are the analogues in the category of p-compact groups of p-toral groups). The main point in §6 is that p-compact toral groups can be approximated well enough by discrete groups or even finite p-groups that the results of §5 can be applied to them. Section 7 analyzes in a special case the notion of the *kernel* of a homomorphism between p-compact groups. Finally, §8 contains a proof that any p-compact group has a maximal torus, and §9 uses the existence of a maximal torus to prove the desired finite generation result. There are three appendices: one (§10) describing some basic properties of homotopy fixed point sets, a second (§11) assembling technical information about the $\mathbf{F}_p$-completion functor of Bousfield and Kan, and a final one (§12) which contains a direct proof that certain discrete groups related to p-toral groups have finitely generated cohomology rings.

1.5 Notation and terminology. For the rest of the paper p will denote a fixed prime number, $\mathbf{F}_p$ the field with p elements, $\mathbf{Z}_p$ the ring of p-adic integers and $\mathbf{Q}_p=\mathbf{Q}\otimes\mathbf{Z}_p$ the quotient field of $\mathbf{Z}_p$. All unspecified homology and cohomology is with coefficients in $\mathbf{F}_p$; we will use the notation $H^*_{\mathbf{Q}_p}(X)$ for $\mathbf{Q}\otimes H^*(X,\mathbf{Z}_p)$. A graded vector space H^* over a field $\mathbf{F}$ is of *finite type* if each H^i is finite dimensional over $\mathbf{F}$ and is *finite dimensional* if in addition $H^i=0$ for all but a finite number of i. A space X (resp. pair (X,Y)) is $\mathbf{F}_p$*-finite* if H^*X (resp. $H^*(X,Y)$) is finite dimensional over the field $\mathbf{F}_p$. A map is *an* $\mathbf{F}_p$*-equivalence* if it induces an isomorphism on $H^*(-,\mathbf{F}_p)$.

If G is a group acting on a space X we write $X_{\mathrm{h}G}$ for the corresponding Borel construction and $X^{\mathrm{h}G}$ for the homotopy fixed point set (see §10 for details). If X and Y are spaces then $\mathrm{Map}(X,Y)$ denotes the function space of maps from X to Y; the component containing a particular map or homotopy class f is $\mathrm{Map}(X,Y)_f$. If f is a specific map then $\mathrm{Map}(X,Y)_f$ is a pointed space with basepoint f. If X and Y are pointed spaces then $\mathrm{Map}_*(X,Y)$ denotes the subspace of $\mathrm{Map}(X,Y)$ consisting of basepoint preserving maps, and, for $f \in \mathrm{Map}_*(X,Y)$, $\mathrm{Map}_*(X,Y)_f$ denotes the component of $\mathrm{Map}_*(X,Y)$ containing f. We will assume that any space in question has the homotopy type of a CW–complex; in dubious cases, a space can be replaced for our purposes by the realization of its singular complex [**16**].

§2. p-COMPACT GROUPS

For reasons that can be traced mostly to the fact that the theorems of §4 apply only to $\mathbf{F}_p$-complete spaces, we focus our attention on objects that are slightly different from finite loop spaces. Recall that p is a fixed prime. In what follows the term "$\mathbf{F}_p$-complete" means $\mathbf{F}_p$-complete in the sense of [**5**] (see §11). We denote the $\mathbf{F}_p$-completion of a space X by $\mathbf{C}_{\mathbf{F}_p}(X)$.

2.1 LEMMA. *For a loop space X the following two conditions are equivalent:*

(1) *X is $\mathbf{F}_p$-finite, $\mathbf{F}_p$-complete and $\pi_0 X$ is a finite p-group.*
(2) *X is $\mathbf{F}_p$-finite and BX is $\mathbf{F}_p$-complete,*

REMARK: Another equivalent condition would be: X is $\mathbf{F}_p$-finite, $\pi_0 X$ is a finite p-group, and $\pi_i X$ $(i \geq 1)$ is a finitely generated module over $\mathbf{Z}_p$.

PROOF OF 2.1: The fact that (1) implies (2) is 11.9, the opposite implication follows from combining the same result with the fact (11.12, 11.14) that if the space BX is $\mathbf{F}_p$-complete and $\pi_1 BX$ is finite, then $\pi_1 BX$ is a finite p-group.

2.2 DEFINITION: *A p-compact group* is a loop space X which satisfies the conditions of 2.1.

We have chosen the term "p-compact group" because the results of this paper suggest that the structure of these objects is uncannily similar to the structure of compact Lie groups.; the term also suggests "$\mathbf{F}_p$-complete and $\mathbf{F}_p$-finite". There are some interesting things already known about p-compact groups; for instance, although up to equivalence (3.1) there are an uncountable number of loop spaces (X, BX, e) with underlying space X homotopy equivalent to the three-sphere S^3 [**21**], there is up to equivalence only one p-compact group with underlying space homotopy equivalent to the $\mathbf{F}_p$-completion of S^3 [**7**]. What we aim to prove in this paper is the following.

2.3 THEOREM. *If X is a p-compact group, the H^*BX is finitely generated as an algebra.*

REMARK: It is elementary to check that if Y is a connected space then H^*Y is finitely generated as an algebra if and only if H^*Y is *noetherian* as a graded ring, i.e., if and only if every graded ideal in H^*Y has a finite number of homogeneous generators, equivalently, if and only if every graded submodule of a graded finitely generated H^*Y module is itself finitely generated, or equivalently if and only if every ascending chain of graded ideals in H^*Y is eventually constant.

Theorem 2.3 implies Theorem 1.1. The proof of this depends on an algebraic lemma that we will use for a second time later on (§9).

2.4 LEMMA. *Let $f\colon R_1 \to R_2$ be a map of graded-commutative rings such that R_2 is noetherian. Suppose that there exists a map $g\colon R_2 \to R_1$ of R_1 modules such that $g \cdot f$ is the identity map of R_1. Then R_1 is noetherian.*

PROOF: If $I \subset R_1$ is an ideal let $f(I)$ denote the ideal in R_2 generated by the set $\{f(x) : x \in I\}$. If $J \subset R_2$ is an ideal let $g(J)$ denote the ideal in R_1 consisting of the set $\{g(x) : x \in J\}$. For an ideal $I \subset R_1$ a typical element of $f(I)$ is of the form $\sum_k r_k f(x_k)$ with $r_k \in R_2$ and $x_k \in I$; the image under g of such a sum is $\sum_k g(r_k)x_k$, which clearly belongs to I. It follows at once that $g(f(I)) = I$. If $\{I_n\}$ is an ascending chain of ideals in R_1 then the chain $\{f(I_n)\}$ stabilizes because R_2 is noetherian and hence $\{I_n\} = \{g(f(I_n))\}$ also stabilizes.

PROOF OF 1.1(GIVEN 2.3): Let X be a finite loop space, and let $X' \subset X$ be the inverse image in X of a p-Sylow subgroup of $\pi_0 X$. The space X' is a finite loop space with BX' given by the covering space of BX corresponding to a p-Sylow subgroup of $\pi_1 BX$. By 11.9 the completion $\mathbf{C}_{\mathbf{F}_p}(X')$ is a p-compact group with $B\mathbf{C}_{\mathbf{F}_p}(X') \cong \mathbf{C}_{\mathbf{F}_p}(BX')$ and $H^*B\mathbf{C}_{\mathbf{F}_p}(X') \cong H^*BX'$. By 2.3, then, H^*BX' is noetherian. The covering space projection $BX' \to BX$ gives a ring homomorphism $f\colon H^*BX \to H^*BX'$, and, since the index of $\pi_1 BX'$ in $\pi_1 BX$ is a unit in $\mathbf{F}_p$, the cohomological transfer map for this covering (9.13) leads to a map $g\colon H^*BX' \to H^*BX$ of modules over H^*BX which is a left inverse for f. The fact that H^*BX is noetherian now follows from 2.4.

§3. DICTIONARY

In this section we will begin working out a dictionary to guide us in translating constructions and arguments from the geometric setting of Lie groups to the homotopical setting of loop spaces or of p-compact groups.

3.1 Homomorphisms. A *homomorphism* $f\colon X \to Y$ of loop spaces is a pointed map $Bf\colon BX \to BY$. Two homomorphisms f, $g\colon X \to Y$ are *conjugate* if Bf and Bg are freely homotopic, i.e., homotopic as unpointed maps. A homomorphism f is an *equivalence* if Bf is a homotopy equivalence and *trivial* if Bf is null homotopic. A loop space X is *trivial* if X is contractible, i.e., if the unique homomorphism $X \to \{1\}$ is an equivalence.

If G and H are discrete groups, group homomorphisms $G \to H$ correspond bijectively to homotopy classes of loop space homomorphisms $G \to H$, with H-conjugacy classes of group homomorphisms $G \to H$ corresponding bijectively to conjugacy classes of loop space homomorphisms $G \to H$. In general any homomorphism $G \to H$ of compact Lie groups gives a loop space homomorphism $G \to H$, but the converse is not true unless, for instance, H is discrete, G is a finite p-group [**11**], or more generally G is a p-toral group [**19**].

3.2 Homogeneous spaces, monomorphisms. Suppose that $f\colon X \to Y$ is a homomorphism of loop spaces. The *homogeneous space* $Y/f(X)$ (denoted Y/X if f is understood) is defined to be the homotopy fibre of Bf over the basepoint of BY. The space Y/X is pointed by the basepoint of BX. If X and Y are p-compact groups, the homomorphism f is said to be a *monomorphism* if Y/X is $\mathbf{F}_p$-finite, and an *epimorphism* if $\Omega(Y/X)$ is a p-compact group (more precisely, if the triple $(\Omega(Y/X), Y/X, \mathrm{id})$ is a p-compact group). A *short exact sequence* $X \xrightarrow{f} Y \xrightarrow{g} Z$ of p-compact groups is a sequence such that $BX \to BY \to BZ$ is a fibration sequence; in such a sequence f is a monomorphism and g is an epimorphism.

If $f\colon G \to H$ is a map of compact Lie groups with kernel K, then the homotopy fibre of the induced map $BG \to BH$ is equivalent to the usual homogeneous space $H/f(G)$ if f is an algebraic monomorphism and to BK if f is an algebraic epimorphism.

Remark: By 5.4, a homomorphism $f\colon X \to Y$ of p-compact groups which is both a monomorphism and an epimorphism is an equivalence. For another characterization of monomorphism, see 9.11.

3.3 Actions on homogeneous spaces. If $i\colon X \to Y$ and $f\colon G \to Y$ are homomorphisms of loop spaces, then taking the homotopy pullback of $Bi\colon BX \to BY$ along Bf gives an "action" in the sense of 10.8 of

G on Y/X. The homotopy fixed point set of this action (10.8, 10.4) is denoted by $(Y/X)^{\mathrm{h}f(G)}$, or by $(Y/X)^{\mathrm{h}G}$ if f is understood.

In the above situation, the map f lifts up to conjugacy to a loop space homomorphism $G \to X$ if and only if $(Y/X)^{\mathrm{h}G}$ is nonempty; this amounts more or less to the definition of homotopy fixed pont set (§10). This is parallel to the fact that if $i\colon H \to K$, and $f\colon G \to K$ are homomorphisms of compact Lie groups with i a subgroup inclusion, then $f\colon G \to K$ lifts up to conjugacy to a group homomorphism $G \to H$ if and only if the fixed point set $(K/H)^G$ is nonempty.

3.4 Centralizers. If $f\colon X \to Y$ is a homomorphism of loop spaces, the *centralizer of* $f(X)$ *in* Y (denoted $\mathcal{C}_Y(f(X))$, or $\mathcal{C}_Y(X)$ if f is understood), is the loop space obtained as $\Omega\,\mathrm{Map}(BX, BY)_{Bf}$. Evaluation at the basepoint of BX gives a homomorphism $\mathcal{C}_Y(f(X)) \to Y$; the homomorphism f is said to be *central* if $\mathcal{C}_Y(f(X)) \to Y$ is an equivalence. A loop space A is *abelian* if the identity homomorphism of A is central.

One way to motivate the definition of the centralizer $\mathcal{C}_Y(f(X))$ is to interpret $\mathcal{C}_Y(f(X))$ as the homotopy fixed point set of a conjugation action of X on Y (1.2).

If $f\colon G \to H$ is an ordinary homomorphism between discrete groups, and Z is the algebraic centralizer of $f(G)$ in H, then the group homomorphism $G\times Z \to H$ passes to a map $BG\times BZ \to BH$ which upon adjunction gives a homotopy equivalence $BZ \to \mathrm{Map}(BG, BH)_{Bf}$. Something similar happens [**11**] if $f\colon G \to H$ is a homomorphism from a finite p-group G to a compact Lie group H, although in this case the map $BZ \to \mathrm{Map}(BG, BH)_{Bf}$ gives only a homology isomorphism $H_*Z \to H_*\Omega\,\mathrm{Map}(BG, BH)_{Bf}$.

§4. Cohomology of homotopy fixed point sets

In this section we will establish cohomological properties of homotopy fixed point sets associated to actions of p-groups. These properties are entirely parallel to classical results about actions of p-groups on finite complexes. In this homotopy theoretic situation the properties are contingent upon the spaces involved being $\mathbf{F}_p$-complete (§11), but it will become clear in §5 that the necessary $\mathbf{F}_p$-completeness conditions are always satisfied in examples of interest to us.

4.1 Euler characteristics and Lefschetz numbers. Suppose that H^* is a finite dimensional graded vector space over a field $\mathbf{F}$ and that f is a graded endomorphism of H^*. Let f^i denote the automorphism of H^i induced by f.

4.2 Definition: The *Lefschetz number* of f, denoted $\Lambda(H^*, f)$, is the element of $\mathbf{F}$ given by the alternating sum $\sum_i (-1)^i \operatorname{trace}_{\mathbf{F}} f^i$. The *Euler characteristic* of H^*, denoted $\chi(H^*)$, is the integer $\sum_i (-1)^i \dim_{\mathbf{F}} H^i$.

If $\mathbf{F}$ has characteristic 0 then $\chi(H^*) = \Lambda(H^*, \mathrm{id})$.

The following lemma is straightforward, and is the main technical reason for using the functor $H^*_{\mathbf{Q}_p}(-)$. Recall (1.5) that $H^*_{\mathbf{Q}_p}(X)$ is $\mathbf{Q} \otimes H^*(X, \mathbf{Z}_p) = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} H^*(X, \mathbf{Z}_p)$; this is *not* the same as $H^*(X, \mathbf{Q}_p)$ or the same as $\mathbf{Q}_p \otimes H^*(X, \mathbf{Q})$ (compare them, for instance, on $X = K(\mathbf{Q}, n)$).

4.3 Lemma. *If (X, Y) is an $\mathbf{F}_p$-finite pair of spaces, then $H^*_{\mathbf{Q}_p}(X, Y)$ is finite dimensional (over $\mathbf{Q}_p$) and $\chi(H^*(X, Y)) = \chi(H^*_{\mathbf{Q}_p}(X, Y))$.*

If (X, Y) is an $\mathbf{F}_p$-finite pair of spaces and f is a self-map of (X, Y), we will write $\chi(X, Y)$ for the integer which appears in 4.3 and $\Lambda(X, Y; f)$ for $\Lambda(H^*_{\mathbf{Q}_p}(X, Y), f^*)$. If X is an $\mathbf{F}_p$-finite space with an action of the group $G = \mathbf{Z}/p^r$, $r \geq 1$, we will write $\Lambda(X; G)$ for the Lefschetz number $\Lambda(X; g)$, where g is some chosen generator of G. It will turn out below (4.16) that $\Lambda(X; g)$ does not depend on the choice of g.

4.4 Four fixed point formulas. The main theorems we are aiming for are parallel to the following three results from the classical theory of transformation groups (in each of the statements "acting nicely" can be taken to mean "acting smoothly on a compact manifold"):

(1) If G is a cyclic group with generator g acting nicely on a finite complex X, then the Euler characteristic of X^G is equal to the Lefschetz number of the action of g on $H^*(X, \mathbf{Q})$.
(2) If G is a finite p-group acting nicely on a finite complex X, then the Euler characteristic of X^G is congruent mod p to the Euler characteristic of X.
(3) If T is a torus acting nicely on a finite complex X, then the Euler characteristic of X^T is equal to the Euler characteristic of X.

4.5 Theorem. *Let X be a space with an action of the group $G = \mathbf{Z}/p^r$, $r \geq 0$. Assume that X is $\mathbf{F}_p$-finite and that for each subgroup $K \subset G$, $X^{\mathrm{h}K}$ is $\mathbf{F}_p$-complete. Then $X^{\mathrm{h}G}$ is $\mathbf{F}_p$-finite and $\chi(X^{\mathrm{h}G}) = \Lambda(X, G)$.*

4.6 Theorem. *Let X be a space with an action of the finite p-group G. Assume that X is $\mathbf{F}_p$-finite and that, for each subgroup $K \subset G$, $X^{\mathrm{h}K}$ is $\mathbf{F}_p$-complete. Then $X^{\mathrm{h}G}$ is $\mathbf{F}_p$-finite, and $\chi(X^{\mathrm{h}G})$ is congruent* mod p *to $\chi(X)$.*

Let $\mathbf{Z}/p^\infty$ denote the group $\mathbf{Z}[\frac{1}{p}]/\mathbf{Z} = \cup_r \mathbf{Z}/p^r$.

4.7 Theorem. *Let X be a space with an action of the discrete group $T = (\mathbf{Z}/p^\infty)^r$, $r \geq 0$. Assume that X is $\mathbf{F}_p$-finite and that for each finite subgroup $K \subset T$, $X^{\mathrm{h}K}$ is $\mathbf{F}_p$-complete. Then for each finite subgroup $K \subset T$, $X^{\mathrm{h}K}$ is $\mathbf{F}_p$-finite and $\chi(X^{\mathrm{h}K}) = \chi(X)$. If $\chi(X) \neq 0$, then $X^{\mathrm{h}T}$ is nonempty.*

Remark: We do not know whether or not, under hypotheses like those of 4.7, $X^{\mathrm{h}T}$ is $\mathbf{F}_p$-finite and has Euler characteristic equal to $\chi(X)$. In the cases in which we apply 4.7 we are usually able to prove that $X^{\mathrm{h}T} \cong X^{\mathrm{h}K}$ for some finite subgroup K of T (see 6.21, or the proof of 8.10). See §6 for a discussion of the homotopy theoretic relationship between discrete groups of the form $(\mathbf{Z}/p^\infty)^r$ and ordinary tori.

Also very important to us is the following theorem, which has a more prosaic classical analogue.

4.8 Theorem. *Let X be a space with a trivial action of the finite p-group G. Assume that X is $\mathbf{F}_p$-complete and $\mathbf{F}_p$-finite. Then the inclusion $\{1\} \to G$ induces an equivalence $X^{\mathrm{h}G} \to X^{\mathrm{h}\{1\}} \cong X$.*

4.9 The key tools. The machinery behind the proofs of the above theorems comes from Lannes [**15**] and from [**9**]. Let V be an elementary abelian p-group. If X is a space with an action of V, then Lannes [**15**, §4] has shown how to use the cohomology of the Borel construction $X_{\mathrm{h}V}$ to construct an algebraic approximation to the cohomology ring $H^*(X^{\mathrm{h}V})$. We will denote this approximation by $\mathrm{Fix}_V^* X$; in [**15**] the notation is $\mathrm{Fix}(H^*(X_{\mathrm{h}V}))$. There is a natural map $\lambda_{X,V}\colon \mathrm{Fix}_V^* X \to H^*(X^{\mathrm{h}V})$. Lannes has proved the following theorem.

4.10 Theorem. *[**15**] Let X be a an $\mathbf{F}_p$-complete space with an action of the elementary abelian p-group V. Assume that H^*X and $\mathrm{Fix}_V^* X$ are of finite type, and that both X and $X^{\mathrm{h}V}$ are $\mathbf{F}_p$-complete. Then the map $\lambda_{X,V}\colon \mathrm{Fix}_V^* X \to H^* X^{\mathrm{h}V}$ is an isomorphism.*

In some cases [**9**] it is possible to calculate $\mathrm{Fix}_V^* X$ and $\lambda_{X,V}$ in a way reminiscent of Smith theory; more precisely, in these cases $\mathrm{Fix}_V^* X$ can be calculated by applying to $H^* X_{\mathrm{h}V}$ the same functor which for V acting nicely on a finite dimensional X would give $H^*(X^V)$ (see [**9**] and the proof of 4.11 below). This leads to the following result. Note that there is an evaluation map $BV \times X^{\mathrm{h}V} \to X_{\mathrm{h}V}$.

4.11 Theorem. *Let X be a space with an action of the rank one elementary abelian p-group V. Assume that X is $\mathbf{F}_p$-finite and that both X and $X^{\mathrm{h}V}$ are $\mathbf{F}_p$-complete. Then the space $X^{\mathrm{h}V}$ and the pair $(X_{\mathrm{h}V}, BV \times X^{\mathrm{h}V})$ are $\mathbf{F}_p$-finite.*

This is parallel to the classical statement that if V is a rank one elementary abelian p-group acting nicely on a finite complex X, then the space X^V and the pair $(X_{\mathrm{h}V}, (X^V)_{\mathrm{h}V})$ are $\mathbf{F}_p$-finite.

Proof of 4.11: We will use notation from [**9**]. Let $M = H^*(X_{\mathrm{h}V})$ and let $H = H^*BV$. The graded vector space M is an object of the category $\mathcal{U}(H)$, i.e., the category of unstable modules over the mod p Steenrod algebra $\mathcal{A}_p$ which in a compatible way are modules over H. By the Serre spectral sequence M is finitely generated as a module over H. Let N denote the finitely generated H module $\mathrm{Un}(S^{-1}M)$ (here S is the multiplicative subset of H generated by the nonzero elements of degree 2 and $\mathrm{Un}(S^{-1}M)$ denotes the maximal unstable $\mathcal{A}_p$ submodule of $S^{-1}M$). By [**9**, 1.1] and [**15**, 4.5] (cf. [**9**, 2.4]) there is an isomorphism $N \cong H \otimes \mathrm{Fix}_V^* X$ and it follows that $\mathrm{Fix}_V^* X$ is finite and hence (4.10) that $X^{\mathrm{h}V}$ is $\mathbf{F}_p$-finite. By a naturality argument the map on cohomology induced by $BV \times X^{\mathrm{h}V} \to X_{\mathrm{h}V}$ can be identified with the evident map $M \to N$; the kernel and cokernel of this map are S-torsion and thus finite, so the long exact homology sequence of a pair implies that $(X_{\mathrm{h}V}, BV \times X^{\mathrm{h}V})$ is $\mathbf{F}_p$-finite.

Proofs. The following lemma is a consequence of the additivity of traces.

4.12 Lemma. *Let f be an endomorphism of the long exact sequence*

$$\cdots \to A^i \to B^i \to C^i \to A^{i+1} \to \cdots$$

of vector spaces over a field. If two of the graded vector spaces A^*, B^* and C^* are finite dimensional then so is the third, and in this case $\chi(B^*) = \chi(A^*) + \chi(C^*)$ and $\Lambda(B^*, f) = \Lambda(A^*, f) + \Lambda(C^*, f)$.

The next proposition is proved by an inductive long exact cohomology sequence argument that uses 4.12.

4.13 Proposition. *Let X be a connected $\mathbf{F}_p$-finite space and M a finite dimensional vector space over $\mathbf{F}_p$ with a nilpotent (11.5) action of $\pi_1(X)$. Then $H^*(X; M)$ is finite dimensional and $\chi(H^*(X; M)) = \chi(X) \cdot \dim_{\mathbf{F}_p}(M)$.*

If G is a finite p-group then the left action of G on $\mathbf{F}_p[G]$ is nilpotent (11.6). In this case the above proposition implies:

4.14 Corollary. *Suppose that X is $\mathbf{F}_p$-finite and that $\tilde{X} \to X$ is a regular covering space in which the group of covering transformations is a finite p-group G. Then $\tilde{X}$ is $\mathbf{F}_p$-finite and $\chi(\tilde{X}) = |G| \cdot \chi(X)$.*

Proof: By Shapiro's lemma, $H^*\tilde{X}$ is isomorphic to $H^*(X, \mathbf{F}_p[G])$.

4.15 Lemma. *Let $G = \mathbf{Z}/p^r$, let K be a subgroup of G, and let (X,Y) be a G-pair.*

(1) *If $(X_{\mathrm{h}G}, Y_{\mathrm{h}G})$ is $\mathbf{F}_p$-finite, then $(X_{\mathrm{h}K}, Y_{\mathrm{h}K})$ is $\mathbf{F}_p$-finite and in addition $\chi(X_{\mathrm{h}K}, Y_{\mathrm{h}K}) = |G/K| \cdot \chi(X_{\mathrm{h}G}, Y_{\mathrm{h}G})$.*

(2) *If $K \neq \{0\}$ and $(X_{\mathrm{h}K}, Y_{\mathrm{h}K})$ is $\mathbf{F}_p$-finite, then $(X_{\mathrm{h}G}, Y_{\mathrm{h}G})$ is $\mathbf{F}_p$-finite.*

For Y empty this could be considered a special case of 7.4.

Proof of 4.15: Part (1) is from 4.14. Suppose in (2) that $(X_{\mathrm{h}K}, Y_{\mathrm{h}K})$ is $\mathbf{F}_p$-finite. Up to homotopy there is a fibration sequence

$$(X_{\mathrm{h}K}, Y_{\mathrm{h}K}) \to (X_{\mathrm{h}G}, Y_{\mathrm{h}G}) \to B(G/K)$$

and, because the augmentation ideal of $\mathbf{F}_p[G/K]$ is nilpotent (11.6), each cohomology group $H^i(X_{\mathrm{h}K}, Y_{\mathrm{h}K})$ has a composition series in which the factors are trivial G/K modules. Since $H^*B(G/K)$ is a noetherian ring it follows easily that the E_2 term of the mod p cohomology Serre spectral sequence of this fibration is finitely generated over $H^*B(G/K)$, and hence that $H^*(X_{\mathrm{h}G}, Y_{\mathrm{h}G})$ is also finitely generated over $H^*B(G/K)$. The desired result now follows from the fact that the action $H^*B(G/K)$ on $H^*(X_{\mathrm{h}G}, Y_{\mathrm{h}G})$ factors through the restriction map $H^*B(G/K) \to H^*BG$ and hence, if $K \neq \{0\}$, factors through a finite quotient ring of $H^*B(G/K)$.

Let G denote the additive group $\mathbf{Z}/p^r$ with some fixed generator g. The group G has up to isomorphism $r+1$ distinct irreducible representations $\rho_0, \ldots, \rho_r$ over $\mathbf{Q}_p$; the representation ρ_i can be constructed by composing the quotient map $\mathbf{Z}/p^r \to \mathbf{Z}/p^i$ with the action of $\mathbf{Z}/p^i$ on $\mathbf{Q}_p[\sqrt[p^i]{1}]$ obtained by chosing an isomorphism between $\mathbf{Z}/p^i$ and the multiplicative group of p^i'th roots of unity. The representation ρ_0 is the trivial representation, and the dimension of ρ_i, $i \geq 1$ is $p^{i-1}(p-1)$. An inductive calculation shows that the trace of the action of g on ρ_i is equal to 1 for $i = 0$, equal to -1 for $i = 1$, and zero otherwise.

If (X,Y) is an $\mathbf{F}_p$-finite G-pair, write $H^*_{\mathbf{Q}_p}(X,Y)$ as a direct sum $\oplus H^{*,i}$, where $H^{*,i}$ is a sum of copies of ρ_i, and let $\chi_i(X,Y)$ denote $\chi(H^{*,i})$. The following lemma is clear.

4.16 Lemma. *Let $G = \mathbf{Z}/p^r$ and let g be a generator of G. Suppose that (X,Y) is an $\mathbf{F}_p$-finite G-pair. Then there are equalities*

$$\chi(X,Y) = \sum_{i=o}^{r} \chi_i(X,Y)$$

$$\Lambda(X,Y;g) = \chi_0(X,Y) - \frac{1}{p-1}\chi_1(X,Y)$$

and in particular $\Lambda(X,Y;g)$ is an integer which does not depend on the choice of g.

If G is an abelian group, let ${}_pG$ denote the subgroup $\{g \in G \mid p{\cdot}g = 0\}$.

4.17 LEMMA. *Let $G = \mathbf{Z}/p^r$ $(r > 0)$, let $K = {}_pG$, and let (X,Y) be an $\mathbf{F}_p$-finite G-pair such that $(X_{\mathrm{h}K}, Y_{\mathrm{h}K})$ is $\mathbf{F}_p$-finite. Then $\Lambda(X,Y;G) = 0$.*

PROOF: Let $L = \{g \in G \mid p^{r-1}\cdot g = 0\}$. By 4.15 the pair $(X_{\mathrm{h}L}, Y_{\mathrm{h}L})$ is $\mathbf{F}_p$-finite, and there are equalities

$$p^{r-1}\cdot \chi(X_{\mathrm{h}L}, Y_{\mathrm{h}L}) = \chi(X,Y) = p^r \cdot \chi(X_{\mathrm{h}G}, Y_{\mathrm{h}G})\,.$$

On the other hand, by the Serre spectral sequence (or a transfer argument) $H^*_{\mathbf{Q}_p}(X_{\mathrm{h}G}, Y_{\mathrm{h}G})$ is isomorphic to the fixed point set of the action of G on $H^*_{\mathbf{Q}_p}(X,Y)$, and a similar isomorphism holds for $H^*_{\mathbf{Q}_p}(X_{\mathrm{h}L}, Y_{\mathrm{h}L})$. In the notation of 4.16, this gives formulas

$$\chi(X_{\mathrm{h}G}, Y_{\mathrm{h}G}) = \chi_0(X,Y) \quad \text{and} \quad \chi(X_{\mathrm{h}L}, Y_{\mathrm{h}L}) = \chi_0(X,Y) + \chi_1(X,Y)\,;$$

Combining these equations yields

$$p\cdot \chi_0(X,Y) = \chi_0(X,Y) + \chi_1(X,Y)$$

or $\chi_0(X,Y) - \chi_1(X,Y)/(p-1) = 0$. By 4.16 this is the desired result.

PROOF OF 4.5: The proof is by induction on r, beginning with the trivial case $r = 0$. Let $V = {}_pG$, let A be the G-space $\mathrm{Map}(EG,X)$ and let $B = \mathrm{Map}^V(EG,X)$ be the fixed point set of V acting on A. The space $A^{\mathrm{h}G}$ is equivalent to $X^{\mathrm{h}G}$ (10.2), and it is easy to see that the pair $(A_{\mathrm{h}V}, B_{\mathrm{h}V})$ is equivalent to the pair $(X_{\mathrm{h}V}, BV \times X^{\mathrm{h}V})$ of 4.11. It follows from 4.11 that $(A_{\mathrm{h}V}, B_{\mathrm{h}V})$ is $\mathbf{F}_p$-finite and hence from 4.17 that $\Lambda(A,B;G) = 0$. By additivity (4.12), $\Lambda(A;G) = \Lambda(B;G)$ (note that B is $\mathbf{F}_p$-finite). Now the action of G on B factors through an action of G/V on B, and it is clear that $\Lambda(B;G/V) = \Lambda(B;G)$. For any nontrivial subgroup $L \subset G$, $B^{\mathrm{h}(L/V)}$ is equivalent (10.5) to $X^{\mathrm{h}L}$ and is consequently $\mathbf{F}_p$-complete by assumption. It follows from the inductive hypothesis that $X^{\mathrm{h}G} \sim B^{\mathrm{h}(G/V)}$ is $\mathbf{F}_p$-finite and that the Euler characteristic of this space is equal to $\Lambda(B;G/V)$, which, as above, equals $\Lambda(A;G)$ or $\Lambda(X;G)$.

PROOF OF 4.6: The proof is by induction on the order of G; the theorem is true if G is the trivial group. Assume that G is not trivial and that the theorem is true for all proper subgroups of G. Since G is a nilpotent group, there exists some normal subgroup $K \subset G$ such that G/K is cyclic of order p. By induction the space $X^{\mathrm{h}K}$ is $\mathbf{F}_p$-finite and $\chi(X^{\mathrm{h}K})$ is congruent mod p to $\chi(X)$. The space $X^{\mathrm{h}K}$ is $\mathbf{F}_p$-complete by assumption and (10.5) there is a proxy action (10.3) of G/K on $X^{\mathrm{h}K}$ such that $X^{\mathrm{h}G}$ is equivalent to $(X^{\mathrm{h}K})^{\mathrm{h}(G/K)}$. By 4.5, then, $X^{\mathrm{h}G}$ is $\mathbf{F}_p$-finite and has Euler characteristic equal to the Lefschetz number $\Lambda(X^{\mathrm{h}K}, G/K)$. Write $\chi = \chi(X^{\mathrm{h}K})$, $\Lambda = \Lambda(X^{\mathrm{h}K}, G/K)$ and, in the notation of 4.16, $\chi_i = \chi_i(X^{\mathrm{h}K})$, $i = 0$, 1. Then χ_1 is divisible by $(p-1)$, $\chi = \chi_0 + \chi_1$ and $\Lambda = \chi_0 - \chi_1/(p-1)$. It is immediate that Λ is congruent mod p to χ.

4.18 Lemma. *Any homomorphism* $\mathbf{Z}/p^\infty \to \mathrm{GL}_n(\mathbf{Q}_p)$ $(0 < n < \infty)$ *is trivial.*

Proof: Since any nontrivial quotient of $\mathbf{Z}/p^\infty$ is isomorphic to $\mathbf{Z}/p^\infty$ itself, a nontrivial homomorphism of the indicated type would give an embedding $\mathbf{Z}/p^\infty \to \mathrm{GL}_n(\mathbf{Q}_p)$. Such an embedding is impossible, because as indicated above the smallest faithful representation of $\mathbf{Z}/p^r$ $(r \geq 1)$ over $\mathbf{Q}_p$ has dimension $p^{r-1}(p-1)$.

Proof of 4.7: Let L be a subgroup of T. The proof of the first statement is by induction on the order of L, and uses an argument very similar to the one in the proof of 4.6 above. What differs is the following. Suppose that $K \subset L$ is a subgroup of index p and $X^{\mathrm{h}K}$ is known by induction to be $\mathbf{F}_p$-finite and have the same Euler characteristic as X. Then the action (10.5) of L/K on $X^{\mathrm{h}K}$ extends to an action of the group T/K and thus to an action of T; consequently, by 4.18 the action of L/K on $H^*_{\mathbf{Q}_p}(X^{\mathrm{h}K})$ is trivial and $\Lambda(X^{\mathrm{h}K}; L/K) = \chi(X^{\mathrm{h}K})$.

It is possible to write $T = \cup_n T_n$ as an increasing union of finite subgroups. This gives BT up to equivalence as the sequential homotopy direct limit of the spaces BT_n [**5**, p. 332]; it follows easily [**5**, XII 4.1 and XI 4.1 and 4.3] that $X^{\mathrm{h}T}$ is equivalent to the homotopy inverse limit of the tower $\{X^{\mathrm{h}T_n}\}_n$. In particular [**5**, p. 254], $\pi_0 X^{\mathrm{h}T}$ surjects onto the inverse limit of the tower $\{\pi_0 X^{\mathrm{h}T_n}\}_n$. If $\chi(X) \neq 0$ then the sets in this last tower are finite and nonempty, so the inverse limit of the tower is nonempty.

Proof of 4.8: In this case $X^{\mathrm{h}G}$ is the space $\mathrm{Map}(BG, X)$, so the desired result follows from Miller's proof [**17**] of the Sullivan conjecture; he shows that if K is a finite group and Y is connected, $\mathbf{F}_p$-finite and $\mathbf{F}_p$-complete, then the space $\mathrm{Map}_*(BK, Y)$ is contractible.

§5. Finite p-groups

In this section we will establish the following fundamental properties of representations of a finite p-group in a p-compact group, that is, of homomorphisms of a finite p-group into a p-compact group. (Recall from 3.1 that such homomorphisms are defined in terms of maps between classifying spaces.) Each of the properties is a direct transcription of some fact from the theory of representations of a finite p-group in a compact Lie group. In each of the five propositions G denotes a finite p-group and X a p-compact group.

5.1 PROPOSITION. *For any homomorphism $f\colon G \to X$ the centralizer $\mathcal{C}_X(f(G))$ (3.4) is a p-compact group.*

5.2 PROPOSITION. *For any homomorphism $f\colon G \to X$ and subgroup K of G, the natural homomorphism (3.4)*

$$\mathcal{C}_X(f(G)) \to \mathcal{C}_X(f(K))$$

is a monomorphism (3.2).

REMARK: In the context of 5.2 let $i\colon K \to G$ be the inclusion and let $h = f \cdot i$. The notation $\mathcal{C}_X(f(K))$ in the statement of 5.2 is a shorthand form $\mathcal{C}_X(h(K))$; the indicated "natural map" is induced by the map

$$\mathrm{Map}(BG, BX)_{Bf} \xrightarrow{(-)\cdot Bi} \mathrm{Map}(BK, BX)_{Bh}$$

of function spaces (3.4). Proposition 5.2 applies when K is the trivial subgroup of G and states in this case that $\mathcal{C}_X(f(G)) \to X$ is a monomorphism of p-compact groups.

5.3 PROPOSITION. *If $f\colon G \to X$ is a trivial homomorphism, then the monomorphism (5.2) $\mathcal{C}_X(f(G)) \to X$ is an equivalence.*

The next two propositions parallel the classical statements that if H is a connected compact Lie group then every element in H has at least one p'th root and, if H is nontrivial, H contains at least one nontrivial element of order p.

5.4 PROPOSITION. *If G is the group $\mathbf{Z}/p$ and X is nontrivial and connected, then there exists a nontrivial homomorphism $f\colon G \to X$. If f is any such nontrivial homomorphism, then the map $(Bf)^*\colon H^iBX \to H^iBG$ is nonzero for infinitely many i.*

5.5 PROPOSITION. *Suppose that X is connected and that $f\colon \mathbf{Z}/p^n \to X$ $(n \geq 0)$ is a homomorphism. Let $i\colon \mathbf{Z}/p^n \to \mathbf{Z}/p^{n+1}$ be the group homomorphism given by $i(1) = p$. Then there exists a homomorphism $g\colon \mathbf{Z}/p^{n+1} \to X$ such that $g \cdot i$ is conjugate (3.1) to f.*

5.6 The $\mathbf{F}_p$-completeness problem. Our first task is to dispose once and for all of the $\mathbf{F}_p$-completeness hypotheses in the theorems of 4.4.

5.7 Proposition. *Suppose that X and Y are p-compact groups, that G is a finite p-group acting on BX and BY (not necessarily in a basepoint preserving way), and that $f\colon BX \to BY$ is an equivariant map. For each $y \in (BY)^{\mathrm{h}G}$ let $\alpha(y)$ be the proxy action (10.3) of G on Y/X provided by 10.6. Then for each such y the space $(Y/X)^{\mathrm{h}G}_{\alpha(y)}$ is $\mathbf{F}_p$-complete.*

The proof of 5.7 depends on a lemma which was the starting point of our work.

5.8 Lemma. *Let X be a p-compact group and G a finite p-group acting on BX (not necessarily in a basepoint preserving way). Then each component of $(BX)^{\mathrm{h}G}$ is the classifying space of a p-compact group.*

Proof: Choose a point $a\colon BG \to (BX)_{\mathrm{h}G}$ of $(BX)^{\mathrm{h}G}$ and construct a (homotopy) fibre square

$$\begin{array}{ccc} E & \longrightarrow & \mathrm{Map}(S^1, BX)_{\mathrm{h}G} \\ {\scriptstyle q}\big\downarrow & & \big\downarrow \\ BG & \xrightarrow{\ a\ } & (BX)_{\mathrm{h}G} \end{array}$$

in which the right hand vertical map is derived from the free loop space fibration over BX. The fibration q has fibre equivalent to X and gives a proxy action (10.3) of G on X. If K is any subgroup of G, the space $X^{\mathrm{h}K}$ is $\mathbf{F}_p$-local by 11.13 (because X is $\mathbf{F}_p$-complete and hence $\mathbf{F}_p$-local); this space is also equivalent (10.4) to the loop space of the component of $(BX)^{\mathrm{h}K}$ corresponding to $a|_{BK}$ and so is $\mathbf{F}_p$-good by 11.8 (because the fundamental group of any component of this loop space is abelian and acts trivially on the homology of the universal cover of the component). By 11.13, $X^{\mathrm{h}K}$ is $\mathbf{F}_p$-complete. It follows from 4.6 that the space $X^{\mathrm{h}G}$ is $\mathbf{F}_p$-finite and in particular has a finite group of components. This finite group is the fundamental group of the component of $(BX)^{\mathrm{h}G}$ containing a; by 11.13 this component is $\mathbf{F}_p$-local, so by 11.14 the fundamental group of this component is a p-group. This implies (2.1) that the indicated component is the classifying space of a p-compact group.

Proof of 5.7: By 10.6 there is a homotopy fibre square

$$\begin{array}{ccc} (Y/X)^{\mathrm{h}G}_{\alpha(y)} & \longrightarrow & * \\ \big\downarrow & & \big\downarrow{\scriptstyle y} \\ (BX)^{\mathrm{h}G} & \xrightarrow{\ f^{\mathrm{h}G}\ } & (BY)^{\mathrm{h}G} \end{array}$$

and by 5.8 the spaces on the bottom of this square have finite p-groups as the fundamental groups of their components. These bottom spaces are $\mathbf{F}_p$-local by 11.13, $\mathbf{F}_p$-good by 11.9, hence (11.13) $\mathbf{F}_p$-complete. The vertical fibres are either empty or $\mathbf{F}_p$-local (11.13) loop spaces, so they are also $\mathbf{F}_p$-complete. By 11.7 the space $(Y/X)^{\mathrm{h}G}_{\alpha(y)}$ is $\mathbf{F}_p$-complete.

Proofs (part I). Proposition 5.1 is the special case of 5.8 in which the action of the group G on BX is trivial. Note that the argument used to prove 5.8 depended on the homotopy theoretic analogue of the fact that if $f: G \to H$ is a homomorphism of compact Lie groups, then the centralizer of $f(G)$ in H is the fixed point set of the action of G on H via f by conjugation. The proof below of 5.2 depends on a corresponding analogue of a slightly more arcane geometric fact, which we will now explain. Let G and K be as in 5.2 and let H be a compact Lie group (playing the role of X). Let $n = |G/K|$ and let $G \ltimes H^n$ denote the semidirect product obtained by letting G act on $H^n = \mathrm{Map}(G/K, H)$ in the evident way. There is an injection $G \times H \to G \ltimes H^n$ coming from the diagonal map $H \to H^n$ and a projection $G \ltimes H^n \to G$; a section s of this projection corresponds to a group homomorphism $h_s: K \to H$. Any such section $s: G \to G \ltimes H^n$ gives an action of G on the homogeneous space $(G \ltimes H^n)/(G \times H) \cong H^{n-1}$ (this homogeneous space plays the role of F in the proof below); the fixed point set of this action is a disjoint union, indexed by homomorphisms $h^i_s: G \to H$ which extend h_s, of homogeneous spaces. The homogeneous space corresponding to h^i_s is obtained by dividing the centralizer of $h_s(K)$ in H by the centralizer of $h^i_s(G)$ in H.

Proof of 5.2: By 10.7 and 10.6, the homotopy fibre of the restriction map $B\mathcal{C}_X(G) \to B\mathcal{C}_X(K)$ is equivalent to one component of $F^{\mathrm{h}G}$, where F is the homotopy fibre of the diagonal map $BX \to \mathrm{Map}(G/K, BX)$, (provided with a suitable proxy action of G). The space F is $\mathbf{F}_p$-finite, in fact, equivalent to X^n for $n = |G/K| - 1$, so the fact that $F^{\mathrm{h}G}$ is also $\mathbf{F}_p$-finite is a consequence of 5.7 and 4.6.

Proof of 5.3: This follows from 4.8, since the loop space of the trivial component of $\mathrm{Map}(BG, BX)$ is $\mathrm{Map}(BX, X)$.

5.9 A Lefschetz number calculation. Suppose that X is a connected p-compact group. By [**18**, 7.20] the cohomology algebra $H^*_{\mathbf{Q}_p}(X)$ is an exterior algebra over $\mathbf{Q}_p$ on r odd dimensional generators, $r \geq 0$. The number r is called the *rational rank* of X (note that $H^*_{\mathbf{Q}_p}(BX)$ is a polynomial algebra over $\mathbf{Q}_p$ on r generators.)

5.10 Lemma. *If X is a connected p-compact group which is not contractible, then the rational rank of X is greater than zero.*

Proof: Assume p odd; the case $p = 2$ is similar. The cohomology H^*X is a nontrivial finite connected Hopf algebra over $\mathbf{F}_p$ and so is isomorphic to a tensor product of exterior algebras on odd dimensional generators and truncated polynomial algebras on even dimensional generators (where each truncation height is a power of p). It follows that $\chi(X)$ is divisible by p. On the other hand (4.3), if $H^*_{\mathbf{Q}_p}(X)$ were trivial, $\chi(X)$ would be equal to 1. Since 1 is not divisible by p, it follows that $H^*_{\mathbf{Q}_p}(X)$ is a an exterior algebra on $r > 0$ generators and $\chi(X) = 0$.

We will need one particular Lefschetz number calculation connected with the rational rank. Let X be a connected p-compact group. Let $n \geq 1$ be an integer, let G denote the group $\mathbf{Z}/p^n$ with subgroup $H = \mathbf{Z}/p^{n-1} \subset G$, and let G act on $(BX)^p = \mathrm{Map}(G/H, BX)$ by permuting factors in the obvious way. The diagonal map $\Delta\colon BX \to (BX)^p$ is equivariant; consequently, by 10.6 each point $x \in (BX^p)^{\mathrm{h}G}$ gives a proxy action $\alpha(x)$ of G on the homotopy fibre of this diagonal map. This homotopy fibre is an $\mathbf{F}_p$-finite space equivalent to $X^p/\Delta(X) \cong X^{p-1}$, and so (4.3) a Lefschetz number $\Lambda(X^{p-1}_{\alpha(x)}; G)$ is defined.

5.11 Lemma. *Let X be a connected p-compact group and G the group $\mathbf{Z}/p^n$, $n \geq 1$. Let r denote the rational rank of X. Then for any $x \in (BX^p)^{\mathrm{h}G}$, $\Lambda(X^{p-1}_{\alpha(x)}; G) = p^r$.*

Proof: The point $x \in (BX^p)^{\mathrm{h}G}$ amounts to a map $x\colon BG \to (BX^p)_{\mathrm{h}G}$ which lifts the natural map $(BX^p)_{\mathrm{h}G} \to BG$; the space $(X^{p-1}_{\alpha(x)})_{\mathrm{h}G}$ then belongs to the homotopy pullback square

$$\begin{array}{ccc} (X^{p-1}_{\alpha(x)})_{\mathrm{h}G} & \longrightarrow & (BX)_{\mathrm{h}G} \\ \downarrow & & \downarrow{\scriptstyle (\Delta)_{\mathrm{h}G}} \\ BG & \xrightarrow{x} & (BX^p)_{\mathrm{h}G} \end{array} .$$

It follows from this pullback square that the (monodromy) action of G on $H^*_{\mathbf{Q}_p}(X^{p-1}_{\alpha(x)})$ depends only on the map

$$G = \pi_1(BG) \to \pi_1((BX^p)_{\mathrm{h}G}) = G$$

induced by x (the last equality here depends on the fact that X is connected). In fact, this fundamental group map is constrained to be the identity and so the cohomology action in question and the resulting Lefschetz number $\Lambda(X^{p-1}_{\alpha(x)}; G)$ do not depend on x at all. We will compute this Lefschetz number in the special case in which x is the composite

$$BG \xrightarrow{(*,\mathrm{id})} BX \times BG = (BX)_{\mathrm{h}G} \xrightarrow{(\Delta)_{\mathrm{h}G}} (BX^p)_{\mathrm{h}G}\,,$$

where $*\colon BG \to BX$ is the constant map.

In this special case the G-space $X^{p-1}_{\alpha(x)}$ (which we will write simply X^{p-1}) lies equivariantly in a fibration sequence

$$X \xrightarrow{\Delta} X^p \to X^{p-1}$$

in which G acts on X trivially and on X^p by permuting factors. If Y is a G-space such that $H^*_{\mathbf{Q}_p}(Y)$ is of finite type, define $\Lambda(Y;G)[t]$ to be the formal power series $\sum_i(\mathrm{trace}_{\mathbf{Q}_p}\, g^{(i)})t^i$, where $g^{(i)}$ is the action of a generator g of G in dimension i (cf. 4.16; this is a power series with integral coefficients which does not depend on the choice of g). Define $\chi(Y)[t]$ as the analogous sum $\sum_i(\dim_{\mathbf{Q}_p} H^i_{\mathbf{Q}_p}(Y))t^i$. Direct calculation shows that $\Lambda(X^p;G)[t] = \chi(X)[(-1)^{p-1}t^p]$. The Serre spectral sequence of the above fibration collapses and an inspection of the E^2 term shows that

$$\Lambda(X^{p-1};G)[t] = \frac{\Lambda(X^p;G)[t]}{\chi(X)[t]} = \prod_{i=1}^{r} \frac{1+(-1)^{m_i}(-t)^{pm_i}}{1+t^{m_i}}$$

where the m_i are the degrees of the exterior generators of $H^*_{\mathbf{Q}_p}(X)$. Substituting $t=-1$ gives the desired result.

5.12 Proofs (part II). We will finish this section with proofs of 5.4 and of 5.5. The proof of 5.4 is a homotopy theoretic translation of the following argument. Let H be a finite group (playing the role of X) such the $|H|$ is divisible by p (the corresponding assumption on X is that X is connected and not contractible). Let $U = H^p/\Delta(H)$ be the quotient of H^p by the diagonal subgroup and let $V \subset H^p$ be the collection $\{(x_1,\ldots,x_p) | \prod_i x_i = 1\}$. The group $G = \mathbf{Z}/p$ acts on both U and V by cyclic permutation, and there is an equivariant bijection $U \to V$ sending the coset of $(x_1,\ldots,x_p)$ to $(x_1x_2^{-1},\ldots,x_px_1^{-1})$; the G-set U ($\cong V$) is analogous to X^{p-1} below. It is clear that V^G is the set of elements in G of order p (cf. the discussion at the beginning of 5.8, in the special case $G=\mathbf{Z}/p$, $K=\{1\}$). Since $|V|$ is divisible by p and G is a p-group, elementary counting shows that $|V^G|$ is also divisible by p. Therefore the number of elements in G of order p is divisible by p; since there is only one trivial element of order p, there must be at least $(p-1)$ nontrivial ones. If H is a connected compact Lie group, essentially the same argument (in combination with the Lefschetz number calculation of 5.11) shows that p^r (where r is the rational rank of H) is the Euler characteristic of the *space* of elements in H of order p; again there is only one trivial element of order p, so if the rational rank of H is positive

(5.10) there must be nontrivial ones. It is interesting to examine the sense (see below) in which there is only "one" trivial element of order p in a p-compact group X. Essentially, the centralizer $\mathcal{C}_X(f(\mathbf{Z}/p))$ of the trivial homomorphism $f\colon \mathbf{Z}/p \to X$ is X itself (5.3), so the "conjugacy class" $X/\mathcal{C}_X(f(\mathbf{Z}/p))$ of the trivial element is contractible, i.e., has only one point.

Proof of 5.4: Let $G = \mathbf{Z}/p$. By 10.7 the basepoint evaluation map $e\colon \mathrm{Map}(BG, BX) \to BX$ can be identified up to homotopy with the map $(\Delta)^{\mathrm{h}G}\colon (BX)^{\mathrm{h}G} \to (BX^p)^{\mathrm{h}G}$ induced by the diagonal map Δ. It follows that the homotopy fibre of e, which is the space $\mathrm{Map}_*(BG, BX)$ of pointed maps $BG \to BX$, is equivalent in the notation of 5.11 to $(X^{p-1}_{\alpha(x)})^{\mathrm{h}G}$, where x is any point in the connected space $BG \cong (BX^p)^{\mathrm{h}G}$. In particular, $\mathrm{Map}_*(BG, BX)$ is $\mathbf{F}_p$-finite and has Euler characteristic p^r for $r > 0$. Let $\tau\colon G \to X$ be the trivial homomorphism. The component of $\mathrm{Map}_*(BG, BX)$ consisting of null homotopic maps is $X/\mathcal{C}_X(\tau(G))$; this is contractible (5.3) and has Euler characteristic 1. It follows that $\mathrm{Map}_*(BG, BX)$ must have components corresponding to essential maps $BG \to BX$.

It is clear by a Serre spectral sequence argument that H^*BX has finite type (1.5); since BX is $\mathbf{F}_p$-complete (2.1) it follows from results of Lannes [**15**] that any essential map $f\colon BG \to BX$ induces a nontrivial map on $\mathbf{F}_p$ cohomology in infinitely many dimensions.

Proof of 5.5: This is very similar to the proof above of 5.4. Let $G = \mathbf{Z}/p^{n+1}$ and $H \subset G$ the subgroup $\mathbf{Z}/p^n$. Let $f\colon H \to X$ be a homomorphism. By 10.7 the restriction map $e\colon \mathrm{Map}(BG, BX) \to \mathrm{Map}(BH, BX)$ is equivalent to the map $(\Delta)^{\mathrm{h}G}\colon (BX)^{\mathrm{h}G} \to (BX^p)^{\mathrm{h}G}$ induced by the diagonal map Δ, and so the homotopy fibre of e over Bf is equivalent, in the notation of 5.11, to $(X^{p-1}_{\alpha(x)})^{\mathrm{h}G}$ for a point $x \in (BX^p)^{\mathrm{h}G} \cong \mathrm{Map}(BH, BX)$ corresponding to Bf. By 5.11 this homotopy fibre is nonempty, since it has a nonzero Euler characteristic.

§6. p-compact toral groups and their approximations

In this section we will introduce and study p-compact toral groups; these are p-compact groups that are analogues of *p-toral groups* [**19**]. The maximal torus T of a p-compact group X (8.9) will be one of these objects, as will the inverse image in the normalizer $\mathcal{N}(T)$ of T of a p-Sylow subgroup of $\mathcal{N}(T)/T$ (9.8). Our goal is to generalize some of the results of §5 from finite p-groups to p-compact toral groups. The technique is to approximate p-compact toral groups first by discrete groups and then by finite p-groups, and finally to show that the limiting process that arises in the approximating procedure is a trivial one. The conclusion of this is the following analogue of the fact that if $G \subset H$ is a p-toral subgroup of the compact Lie group H, then there is a finite subgroup K of G such that the centralizer of K in H is equal to the centralizer of G in H.

6.1 THEOREM. *Let X be a p-compact group, G a p-compact toral group, and $f\colon G \to X$ a homomorphism. Then there exists a finite p-group K and homomorphism $K \to G$ such that the restriction map $\mathcal{C}_X(G) \to \mathcal{C}_X(K)$ is an equivalence.*

6.2 p-compact toral groups. Recall that a *torus* T of rank r is a compact Lie group which is isomorphic to the r-fold cartesian power $\mathrm{SO}(2)^r$ of the circle group $\mathrm{SO}(2)$; the space BT is equivalent to an Eilenberg-Mac Lane space of type $K(\mathbf{Z}^r, 2)$. A *p-toral group* G is a compact Lie group group which is an extension of a torus T by a finite p-group P [**19**]; the space BG is equivalent to the total space of a fibration over BP with fibre BT.

6.3 DEFINITION: A *p-compact torus* T of rank r is a p-compact group such that BT is an Eilenberg-Mac Lane space of type $K((\mathbf{Z}_p)^r, 2)$. A *p-compact toral group* G is a p-compact group which is an extension of a p-compact torus by a finite p-group, in the sense (§3) that there is a short exact sequence $T \to G \to P$ (equivalently, a fibration sequence $BT \to BG \to BP$) with T a p-compact torus and P a finite p-group.

REMARK: If G is a p-compact toral group then the identity component G_0 of G is a p-compact torus.

6.4 Discrete approximations. One example of the kind of discrete approximation we have in mind is given by the torus T of rank r. Associated to T is the discrete group $\breve{T} = \{x \in T \mid x^{p^n} = 1 \text{ for some } n\}$. This is a group, isomorphic to $(\mathbf{Z}/p^\infty)^r$, which is an approximation to T in the sense that the natural map $\breve{T} \to T$ induces an $\mathbf{F}_p$-equivalence $B\breve{T} \cong BT$.

6.5 DEFINITION: A *p-discrete torus* T of rank r is a discrete group isomorphic to $(\mathbf{Z}/p^\infty)^r$. A *p-discrete toral group* G is a discrete group which is an extension of a p-discrete torus by a finite p-group.

6.6 DEFINITION: Suppose that $f\colon \breve{G} \to \bar{G}$ is a (loop space) homomorphism, where $\breve{G}$ is a p-discrete toral group and $\bar{G}$ is a p-compact toral group. If Bf is an $\mathbf{F}_p$-equivalence, then $\breve{G}$ is said to be a *discrete approximation* to $\bar{G}$ and $\bar{G}$ is said to be a *closure* of $\breve{G}$.

The next proposition explains indicates why discrete approximations of a p-compact toral group might be useful.

6.7 PROPOSITION. *Let $f\colon \breve{G} \to G$ be a discrete approximation of the p-compact toral group G, and let X be an $\mathbf{F}_p$-complete space with an action (10.8) of G. Then f induces a homotopy equivalence $X^{\mathrm{h}G} \to X^{\mathrm{h}\breve{G}}$.*

Taking the trivial action of G on X above gives in particular an equivalence $\mathrm{Map}(BG, X) \cong \mathrm{Map}(B\breve{G}, BX)$.

PROOF OF 6.7: The space BG is $\mathbf{F}_p$-complete by assumption (2.1) and hence $\mathbf{F}_p$-local (11.12). The Borel construction $X_{\mathrm{h}G}$ is therefore $\mathbf{F}_p$-complete by the fibre lemma (11.7) and hence similarly $\mathbf{F}_p$-local. It then follows from the definition of an $\mathbf{F}_p$-local space that in the commutative square

$$\begin{array}{ccc} \mathrm{Map}(BG, X_{\mathrm{h}G}) & \longrightarrow & \mathrm{Map}(B\breve{G}, X_{\mathrm{h}G}) \\ \downarrow & & \downarrow \\ \mathrm{Map}(BG, BG) & \longrightarrow & \mathrm{Map}(B\breve{G}, BG) \end{array}$$

the horizontal arrows are homotopy equivalences. The homotopy fibre of the left hand vertical map over the identity map of BG is by definition $X^{\mathrm{h}G}$ and it is not hard to identify the homotopy fibre of the right hand vertical map over Bf as $X^{\mathrm{h}\breve{G}}$. The desired result follows at once.

6.8 PROPOSITION. *Any p-compact toral group G has a discrete approximation $\breve{G} \to G$.*

PROOF: Let $\mathbf{C}_{\mathbf{Q}}(-)$ denote the $\mathbf{Q}$-completion functor of [**5**]. For simply connected X the rational completion map $X \to \mathbf{C}_{\mathbf{Q}}(X)$ induces isomorphisms $\mathbf{Q} \otimes \pi_i X \cong \pi_i \mathbf{C}_{\mathbf{Q}}(X)$. Construct a fibration $BT \to BG \to BP$ in which P is a finite p-group and BT is $K((\mathbf{Z}_p)^r, 1)$ $(r \geq 0)$. Let F be the homotopy fibre of the rational completion map $BT \to \mathbf{C}_{\mathbf{Q}}(BT)$; the space F is a space of type $K((\mathbf{Q}_p/\mathbf{Z}_p)^r, 1) \cong K((\mathbf{Z}/p^\infty)^r, 1)$, and, because $\mathbf{C}_{\mathbf{Q}}(BT)$ is acyclic with respect to $\mathbf{F}_p$ cohomology, the map $F \to BT$ is an $\mathbf{F}_p$-equivalence. Now let $f\colon \mathbf{C}^{\bullet}_{\mathbf{Q}}(BG) \to BP$ be the fibrewise $\mathbf{Q}$-completion [**5**] of $BG \to BP$. The homotopy fibre of the map f is $\mathbf{C}_{\mathbf{Q}}(BT) \cong K((\mathbf{Q}_p)^r, 2)$. so by obstruction theory the map f has a section s. If $B\breve{G}$ is defined by the homotopy fibre square

$$\begin{array}{ccc} B\breve{G} & \longrightarrow & BG \\ \downarrow & & \downarrow \\ BP & \xrightarrow{\ s\ } & \mathbf{C}^{\bullet}_{\mathbf{Q}}(BG) \end{array}$$

(note that the vertical homotopy fibre here is F) it is not hard to check that $B\breve{G}$ is the classifying space of a p-discrete toral group and that $B\breve{G} \to BG$ is a discrete approximation.

6.9 Proposition. *Any p-discrete toral group G has a (functorial) closure $G \to \mathrm{Cl}(G)$.*

Proof: By definition there is a normal subgroup $A \subset G$ isomorphic to $(\mathbf{Z}/p^\infty)^r$ $(r \geq 0)$ such that G/A is a finite p-group P. The subgroup A is uniquely determined by these conditions. The space $\mathbf{C}_{\mathbf{F}_p}(BA)$ is $\mathbf{F}_p$-good (11.3) and is homotopy equivalent to $K((\mathbf{Z}_p)^r, 2)$ (to see this embed A in $\mathrm{SO}(2)^r$, observe that $BA \to B\,\mathrm{SO}(2)^r \cong K(\mathbf{Z}^r, 2)$ is an $\mathbf{F}_p$-equivalence, and conclude (§11) that $\mathbf{C}_{\mathbf{F}_p}(BA)$ is equivalent to $\mathbf{C}_{\mathbf{F}_p}K(\mathbf{Z}^r,2) \cong K((\mathbf{Z}_p)^r, 2)$). The space BP is $\mathbf{F}_p$-complete (11.9), so the fibre lemma (11.7) now shows (see 11.6) that $\mathbf{C}_{\mathbf{F}_p}(BG)$ is the classifying space of a p-compact toral group. It is easy to argue as in the proof of 11.9 that BG is $\mathbf{F}_p$-good and hence that the completion map $BG \to \mathbf{C}_{\mathbf{F}_p}(BG)$ gives the desired functorial closure homomorphism, with $\mathrm{Cl}(G) = \Omega\mathbf{C}_{\mathbf{F}_p}(BG)$.

6.10 Remark: If G is a p-discrete toral group and $f\colon G \to \bar{G}$ is a closure homomorphism, then $f\colon BG \to B\bar{G}$ is an $\mathbf{F}_p$-equivalence and hence $\mathbf{C}_{\mathbf{F}_p}(BG) \to \mathbf{C}_{\mathbf{F}_p}(B\bar{G}) \cong B\bar{G}$ is an equivalence. This shows that $\bar{G}$ is equivalent to $\mathrm{Cl}(G)$.

6.11 Remark: If G is as in 6.9, then (in the notation of the proof) the homotopy fibre of the map $BG \to B\,\mathrm{Cl}(G)$ is the same as the homotopy fibre of the completion map $BA \to \mathbf{C}_{\mathbf{F}_p}(BA)$; it is not hard to check that this is $K((\mathbf{Q}_p)^r, 1)$, and, in particular, that it is $K(V, 1)$ for a rational vector space V. By obstruction theory this implies that if K is a p-discrete toral group then any homomorphism $K \to \mathrm{Cl}(G)$ lifts up to conjugacy to a homomorphism $K \to G$.

6.12 Dimension and the descending chain condition. In order to go further with the approximation procedure, we will study a notion of "dimension" for $\mathbf{F}_p$-finite spaces; this will lead directly to a descending chain condition for "subgroups" of a p-compact group.

6.13 Definition: Suppose that X is a space and that M is a local coefficient system on X. The *dimension* of X with coefficients in M, denoted $\mathrm{cd}(X, M)$, is the largest integer i such that $H^i(X, M) \neq 0$, if such an integer exists. If the set of integers i such that $H^i(X, M) \neq 0$ is not bounded above, then $\mathrm{cd}(X, M) = \infty$; if $H^*(X, M)$ vanishes, then $\mathrm{cd}(X, M) = -\infty$. The mod p (cohomological) dimension of X, denoted $\mathrm{cd}_{\mathbf{F}_p}(X)$, is $\mathrm{cd}(X, \mathbf{F}_p)$.

6.14 Proposition. *If $f\colon X \to Y$ is a monomorphism of p-compact groups, then $\mathrm{cd}_{\mathbf{F}_p}(Y) = \mathrm{cd}_{\mathbf{F}_p}(X) + \mathrm{cd}_{\mathbf{F}_p}(Y/X)$. In particular, $\mathrm{cd}_{\mathbf{F}_p}(X) \leq \mathrm{cd}_{\mathbf{F}_p}(Y)$, and equality holds if and only if Y/X is homotopically discrete.*

6.15 Remark: A space is *homotopically discrete* if it is homotopy equivalent to a discrete space. If $f\colon X \to Y$ is a homomorphism of p-compact groups, the space Y/X is homotopically discrete if and only if f gives an equivalence between X and a union of components of Y.

The following is proved by an inductive long exact sequence argument.

6.16 Lemma. *Let X be a connected space and M a finite dimensional vector space over $\mathbf{F}_p$ with a nilpotent (11.5) action of $\pi_1(X)$. Suppose that $\mathrm{cd}_{\mathbf{F}_p}(X) < \infty$. Then $\mathrm{cd}(X, M) = \mathrm{cd}_{\mathbf{F}_p}(X)$.*

Proof of 6.14: Let $a = \mathrm{cd}_{\mathbf{F}_p}(X)$ and $b = \mathrm{cd}_{\mathbf{F}_p}(Y/X)$. In the fibration $X \to Y \to Y/X$ the action of the fundamental group of any component of the base on each mod p cohomology group of the fibre is nilpotent, since (11.6) this action factors through an action of the finite p-group $\pi_1(BX)$. By 4.13 the E_2 term of the associated cohomology Serre spectral sequence is concentrated in a rectangle of height a and width b, with the upper right hand corner group $E_2^{b,a} \neq 0$. This gives the formula for $\mathrm{cd}_{\mathbf{F}_p}(Y)$. If $\mathrm{cd}_{\mathbf{F}_p}(X) = \mathrm{cd}_{\mathbf{F}_p}(Y)$ then $\mathrm{cd}_{\mathbf{F}_p}(Y/X) = 0$, i.e., Y/X has the mod p cohomology of a finite discrete space. Since Y/X is $\mathbf{F}_p$-complete (5.7), this implies that Y/X is in fact homotopically discrete.

6.17 Proposition. *Given an inverse system*

$$\cdots \to X_n \xrightarrow{f_n} \cdots \to X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0$$

of p-compact groups and monomorphisms, there exists $N > 0$ such that for $i > N$ the map f_i is an equivalence.

Proof: By 6.14 the sequence $\mathrm{cd}_{\mathbf{F}_p}(X_0)$, $\mathrm{cd}_{\mathbf{F}_p}(X_1), \ldots$ is a weakly decreasing sequence of nonnegative integers. At some point M this sequence must stabilize, i.e., $\mathrm{cd}_{\mathbf{F}_p}(X_i) = \mathrm{cd}_{\mathbf{F}_p}(X_M)$ for $i > M$. Past this point each space X_i is a union of components of the space X_M; the proposition now follows from the fact that a weakly decreasing sequence of subsets of the finite set $\pi_0(X_M)$ must eventually stabilize.

6.18 Finite approximations. We conclude by expressing a p-discrete toral group as a union of finite groups, and drawing out mapping space consequences.

6.19 Proposition. *If G is a p-discrete toral group, then there exists an increasing chain $G_n \subset G_{n+1} \subset \cdots$ of finite subgroups of G such that $G = \cup_{m \geq n} G_m$.*

Proof: The group G lies in a short exact sequence

$$\{1\} \to A \to G \to P \to \{1\},$$

where P is a finite p-group and $A \cong (\mathbf{Z}/p^\infty)^s$ for some $s \geq 0$. For $n \geq 0$ define $A_n \subset A$ to be the subgroup $\{x \in A \mid p^n x = 0\}$ where we have used additive notation for the group operation in A. The group A_n is isomorphic to $(\mathbf{Z}/p^n)^s$. Let $c \in C^2(P, A)$ be some cocycle determining the above extension. Since P is finite, there is an integer n such that the $c \in C^2(P, A_n)$. This makes it simple to find a sequence G_m of groups, $m \geq n$ such that $G = \cup_{m \geq n} G_m$ and such that in fact for each $m \geq n$ there is a map of short exact sequences

$$\begin{array}{ccccccccc}
\{1\} & \longrightarrow & A_m & \longrightarrow & G_m & \longrightarrow & P & \longrightarrow & \{1\} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\{1\} & \longrightarrow & A_{m+1} & \longrightarrow & G_{m+1} & \longrightarrow & P & \longrightarrow & \{1\}
\end{array}$$

inducing the identity map on P and the evident inclusion $A_m \subset A_{m+1}$.

6.20 Remark: The expression $G = \cup_{m \geq n} G_m$ in 6.19 allows BG to be described up to homotopy as the mapping telescope of the direct system $BG_n \to BG_{n+1} \to \cdots$. This description implies that for any space X the space $\mathrm{Map}(BG, X)$ is equivalent to the homotopy inverse limit of the tower $\{\mathrm{Map}(BG_m, X)\}_{m \geq n}$, or more generally that for any space X with an action of G the space $X^{\mathrm{h}G}$ is equivalent to the homotopy inverse limit of the tower $\{X^{\mathrm{h}G_m}\}_{m \geq n}$ (cf. proof of 4.7).

6.21 Proposition. *Let X be a p-compact group, G a p-discrete toral group, and $f: G \to X$ a homomorphism. Then there is a finite subgroup $K \subset G$ such that the restriction map $\mathcal{C}_X(G) \to \mathcal{C}_X(K)$ is an equivalence.*

Proof: Write $G = \cup_m G_m$ as in 6.19. By 5.1, 5.2 and 6.17 the tower $\{B\mathcal{C}_X(G_m)\}_m$ is eventually constant up to homotopy, i.e., there is an integer k such that the restriction map $\mathcal{C}_X(G_i) \to \mathcal{C}_X(G_k)$ is an equivalence for $i \geq k$. Elementary properties of the homotopy inverse limit of a tower give (6.20) that the map $\mathcal{C}_X(G) \to \mathcal{C}_X(G_k)$ is an equivalence.

Proof of 6.1: This is a combination of 6.21, 6.8, and 6.7.

§7. Kernels and monomorphisms

Let X be a p-compact group and G a p-compact toral group. The aim of this section is to show that there is a simple way to test in terms of a "kernel" whether or not a homomorphism $G \to X$ is a monomorphism (3.2). We will use this result in §9 to prove that $\mathcal{N}_p(T) \to X$ is a monomorphism.

If G is a p-discrete toral group and $g \in G$ is an element, let $o(g)$ denote the order of g and $\kappa_g: \mathbf{Z}/o(g) \to G$ the homomorphism with $\kappa_g(1) = g$.

7.1 Definition: Let G be a p-discrete toral group, X a p-compact group, and $f\colon G \to X$ a homomorphism. The *kernel* of f, denoted $\ker(f)$, is the subset of G consisting of all $g \in G$ such that $f \cdot \kappa_g$ is trivial (3.1).

We will say that $\ker(f)$ itself is *trivial* if it contains only the identity element. There are two theorems about kernels that we are particularly interested in.

7.2 Theorem. *Suppose that G is a p-discrete toral group, X is a p-compact group, and $f\colon G \to X$ is a homomorphism. Then $\ker(f)$ is a normal subgroup of G and f factors uniquely up to homotopy through a homomorphism $f'\colon G/\ker(f) \to X$.*

7.3 Theorem. *Suppose that G is a p-compact toral group, X is a p-compact group, and $f\colon G \to X$ is a homomorphism. Let $i\colon \breve{G} \to G$ be a discrete approximation. Then f is a monomorphism if and only if $\ker(f \cdot i)$ is trivial.*

The proof of 7.3 depends on the following statement, which is analogous to the observation that an action of a finite p-group G on a space X is *free* if and only if each subgroup of G of order p acts without fixed points.

7.4 Theorem. *Let X be an $\mathbf{F}_p$-complete space with an action of the finite p-group G. Assume that X is $\mathbf{F}_p$-finite. The $X_{\mathrm{h}G}$ is $\mathbf{F}_p$-finite if and only if $X^{\mathrm{h}K}$ is empty for each subgroup $K \subset G$ of order p.*

The rest of this section is devoted to proofs of these results.

7.5 Lemma. *Let G be a p-discrete toral group, $K \subset G$ a normal subgroup, X a p-compact group, and $f\colon G \to X$ a homomorphism. If the restriction of f to K is trivial, then f factors uniquely up to homotopy through a homomorphism $g\colon G/K \to X$. Moreover, the induced homomorphism $\mathcal{C}_X(g(G/K)) \to \mathcal{C}_X(f(G))$ is an equivalence.*

Proof: Let U denote the subspace of $\mathrm{Map}(BG, BX)$ consisting of maps which are null homotopic when restricted to BK and let V be the null component $\mathrm{Map}(BK, BX)$. The space U is up to homotopy the space of sections of a fibration $E \to B(G/K)$ with V as fibre. Combining 6.21 and 5.3 shows that the inclusion $BX \to V$ of constant maps is a homotopy equivalence, so that E is equivalent to $BX \times B(G/K)$ and U is equivalent to $\mathrm{Map}(B(G/K), BX)$. This gives the desired result.

7.6 Remark: Let G and X be as in 7.5. The uniqueness provision in 7.5 shows that if K is a normal subgroup of G and $h\colon G/K \to X$ a homomorphism which is trivial when restricted to G, then h itself is trivial.

7.7 Lemma. *Let G be a p-discrete toral group, X a p-compact group, and $f\colon G \to X$ a homomorphism. If* $\ker(f) = G$, *then f is trivial.*

Proof: Write $G = \cup G_m$ as a union of finite p-groups (6.19). By 6.20, the subspace F_∞ of $\mathrm{Map}(BG, BX)$ consisting of maps which are null homotopic on BK is the homotopy inverse limit of the tower $\{F_m\}$, where F_m is the null component of $\mathrm{Map}(BG_m, BX)$. By 5.3 each F_m is equivalent to BX, so the tower is homotopically constant and $F_\infty \cong BX$. In particular F_∞ is connected; this implies that a homomorphism $G \to X$ is trivial if and only if its restriction to each G_m is trivial. Consequently, in proving the lemma we can assume that G is a finite p-group and work by induction on $|G|$. If $|G| > 1$ and $\ker(f) = G$ then there exists a nontrivial cyclic subgroup K of the center of G and by the definition of kernel the restriction of f to K is trivial. By 7.5 f extends to a homomorphism $g\colon G/K \to X$. By 7.6 and the definition of kernel, $\ker(g) = G/K$. It follows from induction that g is trivial and thus that f is.

Proof of 7.2: It is only necessary to prove that $\ker(f)$ is a normal subgroup of G; the indicated factorization follows from 7.5 and 7.7. Since G is a union of finite groups (6.19) it is enough to consider the special case in which G is a finite p-group. The set $\ker(f)$ is by definition closed under conjugation by elements of G, so it is enough to prove that this set is a subgroup. The proof is then by induction on the order of G. If $\ker(f)$ is trivial there is nothing to prove. Otherwise, suppose that there is some proper normal subgroup $K \subset G$ such that $K \cap \ker(f)$ is non-trivial and let $L = K \cap \ker(f)$. By induction L is a normal subgroup of K, and it follows that L is a normal subgroup of G since L is preserved by conjugation with elements of G. By 7.5 and 7.7 the homomorphism f factors through a homomorphism $f'\colon G/L \to X$. By induction $\ker(f')$ is a normal subgroup of G/L so that $\ker(f)$, which is (7.6) the inverse image in G of $\ker(f')$, is a normal subgroup of G. Suppose that no such proper normal subgroup K exists. Pick a nontrivial element $g \in \ker(f)$. By assumption the normal closure of g is G itself, so that in particular the image of g generates the abelianization G_{ab} of G. The fact that G_{ab} is cyclic easily implies that G is abelian and hence that $G = G_{\mathrm{ab}}$ is generated by g. Since the set $\ker(f)$ is clearly closed under taking powers, it follows that $\ker(f) = G$ and in particular that $\ker(f)$ is a normal subgroup of G.

Proof of 7.4: Suppose that there is some subgroup $K \subset G$ of order p such that $X^{\mathrm{h}K}$ is nonempty. This implies (§10)that the map $BK \to BG$ lifts to a map $BK \to X_{\mathrm{h}G}$. Such a lift implies that $X_{\mathrm{h}G}$ is not $\mathbf{F}_p$-finite, since the restriction map $H^*BG \to H^*BK$ is nonzero in infinitely many dimensions (12.6).

In the other direction the proof will be by induction on the order of G, beginning with the trivial group. Suppose that G is nontrivial and that $X^{\mathrm{h}K}$ is empty for each nontrivial subgroup $K \subset G$ of order p. There exists a normal subgroup $L \subset G$ of index p and the analogue of 10.5 for homotopy orbit spaces reads that $X_{\mathrm{h}G}$ is equivalent to $Y_{\mathrm{h}(G/L)}$ where $Y = X_{\mathrm{h}L}$. By induction Y is $\mathbf{F}_p$-finite and by 11.7 and 11.4 it is $\mathbf{F}_p$-complete, so the fact that $Y_{\mathrm{h}(G/L)}$ is $\mathbf{F}_p$-finite will follow immediately from 4.11 if we can show that $Y^{\mathrm{h}(G/L)}$ is empty. By 10.6 the space $Y^{\mathrm{h}(G/L)}$ is the total space of a fibration over $(BL)^{\mathrm{h}(G/L)}$ in which the fibre over any point $b \in (BL)^{\mathrm{h}(G/L)}$ is the homotopy fixed point set $X_{\alpha(b)}^{\mathrm{h}(G/L)}$ of a proxy action $\alpha(b)$ of G/L on X. However, $(BL)^{\mathrm{h}(G/L)}$ is the space of sections of the fibration $BL_{\mathrm{h}G/L} \cong BG \to BL$. Each such section b gives after choice of basepoint a section $\tilde{b}\colon L \to G$ of the projection $G \to G/L$, and the proxy action $\alpha(b)$ of L on X is essentially obtained by compsing the action of G on X with $\tilde{b}$. Since $\tilde{b}(L)$ is a subgroup of G of order p, it follows from the assumptions that the $X_{\alpha(b)}^{\mathrm{h}(G/L)}$ is empty.

Proof of 7.3: The map $X/\breve{G} \to X/G$ is an $\mathbf{F}_p$-equivalence (6.6). If $\ker(f \cdot i)$ is non-trivial, then it contains a nontrivial subgroup K of order p. The composite $BK \to B\breve{G} \to BX$ is null homotopic, so that the map $BK \to B\breve{G}$ lifts to a map $BK \to X/\breve{G}$. Since the restriction map $H^*B\breve{G} \to H^*BK$ is nonzero in infinitely many degrees (12.6), this implies that $X/\breve{G}$ is not $\mathbf{F}_p$-finite and hence that f is not a monomorphism.

Suppose on the other hand that $\ker(f \cdot i)$ is trivial. Write $\breve{G} = \cup \breve{G}_m$ as in 6.19. Suppose that $X/\breve{G}_m$ is $\mathbf{F}_p$-finite for each m. Since H^*X is isomorphic to $H^*(X/\breve{G}_m, \mathbf{F}_p[\breve{G}_m])$, 11.6 and 6.16 imply that $\mathrm{cd}_{\mathbf{F}_p}(X/\breve{G}_m) = \mathrm{cd}_{\mathbf{F}_p}(X)$; in particular, these cohomological dimensions are bounded as m varies, so that $\mathrm{cd}_{\mathbf{F}_p}(X/\breve{G}) = \mathrm{cd}_{\mathbf{F}_p} \operatorname{hocolim}(X/\breve{G}(m)) < \infty$. However, the Serre spectral sequence of the fibration $X \to X/\breve{G} \to B\breve{G}$, in combination with the fact that $H^*B\breve{G}$ is noetherian (12.1), shows that $H^*(X/\breve{G})$ is finitely generated as a module over $H^*B\breve{G}$. Consequently, $X/\breve{G}$ must be $\mathbf{F}_p$-finite.

It is thus enough to show that X/P is $\mathbf{F}_p$-finite for each finite subgroup $P \subset \breve{G}$. By 7.4 (cf. 10.4) this will be the case if and only if there exists no subgroup $K \subset P$ of order p such that the map $BK \to BP$ lifts up to homotopy to a map $BK \to X/P$. However, if such a lift were to exist, then K would represent a nontrivial subgroup of order p in $\ker(f \cdot i)$; by hypothesis there are no such subgroups.

§8. Maximal tori

In this section we will define what it means for a p-compact group X to have a maximal torus (8.9) and prove that every X has one (8.13). The strategy of proof is exactly as described in 1.2: essentially (unless X is discrete), find a nontrivial connected abelian subgroup A of X, assume by induction on dimension that $\mathcal{C}_X(A)$ has a maximal torus T, and show that the inverse image of T in $\mathcal{C}_X(A)$ gives a maximal torus for X.

8.1 Centralizers and central maps. We first confirm several reasonable suspicions about centralizers and central homomorphisms (3.4).

8.2 Proposition. *Suppose that A is an abelian (3.4) p-compact toral group and that X is a p-compact group. Let $f: A \to X$ be a homomorphism. Then f lifts (3.4) canonically to a central homomorphism $f': A \to \mathcal{C}_X(A)$.*

8.3 Proposition. *Suppose that A is an abelian p-compact toral group, X is a p-compact group, and $f: A \to X$ is a central monomorphism. Then the map f extends naturally to a short exact sequence (3.2) $A \to X \to X/A$ of p-compact groups.*

8.4 Proposition. *Let $f: C \to D$ be a homomorphism of loop spaces, $X \to Y \to Z$ a sequence of loop spaces such that $BX \to BY \to BZ$ is a fibration sequence, and $g: Y \to C$ a homomorphism. If the map $\mathcal{C}_C(X) \to \mathcal{C}_D(X)$ induced by f is an equivalence, then so is the map $\mathcal{C}_C(Y) \to \mathcal{C}_D(Y)$.*

8.5 Remark: An elementary calculation shows that a p-compact toral group G is *abelian* (3.4) if and only if any discrete approximation $\breve{G}$ to G is abelian as a discrete group, i.e, if and only if G is a product of a p-compact torus and a finite abelian p-group. If G is an abelian p-compact toral group and X is a p-compact group, then *any* homomorphism $f: X \to G$ is central.

The proof of the first proposition depends on an interesting lemma.

8.6 Lemma. *Suppose that X is a p-compact group and G is a p-compact toral group. Then a homomorphism $f: G \to X$ is central if and only if there exists a homomorphism $h: G \times X \to X$ such that $h = f$ on $G \times *$ and $h = \mathrm{id}_X$ on $* \times X$.*

Proof: If f is central, the evaluation map

$$BG \times BX \cong BG \times B\mathcal{C}_X(G) = BG \times \mathrm{Map}(BG, BX)_{Bf} \to BX$$

gives the homomorphism h. Conversely, given h it is possible to combine h with id_G to get an equivalence $j: G \times X \to G \times X$ which is (id_G, f) on $G \times *$ and $(*, \mathrm{id}_X)$ on $* \times X$. Let $a: G \to G \times X$ be the inclusion of the first factor and let $b = j \cdot a$. Since j is an equivalence it induces an equivalence $k: \mathcal{C}_{G\times X}(a(G)) \to \mathcal{C}_{G\times X}(b(G))$. The domain of k is $\mathcal{C}_G(G) \times X$ (see 6.1 and 5.3) while the range is $\mathcal{C}_G(G) \times \mathcal{C}_X(f(G))$; the result follows from the fact that k commutes with projections to $\mathcal{C}_G(G)$.

Proof of 8.2: There is a commutative diagram

$$\begin{array}{ccc} \mathrm{Map}(BA, BA)_{\mathrm{id}} & \xrightarrow{Bf\cdot(-)} & \mathrm{Map}(BA, BX)_{Bf} \cong B\mathcal{C}_X(A) \\ \cong\big\downarrow & & \big\downarrow \\ BA & \xrightarrow{Bf} & BX \end{array}$$

in which the vertical arrows are obtained by evaluation at the basepoint of BA. This gives the required lift of f. The existence of the composition map

$$BA \times B\mathcal{C}_X(f(A)) \cong \mathrm{Map}(BA, BA)_{\mathrm{id}} \times \mathrm{Map}(BA, BX)_{Bf} \to B\mathcal{C}_X(f(A))$$

shows by 8.6 (cf. 6.1 and 5.1) that this lift is central.

Proof of 8.3: Consider the square which appears above in the proof of 8.2. In the present case both vertical arrows are equivalences, and the upper horizontal arrow is essentially the inclusion of the orbit of Bf under the composition action of the group-like topological monoid $\mathrm{Map}(BA, BA)_{\mathrm{id}}$ on $\mathrm{Map}(BA, BX)_{Bf}$. Let $B(X/A)$ denote the Borel construction of this action; this is a pointed space with basepoint given by the image of the Bf. It is immediate that the loop space $\Omega B(X/A)$ is equivalent to X/A. The space X/A is $\mathbf{F}_p$-finite because f is a monomorphism and $\mathbf{F}_p$-complete by 11.7; its group of components is a quotient of the p-group $\pi_0 X$. Consequently (2.1) X/A is a p-compact group.

Proof of 8.4: By assumption there is a fibration sequence $BX \to BY \to BZ$. Let Bh denote the restriction of Bg to BX. The mapping space $\mathrm{Map}(BY, BC)_{Bg}$ is one component of the space of sections of a suitable fibration E_1 over BZ with fibre $\mathrm{Map}(BX, BC)_{Bh}$, and $\mathrm{Map}(BY, BD)_{B(f\cdot g)}$ is the corresponding component of a fibration E_2 over the same base with fibre $\mathrm{Map}(BX, BD)_{B(f\cdot h)}$. The proposition follows from the fact that by assumption these two fibrations are equivalent.

8.7 Maximal tori: basic properties. If H is compact Lie group which is not necessarily connected, a *maximal torus* for H is a toral subgroup (1.2) of H which is of finite index in its own centralizer. We will use the corresponding homotopy theoretic condition to define a maximal torus for a p-compact group X and show (8.10) that the homogeneous space of such a torus has strictly positive Euler characteristic (as in the Lie group case). This implies (8.11) that up to conjugacy a maximal torus contains any torus in X.

8.8 DEFINITION: Suppose that A is an abelian p-compact toral group, G a p-compact group, and $f\colon A \to G$ a homomorphism. The homomorphism f is said to be *self-centralizing* if the induced (8.2) homomorphism $f'\colon A \to \mathcal{C}_X(A)$ is an equivalence, and *almost self-centralizing* if $\mathcal{C}_X(A)/f'(A)$ is homotopically discrete (6.15).

Note that if G is a p-compact toral group then the inclusion $G_0 \to G$ of the identity component is almost self-centralizing. Conversely, if T is a p-compact torus, X is a p-compact group, and $T \to X$ is almost self-centralizing, then $\mathcal{C}_X(T)$ is a p-compact toral group with T as its identity component.

8.9 DEFINITION: A *maximal torus* for a p-compact group X is a p-compact torus T together with an almost self-centralizing homomorphism $f\colon T \to X$.

We will show later (9.1) that if X is a *connected* p-compact group then any maximal torus in X is actually self-centralizing.

8.10 PROPOSITION. *Suppose that X is a p-compact group and $i\colon T \to X$ is a maximal torus. Then the space X/T is $\mathbf{F}_p$-finite (i.e., i is a monomorphism), the homotopy fixed point set $(X/T)^{\mathrm{h}T}$ is $\mathbf{F}_p$-finite and homotopically discrete, and $\chi(X/T) = \chi((X/T)^{\mathrm{h}T}) > 0$.*

PROOF: The fact that i is almost self-centralizing implies that T is equivalent to the identity component of $\mathcal{C}_X(T)$, so that $\mathcal{C}_X(T)/T \cong \pi_0\mathcal{C}_X(T)$ is a finite p-group (6.1, 5.1). Since X/T is a regular covering space of the $\mathbf{F}_p$-finite (6.1, 5.2) space $X/\mathcal{C}_X(T)$ with covering group $\pi_0\mathcal{C}_X(T)$, it follows from 4.14 that i is a monomorphism.

Let $\breve{T} \to T$ be a discrete approximation of T, so that (6.7) $\mathcal{C}_X(\breve{T}) \cong \mathcal{C}_X(T)$ and $(X/T)^{\mathrm{h}\breve{T}} \cong (X/T)^{\mathrm{h}T}$ Choose (6.21) a finite subgroup $A \subset \breve{T}$ such that the map $\mathcal{C}_X(\breve{T}) \to \mathcal{C}_X(A)$ is an equivalence. By 10.6 there is a fibration sequence

$$(X/T)^{\mathrm{h}A} \to \coprod_j B\mathcal{C}_T(j(A)) \xrightarrow{q} B\mathcal{C}_X(A) \ \ (\cong B\mathcal{C}_X(T))$$

where the coproduct is taken over all homomorphisms $j\colon A \to T$ such that $i \cdot j$ is conjugate to the given composite $A \to \breve{T} \to T \to X$. The total space of this fibration is equivalent to a disjoint union of copies of BT

(8.5) and the base is equivalent to a space whose universal cover is BT. The restriction q_j of q to the component "j" of its domain corresponds to a homomorphism $\mathcal{C}_T(j(A)) \to \mathcal{C}_X(A)$ which is a monomorphism because the fibre $F = (X/T)^{\mathrm{h}A}$ is $\mathbf{F}_p$-finite (4.6, 5.7). Consequently each restriction q_j induces an epimorphism and hence an isomorphism on π_2 (if $\pi_2 q_j$ were not an epimorphism, then by the long exact homotopy sequence of the above fibration some component of F would have an Eilenberg-MacLane space $K(\mathbf{Z}/p^r, 1)$ as a retract). It follows from the long exact homotopy sequence that F is homotopically discrete; the space F is nonempty because the total space of the above fibration is obviously nonempty (at least one j exists). By 4.7 the Euler characteristic $\chi(F)$ (which in this case is just the number of components of F and is thus strictly positive) equals $\chi(X/T)$. Lemma 10.5 gives that $(X/T)^{\mathrm{h}\breve{T}} \cong F^{\mathrm{h}(\breve{T}/A)}$. Since the divisible group $\breve{T}/A$ must act in an essentially trivial way on F, which up to homotopy is a finite set, the homotopy fixed point set $F^{\mathrm{h}(\breve{T}/A)}$ is equivalent to F. This completes the proof.

8.11 Proposition. *Suppose that X is a p-compact group, $i: T \to X$ a maximal torus, and $f: A \to X$ a homomorphism from a p-compact torus A to X. Then there exists a homomorphism $g: A \to T$ such that $i \cdot g$ is conjugate to f.*

Proof: The components of the space $(X/T)^{\mathrm{h}A}$ correspond (10.6) to conjugacy classes of the desired homomorphisms g. The space X/T is $\mathbf{F}_p$-complete (5.7), so that (6.1) $(X/T)^{\mathrm{h}A}$ is equivalent to $(X/T)^{\mathrm{h}\breve{A}}$, where $\breve{A} \to A$ is a discrete approximation. The space $(X/T)^{\mathrm{h}\breve{A}}$ is nonempty by 5.7 and 4.7, since (8.10) X/T is $\mathbf{F}_p$-finite and $\chi(X/T) \neq 0$.

8.12 Existence of maximal tori. We are finally in a position to complete the existence argument from 1.2.

8.13 Theorem. *Any p-compact group X has a maximal torus.*

8.14 Lemma. *Let A be a p-compact torus, X a p-compact group, and $f: A \to X$ a monomorphism. If $\mathcal{C}_X(A)$ has a maximal torus then so does X.*

Proof: Let $i: T \to \mathcal{C}_X(A)$ be a maximal torus and let $j: T \to X$ be the composite of i with the monomorphism $\mathcal{C}_X(A) \to X$. The homomorphism $f: A \to X$ lifts to a homomorphism $g: A \to \mathcal{C}_X(A)$ or even (8.11) to $h: A \to T$. Since f is a monomorphism, so is h (7.3); the homomorphism g is central by 8.2 and h is central by inspection (8.5) An application of 8.3 now gives a short exact sequence $A \to T \to T/A$, and so 8.4 shows that the centralizer of T in X is equivalent to the centralizer of T in $\mathcal{C}_X(A)$ (because the centralizer of A in X is equivalent to the centralizer of A in $\mathcal{C}_X(A)$). This immediately gives the desired result.

Proof of 8.13: We will work by induction on the dimension (6.13) of X; the theorem is true in the case in which the dimension of X is zero (i.e., X is a discrete group). Assume that $\mathrm{cd}_{\mathbf{F}_p}(X) > 0$, choose (5.4, 7.3) a monomorphism $f\colon \mathbf{Z}/p \to X$, use 5.5 inductively to extend this to a homomorphism $\breve{g}\colon \breve{A} \to X$, where A is the p-discrete torus $\mathbf{Z}/p^\infty$, and then extend further (6.7) to a map $g\colon A \to X$, where A is the p-compact torus $\mathrm{Cl}(\breve{A})$. The homomorphism g is a monomorphism by 7.3. Let Y denote $\mathcal{C}_X(A)$. It is enough (8.14) to show that Y has a maximal torus. By 6.14 the quotient Y/A has smaller dimension than X and so by induction Y/A has a maximal torus $T/A \to Y/A$. Let Z/A be the centralizer of T/A in Y/A, so that Z/A isa p-compact toral group with T/A as its identity component. We can construct p-compact groups T, Z and homomorphisms $T \to Z \to Y$ by building a commutative diagram

$$\begin{array}{ccccc} BT & \longrightarrow & BZ & \longrightarrow & BY \\ \downarrow & & \downarrow & & \downarrow \\ B(T/A) & \longrightarrow & B(Z/A) & \longrightarrow & B(Y/A) \end{array}$$

in which both of the squares are homotopy fibre squares; T (resp. Z) is the "inverse image" of T/A (resp. Z/A) in Y. The fibre of the map $BT \to B(T/A)$ is BA, so it is clear that T is a p-compact torus and similarly that Z is a p-compact toral group with T as its identity component. Let $u\colon BT \to B(Z/A)$ and $v\colon BT \to (Y/A)$ be the maps provided by the above diagram and u/A, v/A the corresponding maps with domain $B(T/A)$. By 7.5 the maps

$$B(Z/A) \cong B\mathcal{C}_{Z/A}((T/A)) = \mathrm{Map}(B(T/A), B(Z/A))_{u/A} \to \mathrm{Map}(BT, B(Z/A))_u$$
$$B(Z/A) \cong B\mathcal{C}_{Y/A}((T/A)) = \mathrm{Map}(B(T/A), B(Y/A))_{v/A} \to \mathrm{Map}(BT, B(Y/A))_v$$

induced by $BT \to B(T/A)$ are equivalences. By using the homotopy fibre square of function spaces obtained by mapping BT into the right hand square above, it is now easy to conclude that $\mathcal{C}_Z(T) \to \mathcal{C}_Y(T)$ is an equivalence, which implies that $T \to Y$ is a maximal torus for Y.

§9. Finite generation of H^*BX

We now complete the project of 1.2 by defining the Weyl group W_X of a p-compact group X, constructing the normalizer $\mathcal{N}(T)$ of a maximal torus in X as well as the p-normalizer $\mathcal{N}_p(T)$ and using transfer in the fibration $B\mathcal{N}_p(T) \to BX$ to prove that H^*BX is noetherian. (It would also be possible to use transfer in the fibration $B\mathcal{N}(T) \to BX$.) Along the way we show that W_X has several properties parallel to those of the Weyl group of a compact Lie group (9.7); one difference is that W_X is in general a finite pseudoreflection group [**8**, §5] over $\mathbf{Q}_p$ rather than a finite reflection group over $\mathbf{Q}$.

There is some similarity between the constructions in this section and the methods used in [**8**] to study spaces Y such that H^*Y is a finitely generated polynomial ring; the connection is that if Y is $\mathbf{F}_p$-complete and H^*Y is a finitely generated polynomial ring, then (by the Eilenberg-Moore spectral sequence) $Y = BX$ for a p-compact group X. From this point of view [**8**] exploits the fact that if X is a p-compact group such that H^*BX is a polynomial ring then for p odd it is possible to use a special method [**8**, proof of 2.5] to construct a maximal torus for X in one step, instead of using the complicated inductive procedure of the present paper.

9.1 PROPOSITION. *If X is a connected p-compact group and $i\colon T \to X$ is a maximal torus then i is self-centralizing.*

PROOF: Let G denote $\mathcal{C}_X(T)$, so that G is a p-compact toral group with T as its identity component; it is necessary to show that the finite p-group $P = \pi_0 G$ is trivial. Choose a discrete approximation $\breve{G} \to G$. It is not hard to see that the p-discrete toral group $\breve{G}$ is a central extension by P of a discrete approximation $\breve{T}$ of T. Suppose that P is not trivial. Pick $g \in \breve{G}$, with $g \notin \breve{T}$, $g^p \in \breve{T}$. After replacing g if necessary by the product $g \cdot t^{-1}$ where $t \in \breve{T}$ satisfies $t^p = g^p$, we can assume that g is of order p. Let A denote the abelian subgroup $\breve{T} \times \langle g \rangle$ of $\breve{G}$. The homomorphism $G \to X$ is a monomorphism (5.2, 6.1) and so induces a map $j\colon A \to X$ with $\ker(j)$ trivial (7.3). The corresponding action (10.6) of A on X/T extends to an action of X and so is trivial up to homology (we are using here the fact that X is connected). In particular, $\Lambda(X/T;\langle g \rangle) = \chi(X/T)$ and so by 8.10, 4.5, and 5.7 the space $F = (X/T)^{\mathrm{h}\langle g \rangle}$ is $\mathbf{F}_p$-finite and has strictly positive Euler characteristic. By 10.5 and 5.7, 4.7 applies to the residual action of $\breve{T}$ on F and shows that the space $F^{\mathrm{h}\breve{T}} \cong (X/T)^{\mathrm{h}A}$ is nonempty. This implies that the monomorphism $A \to X$ lifts to a monomorphism $A \to T$, or even (6.11) to a group-theoretic injection $A \cong \breve{T} \times \langle g \rangle \to \breve{T}$. Such an injection is impossible (for instance, the group of elements of exponent p in $\breve{T} \times \langle g \rangle$ is larger than the group of elements of exponent p in $\breve{T}$).

9.2 DEFINITION: Suppose that X is a p-compact group and $i\colon T \to X$

a maximal torus. The *Weyl space* $\mathcal{W}_T(X)$ is defined to be the space obtained by replacing $Bi\colon BT \to BX$ by an equivalent fibration $BT' \to BX$ and considering the space of self-maps of BT' over BX.

In what follows we will assume that $BT \to BX$ is already a fibration and treat $\mathcal{W}_T(X)$ as the space of self-maps of BT over BX. It will become clear below (9.4) that $\mathcal{W}_T(X)$ does not depend in an essential way upon the choice $T \to X$ of maximal torus.

Observe that composition of maps gives $\mathcal{W}_T(X)$ the structure of an associative topological monoid.

9.3 Lemma. *Let X be a p-compact group and $i\colon T \to X$ a maximal torus. Then any self-map of BT over BX is a homotopy self-equivalence of BT.*

Proof: Let $\breve{T} \to T$ be a discrete approximation (6.8). By 6.11 any map $f\colon BT \to BT$ over BG lifts to a map $B\breve{T} \to B\breve{T}$ over BG, i.e., to a homomorphism $\breve{f}\colon \breve{T} \to \breve{T}$ over G. Such a homomorphism must have a trivial algebraic kernel because (7.3, 8.10) the homomorphism $T \to X$ is a monomorphism. Since any monic self-map of $(\mathbf{Z}/p^\infty)^r$ is an isomorphism, it follows that $\breve{f}$ is an isomorphism and thus (11.2) that f is a self homotopy equivalence.

9.4 Remark: Suppose that X is a p-compact group and that $i\colon T \to X$ and $j\colon T' \to X$ are maximal tori for X. By 8.11 there are maps $f\colon BT \to BT'$ and $g\colon BT' \to BT$ over BX; by 9.3 the composites $f \cdot g$ and $g \cdot f$ are homotopy equivalences. According to the dictionary (§3) this can be expressed by saying that T and T' are conjugate in X.

9.5 Proposition. *If X is a p-compact group with maximal torus $T \to X$, then $\mathcal{W}_T(X)$ is homotopically discrete and $\pi_0\mathcal{W}_T(X)$ is a finite group under composition. The order of $\pi_0\mathcal{W}_T(X)$ is $\chi(X/T)$.*

Proof: By definition $\mathcal{W}_T(X)$ is equivalent to $(X/T)^{\mathrm{h}T}$. Proposition 8.10 states that this space is homotopically discrete and computes the number of its components. Lemma 9.3 implies that $\pi_0\mathcal{W}_T(X)$ is a group.

9.6 Definition: If X is a p-compact group and $T \to X$ is a maximal torus, the finite group $\pi_0\mathcal{W}_T(X)$ is called the *Weyl group* of X and denoted $W_T(X)$, or W_X if T is understood.

Up to inner automorphism $W_T(X)$ does not depend on the choice of T (9.4).

9.7 Theorem. *Suppose that X is a connected p-compact group. Let $i\colon T \to X$ be a maximal torus for X.*

(1) *The rank s of T (6.5) is equal to the rational rank r of X (5.9).*

(2) *The action of* $W_T(X)$ *on* BT *induces a map*

$$W_T(X) \to \mathrm{Aut}(H^2_{\mathbf{Q}_p}BT) \cong \mathrm{GL}_s(\mathbf{Q}_p)$$

which is a monomorphism whose image is a finite subgroup of $\mathrm{GL}_s(\mathbf{Q}_p)$ *generated by pseudoreflections [**8**, §5].*

(3) *The map* $(Bi)^*: H^*_{\mathbf{Q}_p}BX \to (H^*_{\mathbf{Q}_p}BT)^{W_T(X)}$ *is an isomorphism.*

Proof: Let V denote $\mathrm{Map}(BT, BX)_{Bi}$ and let U denote the subspace of $\mathrm{Map}(BT, BT)$ consisting of all maps f such that $(Bi)\cdot f$ is homotopic to Bi. Composition with Bi gives a map $U \to V$, and the homotopy fibre of this map is $\mathcal{W}_T(X)$. The space U is equivalent to a disjoint union of copies of BT (8.5) and, since i is self-centralizing (9.1), the space V is equivalent to a single copy of BT; as in the proof of 8.10 the map $U \to V$ becomes an equivalence when restricted to any component of its domain. It follows that $W_T(X)$ is isomorphic to $\pi_0 U$, and thus (9.3) that the action of $W_T(X)$ on BT gives an injection from $W_T(X)$ to the group $\pi_0\,\mathrm{Aut}(BT)$ of homotopy classes of self homotopy equivalences of BT. This group $\pi_0\,\mathrm{Aut}(BT)$ is isomorphic to $\mathrm{GL}_s(\mathbf{Z}_p)$ and acts faithfully on $H^2_{\mathbf{Q}_p}BT$. This gives the monomorphism $W_T(X) \to \mathrm{GL}_s(\mathbf{Q}_p)$.

Let $R_T = H^*_{\mathbf{Q}_p}BT$ and $R_X = H^*_{\mathbf{Q}_p}BX$. The cohomology ring R_X is a polynomial ring on r generators (5.9) and R_T is a polynomial ring on s generators. The Serre spectral sequence of the fibration $X/T \to BT \to BX$ shows that R_T is finitely generated as a module over R_X (the argument here uses the fact that R_X is noetherian), and transfer in this fibration (9.14, 8.10) shows that the map $R_X \to R_T$ is injective. This implies by a standard algebraic argument involving transcendence degrees [**1**, 11.5] (or Krull dimension) that $r = s$. The fact that $r = s$ and the fact that R_T is finitely generated as a module over R_X together imply that R_T is *free* as a module over R_X on, say, t generators (this freeness condition is a consequence of the formula in [**2**] relating depth and projective dimension). Thus the Eilenberg-Moore spectral sequence of the fibration $BT \to BX$ collapses, $H^*_{\mathbf{Q}_p}(X/T) \cong \mathbf{Q}_p \otimes_{R_X} R_T$ is concentrated in even degrees, and t equals the Euler characteristic $\chi(X/T)$ (which is also by 9.5 equal to $|W|$, where $W = W_T(X)$). The number t is also clearly the degree of the graded fraction field F_T of R_T over the graded fraction field F_X of R_X. Since $F_X \subset (F_T)^W$ and the degree of F_T over F_X equals $|W|$, Galois theory shows that $F_X = (F_T)^W$. Then $R_X \subset (R_T)^W \subset F_X$, then extension $R_X \subset (R_T)^W$ is clearly integral [**1**, 5.1], and the equality $R_X = (R_T)^W$ follows from the fact that as a polynomial algebra R_X is integrally closed in its field of fractions. This proves (3). Statement (2) now comes down to a classical result [**8**, 5.1]:

the ring of polynomial invariants (viz. $(R_T)^W$) of a representation of a finite group W over a field of characteristic zero is itself a polynomial ring (R_X) if and only if W is generated by pseudoreflections.

9.8 Definition: Let X be a p-compact group and $i\colon T \to X$ a maximal torus. The *normalizer* of T denoted $\mathcal{N}(T)$ is the loop space such that $B\mathcal{N}(T)$ is the Borel construction of the action of $\mathcal{W}_T(X)$ on BT. The *p-normalizer* of T is the p-compact group such that $B\mathcal{N}_p(T)$ is the Borel construction of the action of $\mathcal{W}_T(X)_p$ on BT, where $\mathcal{W}_T(X)_p$ is a union of components of $\mathcal{W}_T(X)$ corresponding to a p-Sylow subgroup of $\pi_0\mathcal{W}_T(X) = W_T(X)$.

The space $B\mathcal{N}(T)$ is up to homotopy the total space of a fibration over $BW_T(X)$ with fibre BT; similarly, $B\mathcal{N}_p(T)$ is up to homotopy the total space of a fibration over BW_p with fibre BT, where $W_p \subset W_T(X)$ is a p-Sylow subgroup. Since the action of $\mathcal{W}_T(X)$ on BT respects the map $BT \to X$, the homomorphism $T \to X$ extends to $T \to \mathcal{N}_p(T) \to \mathcal{N}(T) \to X$.

9.9 Proposition. *If X is a p-compact group and $T \to X$ is a maximal torus, then the homomorphism $\mathcal{N}_p(T) \to X$ is a monomorphism.*

Proof: Assume first that X is connected, and choose a discrete approximation $\breve{G} \to \mathcal{N}_p(T)$ of $\mathcal{N}_p(T)$. The group $\breve{G}$ is an extension of $\breve{T} = (\mathbf{Z}/p^\infty)^r$ by a p-Sylow subgroup W_p of $W_T(X)$ (here r is the rank of T, which is also the rational rank of X). By 7.3 the desired result is equivalent to the statement that $\ker(j)$ is trivial, where j is the composite $\breve{G} \to \mathcal{N}_p(T) \to X$. It is clear that $\ker(j) \cap \breve{T}$ is trivial, since (8.10) $T \to X$ is a monomorphism. It follows directly from 9.7(1) that conjugation in $\breve{G}$ provides an injective homomorphism $W_p \to \mathrm{GL}_r(\mathbf{Z}_p) \cong \mathrm{Aut}(\breve{T})$. This implies that any nontrivial normal subgroup of $\breve{G}$ intersects $\breve{T}$ nontrivially; since $\ker(j)$ is a normal subgroup of $\breve{G}$ (7.2), it follows that $\ker(j)$ is trivial.

For the general case, let X_0 be the identity component of X. Because T is connected the homomorphism $T \to X$ lifts to a homomorphism $T \to X_0$. Let N be the p-normalizer of T in X_0. The homomorphism $\mathcal{N}_p(T) \to X$ induces a map $\mathcal{N}_p(T) \to \pi_0 X$ and it is not hard to see that under this map the inverse image in $\mathcal{N}_p(T)$ of the trivial element of $\pi_0 X$ is essentially N. By a kernel argument (7.3) the homomorphism $\mathcal{N}_p(T) \to X$ is a monomorphism if and only if the restricted homomorphism $N \to X$ is a monomorphism , or even (by inspection) if and only if the homomorphism $N \to X_0$ is a monomorphism. This last was shown to be the case above.

Proof of 2.3: Let T be a maximal torus for X and $i: \mathcal{N}_p(T) \to X$ the homomorphism from the p-normalizer of T to X. Since $X/\mathcal{N}_p(T)$ is $\mathbf{F}_p$-finite by 9.9, and X/T is a regular covering space of $X/\mathcal{N}_p(T)$ in which (9.5) the cardinality of the cover is the p-primary part of the Euler characteristic of X/T, multiplicativity of the Euler characteristic (4.14) shows that $\chi(X/\mathcal{N}_p(T))$ is relatively prime to p. Since $H^*B\mathcal{N}_p(T)$ is noetherian by 12.1, the transfer map (9.13) for the fibration $X/\mathcal{N}_p(T) \to B\mathcal{N}_p(T) \to BX$ combines with 2.4 to finish off the proof.

9.10 A few consequences. We note a few consequences of the material in this section, although we will not follow them up now. First of all, if X is a connected p-compact group of rank s, then by 9.7(2) the Weyl group W_X, considered as a subgroups of $\mathrm{GL}_s(\mathbf{Q}_p)$, must be a product of irreducible pseudoreflection groups over $\mathbf{Q}_p$; these irreducible pseudoreflection groups are known [**6**] [**8**, §5]. This places severe restrictions on the cohomology algebra $H^*_{\mathbf{Q}_p}BX$; basically, this algebra must be a tensor product of the algebras enumerated in [**6**].

Secondly, the proof of 2.3 shows more or less immediately that the decomposition theorem of [**10**] applies to a p-compact group X (see [**10**, 1.2 and 6.1]; condition (2) of 6.1 is satisfied for BX because for any elementary abelian p-group V and homomorphism $f: V \to X$, the group $\pi_0\mathcal{C}_X(f(V))$ is by 5.1 a finite p-group). Thus BX can be expressed up to $\mathbf{F}_p$-equivalence as a homotopy direct limit of classifying spaces of centralizers in X of nontrivial elementary abelian "subgroups".

Finally, Theorem 2.3 shows that the notion of monomorphism between p-compact group has a simple algebraic interpretation..

9.11 Proposition. *A homomorphism $f: X \to Y$ of p-compact groups is a monomorphism if and only if under $(Bf)^*$ the ring H^*BX is a finitely generated module over H^*BY.*

Proof: First note that if Z is a p-compact group and M an $\mathbf{F}_p$ vector space which is a module over $\pi_1 BX$, then $H^*(BX, M)$ is finitely generated as a module over H^*BX (cf. proof of 12.5). Now if f is a monomorphism, the Serre spectral sequence of $Y/X \to BX \to BY$ shows that H^*BX is finitely generated over H^*BY: by the above remark the E^2 term is finitely generated over H^*BY and because H^*BY is noetherian this property passes to E^∞. For any f the Serre spectral sequence of $Y \to Y/X \to BX$ similarly shows that $H^*(Y/X)$ is finitely generated as a module over H^*BX. The action of H^*BX on $H^*(Y/X)$ factors through the quotient ring $\mathbf{F}_p \otimes_{H^*BY} H^*BX$, so if this quotient ring is finite, i.e. if H^*BX is finitely generated as a module over H^*BY, $H^*(Y/X)$ must be finite.

9.12 The transfer. We need a version of the Becker-Gottlieb transfer [**3**] which does not seem to appear in the literature, and so we sketch the construction.

9.13 THEOREM. *Let B be a connected space and $f\colon E \to B$ a fibration such that the fibre F is $\mathbf{F}_p$-finite. Then there exists a (transfer) map $f_*\colon H^*E \to H^*B$ of modules over H^*B such that $f_* \cdot f^*$ is multiplication by $\chi(F)$.*

SKETCH OF PROOF: Suppose that C is a non-negatively graded chain complex over $\mathbf{F}_p$. Let D be the non-positively graded dual chain complex $\operatorname{Hom}(C, \mathbf{F}_p)$, and $\operatorname{Hom}(C, C)$ the chain complex with

$$\operatorname{Hom}(C, C)_n = \prod \operatorname{Hom}_{\mathbf{F}_p}(C_i, C_{n+i})$$

and with differential so that $H_i \operatorname{Hom}(C, C)$ is the group of chain homotopy classes of degree i chain maps from C to itself. There is a chain map $h\colon D \otimes C \to \operatorname{Hom}(C, C)$ which is an isomorphism if C is a finite chain complex and therefore (since both constructions involved in h respect chain homotopy) a weak equivalence (i.e., homology isomorphism) if H_*C is finite dimensional. There is a map $u\colon \mathbf{F}_p \to \operatorname{Hom}(C, C)$ determining the identity map of C and an evaluation or trace map $v\colon D \otimes C \to \mathbf{F}_p$ sending $g \otimes c$ to $\pm g(c)$. If C is a finite chain complex with a zero differential, a calculation shows that $v \cdot h^{-1} \cdot u$ is multiplication by the Euler characteristic of H_*C (this is the graded trace of the identity map of C), and the same result therefore holds if H_*C is finite dimensional, as long as h^{-1} is interpreted as a chain homotopy inverse for h. If C is the differential graded coalgebra of singular chains on an $\mathbf{F}_p$-finite space X, with diagonal $\Delta\colon C \to C \otimes C$ and augmentation $\epsilon\colon C \to \mathbf{F}_p$, the composite

$$\mathbf{F}_p \xrightarrow{u} \operatorname{Hom}(C, C) \xrightarrow{h^{-1}} D \otimes C \xrightarrow{\operatorname{id} \otimes \Delta} D \otimes C \otimes C \xrightarrow{v \otimes \operatorname{id}} C \xrightarrow{\epsilon} \mathbf{F}_p$$

is then multiplication by $\chi(X)$. Suppose that G is a discrete group acting on X. Let R be the singular chain complex of EG (this is a free resolution of $\mathbf{F}_p$ over $\mathbf{F}_p[G]$), and let S be a free resolution of $\operatorname{Hom}(C, C)$ over $\mathbf{F}_p[G]$. (By a *free resolution* of a chain complex which is not bounded below we mean a resolution in the usual sense which in addition is a union of finite free chain complexes.) All of the maps in the above display except for h^{-1} are equivariant in a natural way, and this slight deficit can be repaired by replacing $u\colon \mathbf{F}_p \to \operatorname{Hom}(C, C)$ by $u'\colon R \to S$ and exploiting the fact that S is a free resolution to construct an equivariant map $S \to D \otimes C$ which in a suitable sense is a chain homotopy inverse for h.

For any chain complex A over $\mathbf{F}_p[G]$ let $H_*(G,A)$ denote the homology of the chain complex $\mathbf{F}_p\otimes_{\mathbf{F}_p[G]}(R\otimes A)$; the formula shows that $H_*(G,A)$ is naturally a graded cap product module over $H^*BG\cong \mathrm{Ext}^*_{\mathbf{F}_p[G]}(\mathbf{F}_p,\mathbf{F}_p)$. It is clear that $H_*(G,C)$ is naturally isomorphic to the homology of the Borel construction $X_{\mathrm{h}G}$, and so the above composite $R\to C$ induces a map $H_*BG\cong H_*(B,R)\to H_*(B,C)\cong H_*X_{\mathrm{h}G}$ which upon dualization gives the desired transfer for the case in which the base $B=BG$ is a space of type $K(\pi,1)$ (cf. proof of 10.4). The general case can be immediately reduced to this special one by using the theorem of Kan and Thurston [**14**] to find a $K(\pi,1)$ homological approximation to B.

9.14 Remark: We will indicate the modifications that have to be made in the above argument to obtain a transfer for $H^*_{\mathbf{Q}_p}(-)$ under the assumption that F is $\mathbf{F}_p$-finite. The main changes are to work with chain complexes over $\mathbf{Z}_p$ with finitely generated homology instead of with chain complexes over $\mathbf{F}_p$, to take all tensor products over $\mathbf{Z}_p$, and to let $D=\mathrm{Hom}_{\mathbf{Z}_p}(C,\mathbf{Z}_p)$. The chain complex C associated to a space X is then $C=\varprojlim C(n)$ where $C(n)$ is the singular chain complex of X with $\mathbf{Z}/p^n$ coefficients. There is a slight complication in that the diagonal map of X induces a map $C\to C\hat{\otimes}C$, where $C\hat{\otimes}C=\varprojlim C(n)\otimes C(n)$, but this is circumvented by noticing that if X is $\mathbf{F}_p$-finite the map $C\otimes_{\mathbf{Z}_p}C\to C\hat{\otimes}C$ is a weak equivalence.

§10. Appendix: Homotopy fixed point sets

In this section we will recall the notion of *homotopy fixed point set* and describe some of its properties. Throughout this section (except in Remark 10.8) G denotes a discrete group; EG is some functorial CW-complex on which G acts freely and cellularly. Suppose that X is a G-space.

10.1 Definition: The *homotopy fixed point set* of the action of G on X, denoted $X^{\mathrm{h}G}$, is the equivariant mapping space $\mathrm{Map}^G(EG,X)$.

10.2 Remarks: The space $X^{\mathrm{h}G}$ can be interpreted as the *homotopy inverse limit* of the action of G on X [**5**, XI]. A G-map $f\colon X\to Y$ induces a map $f^{\mathrm{h}G}\colon X^{\mathrm{h}G}\to Y^{\mathrm{h}G}$; if f is an ordinary (non-equivariant) homotopy equivalence, then $f^{\mathrm{h}G}$ is a homotopy equivalence. A group homomorphism $K\to G$ induces a restriction map $X^{\mathrm{h}G}\to X^{\mathrm{h}K}$. If G acts trivially on X, then $X^{\mathrm{h}G}$ is $\mathrm{Map}(BG,X)$.

10.3 Remark: A *proxy action* of G on X is a space Y homotopy equivalent to X together with an action of G on Y. Standard homotopy theoretic constructions often give proxy actions of this type. To avoid too much notation, if there is a specific proxy action of G on X under consideration we will usually write X^{hG} for the associated homotopy fixed point set instead of introducing a symbol for the proxy space Y and writing Y^{hG}. If the proxy action is parameterized by some symbol α, we will write X^{hG}_{α}. In many cases we will even refer to the proxy action as an action.

Let X_{hG} denote the Borel construction $EG \times_G X$ (this is sometimes called the *homotopy orbit space* of the action of G on X). It is easy to see that the space X^{hG} is isomorphic to the space of sections of the fibration $X_{hG} \to (*)_{hG} = BG$. The following lemma asserts conversely that the space of sections of any such fibration can be interpreted as a homotopy fixed point set.

10.4 LEMMA. *Let $W \to BG$ be a fibration with fibre F. Then there is a proxy action of G on F such that F^{hG} is homotopy equivalent to the space of sections of $W \to BG$.*

PROOF: Let $\tilde{W}$ denote the pullback of $W \to BG$ over the universal covering fibration $EG \to BG$. The action of G on EG induces an action of G on $\tilde{W}$. It is clear that $\tilde{W}$ is homotopy equivalent to F, and not hard to check that $\tilde{W}^{hG}$ is homotopy equivalent to the space of sections of $W \to BG$.

The following lemma describes a transitivity property of the homotopy fixed point set construction.

10.5 LEMMA. *Let X be a G-space and K a normal subgroup of G. Then there is a proxy action of G/K on X^{hK} such that X^{hG} is homotopy equivalent to $(X^{hK})^{h(G/K)}$.*

PROOF: The actions of G on EG and on X induce an action of G/K on the mapping space $\mathrm{Map}^K(EG, X)$. This mapping space is homotopy equivalent to $\mathrm{Map}^K(EK, X) \cong X^{hK}$, and it is easy to verify that the homotopy fixed point set of the indicated action of G/K is equivalent to X^{hG}.

10.6 LEMMA. *Let $f\colon X \to Y$ be a map of G-spaces which is an ordinary (non-equivariant) fibration. Assume that Y is connected, and that the homotopy fibre of f is F. Then the map*

$$f^{hG}\colon X^{hG} \to Y^{hG}$$

induced by f is a fibration. Moreover, for any point $y \in Y^{hG}$ there is a proxy action $\alpha(y)$ of G on F such that the (homotopy) fibre of f^{hG} over y is equivalent to $F^{hG}_{\alpha(y)}$.

Remark: If $Y^{\mathrm{h}G}$ is empty there is *no* action of G on F of any kind associated to the fibration $X \to Y$. In general, the proxy action $\alpha(y)$ of G on F depends in an essential way on the choice of $y \in Y^{\mathrm{h}G}$.

Proof of 10.6: It is clear for general reasons that $f^{\mathrm{h}G}$ is a fibration. A point $y \in Y^{\mathrm{h}G}$ amounts to an equivariant map $y\colon EG \to Y$. Pulling the fibration f back over this map gives a fibration $y^*(X) \to EG$ with homotopy fibre F. Since EG is contractible the total space $y^*(X)$ is equivalent to F. The actions of G on all of the other spaces involved induce an action of G on $y^*(X)$, and it is straightforward to check that the homotopy fixed point set of this action is equivalent to the (homotopy) fibre of $f^{\mathrm{h}G}$ over y.

The following statement is a homotopy-theoretic analogue of Shapiro's lemma and is easy to derive directly from the definitions.

10.7 Lemma. *Suppose that X is a G-space and that K and L are subgroups of G with $K \subset L$. Let A and B denote the spaces* $\mathrm{Map}(G/K, X)$ *and* $\mathrm{Map}(G/L, X)$*, each provided with the diagonal G action, and $B \to A$ the equivariant map derived from $G/K \to G/L$. Then the induced map $B^{\mathrm{h}G} \to A^{\mathrm{h}G}$ is equivalent to the restriction map $X^{\mathrm{h}L} \to X^{\mathrm{h}K}$.*

10.8 Remark: It will be convenient to generalize the notion of homotopy fixed point set to the case in which G is a loop space. In this situation a "G-space X" is *defined* to be a fibration $X_{\mathrm{h}G} \to BG$ with X as the fibre, and an "equivariant map" $X \to Y$ to be a map of spaces together with an extension to a map $X_{\mathrm{h}G} \to Y_{\mathrm{h}G}$ of spaces over BG. The homotopy fixed point set $X^{\mathrm{h}G}$ of such an "action" is then the space of sections of $X_{\mathrm{h}G} \to BG$. We will make use of the evident extension of Lemma 10.5 to this wider setting; it is not hard to verify it. The other results of this section can also be interpreted in ways that allow them to extend.

§11. Appendix: $\mathbf{F}_p$-complete spaces

Because of the way in which the machinery in §4 works, we have to deal with some technical properties of the $\mathbf{F}_p$-completion functor $\mathbf{C}_{\mathbf{F}_p}(-)$ of [**5**]. The purpose of this section is gather the information required to do this. The references for the material in this section are [**5**] and [**4**].

11.1 The $\mathbf{F}_p$-completion functor. Bousfield and Kan [**5**] construct a functor $\mathbf{C}_{\mathbf{F}_p}(-)$ on the category of spaces, called the *$\mathbf{F}_p$-completion functor*, together with a natural map $\epsilon_X\colon X \to \mathbf{C}_{\mathbf{F}_p}(X)$ for any space X. The functor has the following basic properties.

11.2 Proposition. *The functor $\mathbf{C}_{\mathbf{F}_p}(-)$ preserves disjoint unions up to homotopy. If X is i-connected ($i \geq 0$) then $\mathbf{C}_{\mathbf{F}_p}(X)$ is also i-connected.*

If $f: X \to Y$ induces an isomorphism $H_*X \cong H_*Y$ then $\mathbf{C}_{\mathbf{F}_p}(f)$ is a homotopy equivalence.

11.3 Definition: A space X is $\mathbf{F}_p$-*good* if $H_*\epsilon_X$ is an isomorphism and $\mathbf{F}_p$-*complete* if ϵ_X is a homotopy equivalence.

11.4 Remark: In [**5**] there is a proof that a space X is $\mathbf{F}_p$-good if and only if $\mathbf{C}_{\mathbf{F}_p}(X)$ is $\mathbf{F}_p$-complete, or in fact if and only if $\mathbf{C}_{\mathbf{F}_p}(X)$ is $\mathbf{F}_p$-good. If X is 1-connected and has finitely generated homotopy groups then X is $\mathbf{F}_p$-good, $\mathbf{C}_{\mathbf{F}_p}(X)$ is $\mathbf{F}_p$-complete, and $\pi_i\mathbf{C}_{\mathbf{F}_p}(X) \cong \mathbf{Z}_p \otimes \pi_i X$. More generally, any nilpotent space is $\mathbf{F}_p$-good. If G is a finite p-group then BG is $\mathbf{F}_p$-complete.

The algebraic notion of "nilpotent action" is an important one for understanding the $\mathbf{F}_p$-completion functor in greater generality.

11.5 Definition: Let G be a (discrete) group, $I \subset \mathbf{Z}[G]$ the augmentation ideal, and M an abelian group with an action of G. The action of G on M is said to be *nilpotent* if there exists some integer k such that $I^k \cdot M = 0$, or, equivalently, if and only if there exist a finite G-filtration of M such that G acts trivially on the filtration quotients.

11.6 Remark: If G is a finite p-group then the augmentation ideal of $\mathbf{F}_p[G]$ is nilpotent as an ideal, so that *any* action of G on an $\mathbf{F}_p$ vector space is nilpotent.

11.7 Proposition. *(Fibre Lemma) Let $F \to E \to B$ be a fibration over the connected pointed space B. Assume that the monodromy action of $\pi_1 B$ on $H_i F$ is nilpotent for each $i \geq 0$. Then the induced sequence $\mathbf{C}_{\mathbf{F}_p}(F) \to \mathbf{C}_{\mathbf{F}_p}(E) \to \mathbf{C}_{\mathbf{F}_p}(B)$ is also a fibration sequence.*

By 11.4, Proposition 11.7 has the following immediate consequence.

11.8 Theorem. *Suppose that X is a connected space, and let $\tilde{X}$ denote the universal cover of X. If $\pi_1 X$ is a nilpotent group and and the natural action of $\pi_1 X$ on each of the homology groups $H_i X$ ($i \geq 0$) is nilpotent, then X is $\mathbf{F}_p$-good.*

11.9 Corollary. *Suppose that X is a connected space such that $\pi_1 X$ is a finite p-group. Then X is $\mathbf{F}_p$-good, ΩX is $\mathbf{F}_p$-good, and $\Omega\mathbf{C}_{\mathbf{F}_p}(X) \cong \mathbf{C}_{\mathbf{F}_p}(\Omega X)$.*

Corollary 11.9 implies (11.4) that under the given hypotheses the spaces $\mathbf{C}_{\mathbf{F}_p}(X)$ and $\mathbf{C}_{\mathbf{F}_p}(\Omega X)$ are $\mathbf{F}_p$-complete.

Proof of 11.9: The fact that X is $\mathbf{F}_p$-good is a consequence of 11.8, since the finite p-group $\pi_1 X$ is nilpotent as a group and acts nilpotently (11.6) on the $\mathbf{F}_p$ homology of the universal cover of X. The space ΩX is $\mathbf{F}_p$-good because, again by 11.8, any loop space is $\mathbf{F}_p$-good. The last statement follows from an application of 11.7 to the path fibration over X, since (11.2) the $\mathbf{F}_p$-completion of the path space on X is contractible.

11.10 $\mathbf{F}_p$-local spaces. In order to handle homotopy inverse limits of $\mathbf{F}_p$-complete spaces, it is convenient to introduce a class of spaces studied by Bousfield [**4**], called $\mathbf{F}_p$-local spaces.

11.11 Definition: A space X is $\mathbf{F}_p$*-local* if any $\mathbf{F}_p$-equivalence $A \to B$ induces a homotopy equivalence $\mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$.

It is an immediate consequence of the definition that the following is true.

11.12 Proposition. *A map $X \to Y$ of $\mathbf{F}_p$-local spaces is an equivalence if and only if it induces an isomorphism $H_*X \to H_*Y$. The class of $\mathbf{F}_p$-local spaces is closed under the process of taking homotopy inverse limits. A space X is $\mathbf{F}_p$-local if and only if each component of X is $\mathbf{F}_p$-local.*

11.13 Remark: In particular, if $f\colon E \to B$ is a fibration with $\mathbf{F}_p$-local fibres, then the space of sections of f is $\mathbf{F}_p$-local; if G is a discrete group acting on a $\mathbf{F}_p$-local space X, then $X^{\mathrm{h}G}$ is $\mathbf{F}_p$-local. If X is any space, then $\mathbf{C}_{\mathbf{F}_p}X$ is $\mathbf{F}_p$-local (because of the way $\mathbf{C}_{\mathbf{F}_p}X$ is constructed as a homotopy inverse limit [**5**]). If X is $\mathbf{F}_p$-local and $\mathbf{F}_p$-good, then ϵ_X is an $\mathbf{F}_p$-equivalence between $\mathbf{F}_p$-local spaces and is therefore a homotopy equivalence; in other words, a space X is $\mathbf{F}_p$-complete if and only if X is both $\mathbf{F}_p$-local and $\mathbf{F}_p$-good.

11.14 Proposition. *If X is a connected $\mathbf{F}_p$-local space such that $\pi_1 X$ is finite, then $\pi_1 X$ is a p-group.*

By 11.9, this proposition implies that if X is a connected $\mathbf{F}_p$-local space and $\pi_1 X$ is finite then X is $\mathbf{F}_p$-complete.

Proof of 11.14: Suppose that $\pi_1 X$ is not a finite p-group. Find a nontrivial element $x \in \pi_1 X$ of order q prime to p and represent x by a Moore space map $f\colon S^1 \cup_q e^2 \to X$. Let X' be the mapping cone of f. The map $X \to X'$ is an $\mathbf{F}_p$-equivalence and so (11.11) induces a homotopy equivalence $\mathrm{Map}(X', X) \to \mathrm{Map}(X, X)$. This implies that up to homotopy the identity map of X factors through the map f which is not monic on π_1. This contradiction proves the result.

§12. Appendix: Finite generation for toral groups

In this section we will give a proof of the following two results (see §6 for a discussion of the terms involved).

12.1 Proposition. *If G is a p-discrete toral group then H^*BG is a noetherian ring.*

12.2 Proposition. *If G is a p-discrete toral group, K is a finite p-group and $f\colon K \to G$ is an algebraic monomorphism, then the map $(Bf)^*$ makes H^*BK into a finitely generated module over H^*BG.*

For the convenience of the reader we begin with proofs of some standard results.

12.3Lemma. *Suppose that $R_1 \to R_2$ is a map of graded commutative rings, that R_1 is noetherian, and that R_2 is finitely generated as a module over R_1. Then R_2 is noetherian.*

Proof: Any ascending chain of ideals inside R_2 is in particular an ascending chain of R_1-submodules of R_2, and so must eventually become constant.

12.4 Proposition. *[**12**] [**13**] If G is a finite p-group, then H^*BG is noetherian.*

Proof: To prove that H^*BG is noetherian, we work by induction on $|G|$ and assume that G is nontrivial and that the result is known for all p-groups of order less than $|G|$. By elementary group theory it is possible to construct a short exact sequence

$$\{1\} \to \langle g \rangle \to G \to \bar{G} \to \{1\}$$

where $g \neq 1$ is an element in the center of G with $g^p = 1$. The Lyndon-Hochschild-Serre spectral sequence of this extension is a spectral sequence of bigraded algebras which has the form

$$E_2^{i,j} = H^i(B\bar{G}) \otimes H^j(B\langle g \rangle) \Rightarrow H^{i+j}BG\,.$$

Let $x \in H^1B\langle g \rangle \cong E_2^{0,1}$ be a nonzero element and let $y = \beta(x) \in H^2B\langle g \rangle \cong E_2^{0,2}$ be its Bockstein; the ring $H^*B\langle g \rangle$ is spanned by the set $\{y^i x \mid i \geq 0\}$. For positional reasons x is *transgressive*, i.e., all differentials on x are zero except possibly for the one which maps x to the appropriate group along the $j = 0$ axis of the spectral sequence. By the Kudo transgression theorem all of the classes in $H^*B\langle g \rangle \cong E_2^{0,*}$ which result from applying Steenrod operations to x are transgressive; in particular, $d^r(y^{p^n}) = 0$ for $r \leq 2p^n$ and $d^{2p^n+1}(y^{p^n}) \in E_{2p^n+1}^{2p^n+1,0}$. Let A_r be the graded ring $E_r^{*,0}$ and I_r the kernel of the natural surjection $H^*B\bar{G} \cong A_2 \to A_r$. The sequence $\{I_r\}_{r\geq 2}$ is an ascending chain of ideals inside the noetherian ring A_2 and so must eventually become constant; consequently for n large enough $A_{2p^n+1} \cong A_{2p^n+2}$ and hence $d^{2p^n+1}(y^{p^n}) = 0$. Choose such a large n and let $z = y^{p^n}$ so $z \in E_2$ survives to E_∞. The above E_2 term is a finitely generated module over the noetherian ring $A_2[z]$ and so by induction each E_r term is also a finitely generated module over $A_2[z]$. Let $B_r \subset E_2$ be the (bigraded) submodule of E_2 consisting of elements which become boundaries in E_r. The sequence $\{B_r\}_{r\geq 2}$ is an ascending chain of $A_2[z]$-submodules of E_2 and so must eventually become constant. This implies that for sufficiently large r, $E_r \cong E_{r+1}$ and hence that E_∞ is a finitely generated module over $A_2[z]$. It follows easily that if $\tilde{z} \in H^*BG$ is an element which restricts to z under the edge homomorphism of the spectral sequence, then H^*BG is a finitely generated module over $A_2[\tilde{z}]$. Thus by 12.3 the ring H^*BG is noetherian.

Recall that if G is a group and M a vector space over $\mathbf{F}_p$ with an action of G, then the local coefficient cohomology $H^*(BG, M)$ is a graded module over H^*BG.

12.5 Lemma. *[**12**] If G is a finite p-group and M a finite dimensional vector space over $\mathbf{F}_p$ with an action of G, then $H^*(BG, M)$ is finitely generated as a module over H^*BG.*

Proof: We will use induction on the dimension of M. Assume that M is nontrivial, and that the result is known to be true for all actions of G on $\mathbf{F}_p$ vector spaces of lower dimension. By 11.6 there is a G-submodule $M' \subset M$ such that G acts trivially on M/M' and $\dim_{\mathbf{F}_p} M/M' = 1$. The long exact cohomology sequence of the short exact coefficient sequence

$$\{0\} \to M' \to M \to M/M' \to \{0\}$$

is a sequence of graded modules over H^*BG, so the fact that H^*BG is noetherian immediately gives the desired result.

12.6 Corollary. *If $G_1 \to G_2$ is an algebraic monomorphism of finite p-groups, then the induced cohomology map $H^*BG_2 \to H^*BG_1$ makes H^*BG_1 into a finitely generated module over H^*BG_2.*

Proof: By Shapiro's lemma, there is an isomorphism from H^*BG_1 to $H^*(BG_2, \mathbf{F}_p[G_2/G_1])$, and it is easy to check that under this isomorphism the H^*BG_2 module structure of H^*BG_1 of interest here agrees with the H^*BG_2 module structure of $H^*(BG_2, \mathbf{F}_p[G_2/G_1])$ treated in 12.5. The corollary then follows from 12.5.

Proof of 12.1: Write $G = \cup_{m\geq n} G_m$ as in the proof of 6.19, so that, in the notation of that proof, there are maps of exact sequences

$$\begin{array}{ccccccccc} \{1\} & \longrightarrow & A_m & \longrightarrow & G_m & \longrightarrow & P & \longrightarrow & \{1\} \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \{1\} & \longrightarrow & A_{m+1} & \longrightarrow & G_{m+1} & \longrightarrow & P & \longrightarrow & \{1\} \end{array}$$

where P is a finite p-group, $A_m \cong (\mathbf{Z}/p^m)^s$ and $A = \cup_m A_m \cong (\mathbf{Z}/p^\infty)^s$. Let $\mathrm{Cl}(G)$ denote the closure of G (6.9). By construction the homotopy fibre of the map $BG_m \to B\,\mathrm{Cl}(G) \cong \mathbf{C}_{\mathbf{F}_p}(BG)$ $(m \geq n)$ is $K((\mathbf{Z}_p)^s, 1)$ (i.e., the p-completion of the s-torus) and the commutative diagram

$$\begin{array}{ccc} BG_m & \longrightarrow & BG_{m+1} \\ \downarrow & & \downarrow \\ B\,\mathrm{Cl}(G) & \xrightarrow{=} & B\,\mathrm{Cl}(G) \end{array}$$

induces on fibres a map

$$K((\mathbf{Z}_p)^s, 1) \to K((\mathbf{Z}_p)^s, 1)$$

which is multiplication by p on π_1. Notice in particular that this map on fibres induces the zero map on H_*.

Let $E^*(m) = \{E^r_{i,j}(m)\}$ be the homology Serre spectral sequence of $BG_m \to B\,\mathrm{Cl}(G)$. The above commutative diagrams induce maps

$$E^r_{i,j}(m) \to E^r_{i,j}(m+1) \quad (m \geq n,\ r \geq 2)$$

which are zero unless $j = 0$. At E^2 the map $E^2_{*,0}(m) \to E^2_{*,0}(m+1)$ is the identity map of $H_*B\,\mathrm{Cl}(G) \cong H_*BG$. Pick $x \in H_*B\,\mathrm{Cl}(G)$. It is possible that in $E^2(n)$ the class x supports a non-zero d^2, but naturality shows that $d^2x = 0$ in $E^2(m)$ for $m \geq n+1$. Now in $E^2(n+1)$ the class x might support a non-zero d^3, but naturality shows again that that $d^3x = 0$ in $E^3(m)$ for $m \geq n+2$. By induction, then,

$$d^2x = d^3x = \cdots = d^sx = 0$$

in $E^*(m)$ for $m \geq n+s-1$. Since these spectral sequences are concentrated in a vertical band of height s, (i.e., $E^r_{i,j}(m) = 0$ for $j > s$), the element x survives to $E^\infty(m)$ for $m \geq n+s-1$. In other words, for $m \geq n+s-1$ the map $H_*BG_m \to H_*B\,\mathrm{Cl}(G) \cong H_*BG$ is onto.

Let $F^j(m)$ be the j'th stage in the decreasing filtration of H_*BG_m provided by the spectral sequence $E^r(m)$, so that $F^0(m) = H_*BG_m$, $F^j(m)/F^{j+1}(m) \cong E^\infty_{*,j}(m)$, and $F^j(m) = 0$ for $j > s$. Since the above map $E^\infty_{*,j}(m) \to E^\infty_{*,j}(m+1)$ is zero for $j \geq 1$, the map $H_*BG_m \to H_*BG_{m+1}$ must carry $F^j(m)$ to $F^{j+1}(m+1)$ for $j \geq 1$. This implies that for $m \geq n+s-1$ the map $H_*BG_m \to H_*BG_{m+s-1}$ maps $F^1(m)$ to zero and thus factors (necessarily uniquely) through the surjection $H_*BG_m \to H_*BG$. Dualizing gives maps

$$H^*BG_{m+s-1} \xrightarrow{a^*} H^*BG \xrightarrow{b^*} H^*BG_m$$

where b^* is a monomorphism induced by $G_m \subset G$, a^* is given by the above remarks, and $b^* \cdot a^*$ is induced by $G_m \subset G_{m+s-1}$. The fact that a^* is a ring homomorphism follows from the fact that b^* and $b^* \cdot a^*$ are. Since H^*BG_{m+s-1} is noetherian (12.4) and H^*BG_m is finitely generated as a module over H^*BG_{m+s-1} (12.6), H^*BG is finitely generated as a module over H^*BG_{m+s-1}. An application of 12.3 shows that H^*BG is noetherian.

Proof of 12.2: We will use the notation of the above proof of 12.1. It is clear that $f(K)$ lies in G_m for some $m \geq n$, and it follows from the proof that for some $k > m$ the restriction map $H^*BG_k \to H^*BK$ factors through $(Bf)^*$. The result now follows from 12.6.

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