Maps of BZ/pZ to BG

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The purpose of this note is to give an elementary proof of a special case of the result of [Adams, Lannes$_2$, Miller-Wilkerson] characterizing homotopy classes of maps from the classifying space of an elementary p-group into the classifying space of a connected Lie group. Our result states

Theorem I: If G is a connected compact Lie group, then the natural map

$$\alpha : Hom_{grp}(Z/pZ, G)/(\sim) \longrightarrow [BZ/pZ, BG]$$

is a bijection, where $\sim$ is equivalence up to G - conjugation.

The previous proofs relied on Quillen's description of $H^*(BG, F_p)$ as an inverse limit of certain algebras indexed over the elementary p-groups of G. The present proof does not use this description, but has in common with the other proofs the use of Lannes' identification [Lannes$_1$]

$$Hom_{\mathcal{A}-alg}(H^*(X), H^*(BZ/pZ)) \approx [BZ/pZ, X]$$

for suitable X.

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In spite of its limitation to rank one elementary p-groups, Theorem I is exactly the input needed by the machinery of [Dwyer-Zabrodsky] in the general p-group case:

Theorem II: If π is a finite p-group and G is a connected Lie group, then

$$\alpha : Hom_{grp}(\pi, G)/(\sim) \longrightarrow [B\pi, BG]$$

is a bijection, where $\sim$ denotes equivalence up to G-conjugation.

Finally, in the last section, counterexamples are presented to α being an isomorphism for general finite source groups :

Theorem III: If the target group G is $SU(2)$, then α is not surjective if π is the symmetric group of degree 3, and not injective if π is the cyclic group of order 15.

That is, the behavior observed for $SL(2,5)$ by Milnor and Adams is generic for non-p-groups.

Although the proof of Theorem I avoids Quillen's main result, the study of the action of Z/pZ on the flag manifold G/T is very much in the spirit of [Quillen].

1. Obtaining Homomorphisms from Maps

Fix a prime p, let π be a cyclic group of order p, and let G be a compact connected Lie group. The purpose of this paragraph is to prove the following proposition:

1.1 Proposition Given a map $f : B\pi \longrightarrow BG$, there exists a homomorphism $\varphi : \pi \longrightarrow G$ such that f is homotopic to $B\varphi$.

Notation The symbol $H^*(-)$ will stand for $H^*(-; Z/p)$, $\mathcal{A}$ - alg will denote the category of unstable algebras over the mod p Steenrod algebra $\mathcal{A}$, and $[U, V]$ will stand for the set of homotopy classes of unpointed maps from U to V.

The proof of 1.1 depends mainly on the following result:

1.2 Proposition Let X be a p-complete space with mod p cohomology of finite type, and let $q:\ E \longrightarrow B\pi$ be a fibration over $B\pi$ with fiber X. Then the space of sections $\Gamma(q)$ of q is non-empty iff the map $q^* : H^*(B\pi) \longrightarrow H^*(E)$ is a monomorphism.

Proof It is obvious that q^* is a monomorphism if there is a section. Suppose now that q^* is a monomorphism. The space E is p - complete (since the monodromy action of π on $H^*(X)$ is necessarily nilpotent) so by [Lannes$_1$] the map

$$q_\sharp :\ [B\pi, E] \longrightarrow [B\pi, B\pi]$$

is naturally isomorphic to the induced map

$$Hom_{\mathcal{A}-alg}(H^*E, H^*(B\pi)) \longrightarrow Hom_{\mathcal{A}-alg}(H^*(B\pi), H^*(B\pi)).$$

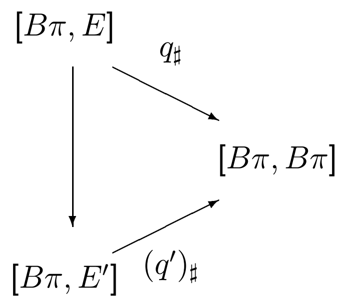
Since q^* is a monomorphism and $H^*(B\pi)$ is an injective object of $\mathcal{A}$-alg, it is possible to find a dotted arrow in $\mathcal{A}$-alg to make the following diagram commute:

$$\begin{array}{ccc} H^*(B\pi) & \xrightarrow{q^*} & H^*E \\ \Big\downarrow \text{id} & \swarrow & \\ H^*(B\pi) & & \end{array}$$

This dotted arrow corresponds to a map $B\pi \longrightarrow E$ which projects under $q_\sharp$ to the identity element of $[B\pi, B\pi]$; such a map immediately gives a section of the fibration q.

1.3 Corollary If X is a simply-connected space of finite type and $q : E \longrightarrow B\pi$ is a fibration over $B\pi$ with fiber X, then the space of sections $\Gamma(q)$ of q is non-empty iff $q^* : H^*(B\pi) \longrightarrow H^*E$ is a monomorphism.

Proof: Consider the commutative diagram

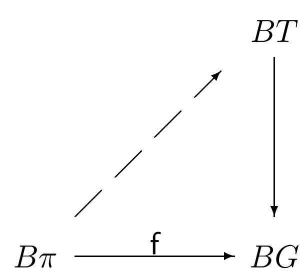


where $E' \xrightarrow{q'} B\pi$ is the fibration with fiber $X_p^\wedge$ obtained from E by fiberwise p-completion. The fiber of the map $E \longrightarrow E'$ is the same as the fiber of the map $X \longrightarrow X_p^\wedge$, and is therefore a connected simple space with uniquely p-divisible homotopy groups. Obstruction theory shows that $[B\pi, E] \longrightarrow [B\pi, E']$ is an isomorphism, so the corollary follows immediately from 1.2.

<u>1.4 Lemma</u> If T is a compact torus and $f : B\pi \longrightarrow BT$ a map, then there exists a homomorphism $\varphi : \pi \longrightarrow T$ such that f is homotopic to $B\varphi$.

<u>Proof</u> This is clear.

<u>Proof of 1.1</u> Let T be the maximal torus of G. By 1.4, it is enough to find a lifting up to the homotopy in the diagram



The homotopy fiber of $BT \longrightarrow BG$ is the flag manifold G/T, so finding the indicated dotted arrow is equivalent to finding a section of the fibration $E \longrightarrow B\pi$ with fiber G/T induced by f. This fibration is simple because BG is simply connected. Since $H^*(G/T; Z)$ is torsion-free and both $H^*(G/T; Z)$ and $H^*(B\pi; Z)$ are concentrated in even dimensions, the integral cohomology Serre spectral sequence of $E \longrightarrow B\pi$ collapses. Reducing mod p shows that the mod p cohomology Serre spectral sequence of $E \longrightarrow B\pi$ also collapses; in effect, the

E_2 term of the spectral sequence is generated over $H^*(B\pi)$ by the reduction of integral classes. It follows that $H^*(B\pi) \longrightarrow H^*(E)$ is a monomorphism, so the proof can be completed by an application of 1.3.

2. Homotopy Implies Conjugacy

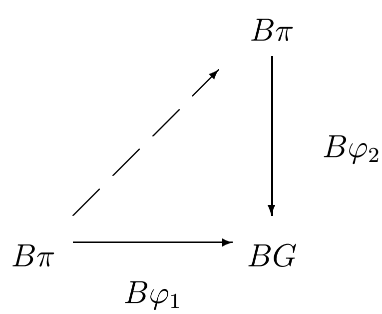
Let p, π and G be as in §1, but drop the assumption that G is connected.

2.1 Proposition If $\varphi_1, \varphi_2 : \pi \longrightarrow G$ are homomorphisms such that $B\varphi_1$ is homotopic to $B\varphi_2$, then φ_1 is conjugate to φ_2.

2.2 Lemma If X is a finite dimensional π-complex, then the fixed-point set X^π is non-empty iff the homotopy fixed-point set $X^{h\pi}$ is non-empty.

Proof of 2.2 If X^π is empty, then $X \times_\pi E\pi$ is a homotopy equivalent to the finite-dimensional space X/π, so that the projection $X \times_\pi E\pi \longrightarrow B\pi$ has no section, i.e., $X^{h\pi} = \phi$. If $X^\pi \neq \phi$, then $X^{h\pi} \neq \phi$ by virtue of the natural map $X^\pi \longrightarrow X^{h\pi}$.

Proof of 2.1 Consider the solid arrow diagram



in which $B\varphi_2$ is thought of as having been replaced by a fibration. If $B\varphi_1$ is homotopic to $B\varphi_2$, then the indicated dotted arrow exists. The fiber of the map from $B\pi$ to BG is, up to homotopy, $G/\varphi_2(\pi)$, so the existence of the dotted arrow indicates that the bundle over $B\pi$ associates to the left translation action of π on $G/\varphi_2(\pi)$ via φ_1 has a section; i.e., that this action of π on $G/\varphi_2(\pi)$ possesses a homotopy fixed point. By 2.2, the action has a genuine fixed point. This fixed point corresponds exactly to an element of G which conjugates $\varphi_1(\pi)$ to a subgroup of $\varphi_2(\pi)$. By symmetry it is possible to find another element of G which conjugates $\varphi_2(\pi)$ to a subgroup of $\varphi_1(\pi)$; this leads easily to the desired conclusion.

3. The Counterexamples D_{2q} and $Z/15Z$

If π is not a p-group, then it is quite possible that α is not onto or monic. Milnor observed that for $\pi = SL(2,5)$ and $G = SU(2), \alpha$ is not onto. A discussion of this case appears in $[Adams_2, Adams_3]$. In fact, in this section it is shown that α is not onto for the dihedral groups $\{D_{2q},$ q an odd prime$\}$ and that α is not monic for $\{Z/pqZ,$ p and q distinct odd primes$\}$, with target group $SU(2)$. Thus $\Sigma_3 \approx D_6$ is the smallest group for which α is not onto. On the other hand while α is not monic for $Z/15Z$, this may not be a minimal example.

The main theoretical tool is Proposition 3.1 below, which states that there is no "coherence" for the mapping sets at various primes. The proof of Proposition 3.1 and Proposition 3.2 are deferred to the end of the section.

3.1 Proposition If π is a finite group, and G is a compact connected Lie group, then

$$[B\pi, BG] = \prod_{p \in P} [(B\pi)^\wedge_p, BG^\wedge_p]$$

where $(\ \_\ )^\wedge_p$ denotes the Bousfield-Kan completion functor.

Proposition 3.2 below allows us to compute $(B\pi)^\wedge_p$ in some favorable cases.

3.2 Proposition If the p-Sylow subgroup $S_p(\pi)$ is abelian, then

$$(B\pi)^\wedge_p \simeq (BN_\pi(S_p(\pi)))^\wedge_p \simeq (BS_p(\pi) \times_W EW)^\wedge_p$$

where $W = N_\pi(S_p(\pi)/\pi$.

3.3 Example For $D_{2q} = \{Z/qZ \rtimes Z/2Z.\}$, for q prime $\neq 2$,

1) $\sharp(Hom(D_{2q}, SU(2))/\sim) = 2$

2) $\sharp[BD_{2q}, BSU(2)] = 2 \cdot (\frac{q+1}{2}) = q+1$

3) $\sharp[(BD_{2q})^\wedge_2, BSU(2)^\wedge_2] = \sharp[BZ/2Z, BSU(2)] = 2$

Thus for D_6, α is not onto.

3.4 Example Let $G = SU(2)$ and $\pi = Z/pqZ$, where p and q are distinct odd primes.

1) $\sharp(Hom(Z/pqZ)/\sim) = 1 + (pq-1)/2 = (pq+1)/2$

2) $[BZ/pqZ, BSU(2)] = [BZ/pZ, BSU(2)] \times [BZ/qZ, BSU(2)]$

so $[BZ/pqZ, BSU(2)] = ((1+p)/2)(1+q)/2$.
Thus for $Z/pq, \alpha$ is not monic.

A purely formal argument allows examples 3.3 and 3.4 to be combined into one common counterexample.

3.5 Proposition For a fixed G, if α is not monic for π, it is not monic for $\pi \times \pi'$. If α is not onto for π', then it is not onto for $\pi \times \pi'$.

3.6 Example : For $G = SU(3)$, and $\pi = \Sigma_3 \times Z/15Z$, α is neither onto nor monic.

Demonstration of Example 3.3

Since the center of $SU(2)$ is $Z/2Z$, and contains all elements of order 2, any homomorphism $\varphi : D_{2q} \longrightarrow SU(2)$ must factor through $Z/2Z$. Thus $\sharp(Hom(D_{2q}, SU(2))/\sim) = 2$.
On the other hand,

$$\begin{aligned} (BD_{2q})^\wedge_2 &\approx BZ/qZ \text{ by Prop. 3.2} \\ \text{and} \quad (BD_{2q})^\wedge_q &\approx (BZ/2Z) \times_{Z/2Z} EZ/2Z \end{aligned}$$

so

$$[BD_{2q}, BSU(2)] = [BZ/2Z, BSU(2)^\wedge_2] \times \pi_0(Map(BZ/qZ, BSU(2)^\wedge_2)^{hZ/2Z}).$$

This latter set of components is by obstruction theory

$$[BZ/qZ, BSU(2)^{\wedge}_{q}]^{Z/2Z}] = [BZ/qZ, BSU(2)]^{Z/2Z}$$

which has (q+1)/2 elements.

Demonstration of Example 3.4

$$Hom(Z/pqZ, SU(2))/\sim = (Z/pqZ)/(action\ of \pm 1)$$

so

$$\begin{aligned} \sharp(Hom(Z/pqZ, SU(2))/\sim) &= 1 + ((pq-1)/2) \\ &= (pq+1)/2 \end{aligned}$$

On the other hand

$$\sharp[BZ/pZ, BSU(2)^{\wedge}_{p}] = \sharp(Hom(Z/pZ, SU(2))/\sim) = 1 + (p-1)/2 = (p+1)/2$$

so by 3.1, $\sharp[BZ/pqZ, BSU(2)] = \dfrac{(p+1)}{2} \cdot \dfrac{(q+1)}{2}$

Proof of 3.1

Apply the "arithmetic square" of Sullivan, Bousfield-Kan to BG.

$$\begin{array}{ccc} BG & \longrightarrow & \Pi(BG)^{\wedge}_{p} \\ \downarrow & & \downarrow \\ BG_0 & \longrightarrow & (\Pi(BG)^{\wedge}_{p})_0 \end{array}$$

After taking the unpointed maps from $B\pi$ into this fiber square, one obtains the fiber square below :

$$\begin{array}{cccc} & BG^{B\pi} & \rightarrow & \prod(BG^{\wedge}_{p})^{B\pi} \\ & \downarrow & & \downarrow \\ BG_0 \simeq & BG_0^{B\pi} & \rightarrow & ((\prod BG^{\wedge}_{p})_0)^{B\pi} \simeq (\prod BG^{\wedge}_{p})_0 \end{array}$$

The Vietoris homotopy sequence of this fiber square has

$$\pi_1((\prod(BG^{\wedge}_{p}))_0) \to \pi_0(BG^{B\pi}) \to \pi_0((\prod BG^{\wedge}_{p})^{B\pi} \times BG_0) \to \pi_0((\prod BG^{\wedge}_{p})_0)$$

But $\prod(BG^{\wedge}_{p})_0$ is connected and simply connected and BG_0 is connected, so 3.1 follows from exactness.

Proof of 3.2

A slightly nontrivial calculation of the stable elements in the sense of [Cartan-Eilenberg, p.257] in these cases with $S_p(\pi)$ abelian shows that

$$H^*(B\pi, Z/pZ) \approx H^*(BS_p(\pi), Z/pZ)^{N_\pi(S_p(\pi))/S_p(\pi)}$$

$$\approx H^*(BN_\pi(S_p), Z/pZ).$$

Proof of 3.5

Since π is a retract of $\pi \times \pi'$, and $B\pi$ is a retract of $B\pi \times B\pi'$, $Hom(\pi, G)/\sim$ is a retract of $Hom(\pi \times \pi', G)/\sim$ and $[B\pi, BG]$ is a retract of $[B\pi \times B\pi', BG]$. Thus

$$\begin{array}{ccc} Hom(\pi, G)/\sim & \xrightarrow{\alpha_1} & [B\pi, BG] \\ \uparrow i^* & & \uparrow Bi^* \\ Hom(\pi \times \pi', G)/\sim & \xrightarrow{\alpha_2} & [B\pi \times B\pi', BG] \\ \uparrow p^* & & \uparrow Bp^* \\ Hom(\pi, G)/\sim & \xrightarrow{\alpha_1} & [B\pi, BG] \end{array}$$

Now, if $\alpha_1\varphi = \alpha_1\varphi'$, then $\alpha_2 p^*\varphi = \alpha_2 p^*\varphi'$. But $p^*\varphi \neq p^*\varphi'$ so α_2 is not monic if α_1 is not. If $f : B\pi \to BG$ is not in the image of α_1, then if $Bp^*f \in im\ \alpha_2$, then $Bi^*Bp^*f = f \in im\ \alpha_1$, which is a contradiction.

References

[Adams 1] J.F. Adams, private communication, Fall 1985.

[Adams 2] J.F. Adams, *Maps between classifying spaces*, Enseigs. Math.(2), 24(1978), no. 1-2, pp.79-85.

[Adams 3] J.F. Adams, *Maps between classifying spaces II*, Invent. Math. 49(1978), no.1, pp 1-65.

[Cartan-Eilenberg] Homological Algebra, Princeton University Press, Princeton, 1956.

[Dwyer-Zabrodsky] W.G. Dwyer and A. Zabrodsky, *Maps between classifying spaces*, 1986 Barcelona Conference Proceedings, Springer Lecture Notes in Mathamatics, to appear.

[Lannes 1] J. Lannes, *Sur la cohomologie modulo p des p-groups abeliens elementaires*, to appear.

[Lannes 2] J. Lannes, *Cohomology of groups and function spaces*, to appear.

[Miller-Wilkerson] H.R. Miller and C.W. Wilkerson, *Maps of elementary p-groups into compact Lie groups,* talk by C.W.Wilkerson at 1985 Yale Conference on Algebraic Topology.