THE CENTER OF A p–COMPACT GROUP

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§1. Introduction

Compact Lie groups appear frequently in algebraic topology, but they are relatively rigid analytic objects, and partially for that reason are a challenge to understand homotopically. Since H. Hopf (and H-spaces) topologists have aspired to escape the analytic straitjacket by finding some homotopy theoretic concept which would capture enough of the idea of "compact Lie group" to lead to rich and interesting structure theorems. In [12] we came up with our own candidate, the notion of *p-compact group*, and studied these objects to the extent of constructing maximal tori, Weyl groups, etc. In this paper we continue the study by looking at the idea of the "center" of a p-compact group and showing that two very different ways of defining the center are equivalent. This leads for instance to a reproof and generalization of a theorem from [15] about the identity component of the space of self homotopy equivalences of $\mathrm{B}G$ (G compact Lie). Along the way we find various familiar-looking elements of internal structure in a p-compact group $\mathcal{X}$, enumerate the $\mathcal{X}$'s which are abelian in the appropriate sense, and construct what might be called the "adjoint form" of $\mathcal{X}$.

Before describing in more detail the main results we are aiming at, we have to introduce some ideas from [12]. A *loop space* $\mathcal{X}$ is by definition a triple $(\mathcal{X}, \mathrm{B}\mathcal{X}, e)$ in which $\mathcal{X}$ is a space, $\mathrm{B}\mathcal{X}$ is a connected pointed space (called the *classifying space* of $\mathcal{X}$), and $e : \mathcal{X} \to \Omega\mathrm{B}\mathcal{X}$ is a homotopy equivalence from $\mathcal{X}$ to the space $\Omega\mathrm{B}\mathcal{X}$ of based loops in $\mathrm{B}\mathcal{X}$. A *p-compact group* is a loop space $\mathcal{X}$ such that

(1) $\mathcal{X}$ is $\mathbf{F}_p$-*finite* (in the sense that $\mathrm{H}^*(\mathcal{X}; \mathbf{F}_p)$ is finite dimensional),
(2) $\pi_0\mathcal{X}$ is a finite p-group, and
(3) $\pi_i\mathcal{X}$ ($i \geq 1$) is a finitely generated module over the ring $\mathbf{Z}_p$ of p-adic integers.

A *homomorphism* $f : \mathcal{X} \to \mathcal{Y}$ between loop spaces is a pointed map $\mathrm{B}f : \mathrm{B}\mathcal{X} \to \mathrm{B}\mathcal{Y}$; if $\mathcal{X}$ and $\mathcal{Y}$ are p-compact groups the homomorphism f is said to be a *monomorphism* if the homotopy fibre $\mathcal{Y}/f(\mathcal{X})$ of $\mathrm{B}f$ is $\mathbf{F}_p$-finite (in this case, if the homomorphism f is understood, we will refer to $\mathcal{X}$ as a *subgroup* of $\mathcal{Y}$). If $f : \mathcal{X} \to \mathcal{Y}$ is a homomorphism of loop spaces, the *centralizer of* $f(\mathcal{X})$ *in* $\mathcal{Y}$, denoted $\mathbf{C}_{\mathcal{Y}}(f(\mathcal{X}))$, is the loop space $\Omega\,\mathrm{Map}(\mathrm{B}\mathcal{X}, \mathrm{B}\mathcal{Y})_{\mathrm{B}f}$. Here $\mathrm{Map}(\mathrm{B}\mathcal{X}, \mathrm{B}\mathcal{Y})_{\mathrm{B}f}$ is the component containing $\mathrm{B}f$ of the space of (unpointed) maps

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from $\mathrm{B}\mathcal{X}$ to $\mathrm{B}\mathcal{Y}$. There is a natural loop space homomorphism $\mathbf{C}_{\mathcal{Y}}(f(\mathcal{X})) \to \mathcal{Y}$, induced by evaluation of maps at the basepoint of $\mathrm{B}\mathcal{X}$.

A p-compact group $\mathcal{X}$ is said to be *abelian* if $\mathbf{C}_{\mathcal{X}}(\mathrm{id}(\mathcal{X})) \to \mathcal{X}$ is an equivalence, where $\mathrm{id} : \mathcal{X} \to \mathcal{X}$ is the identity homomorphism. One of the first things we prove in the paper is

1.1 Theorem. *Any abelian p-compact group is equivalent to the product of a finite abelian p-group and a p-compact torus.*

Note that a p-compact torus is just the $\mathbf{F}_p$-completion of an ordinary torus.

A homomorphism $f : \mathcal{X} \to \mathcal{Y}$ of p-compact groups is said to be *central* if the homomorphism $\mathbf{C}_{\mathcal{Y}}(f(\mathcal{X})) \to \mathcal{Y}$ is an equivalence.

1.2 Theorem. *Let $\mathcal{X}$ be a p-compact group. Then any central subgroup of $\mathcal{X}$ is abelian. Moreover, $\mathcal{X}$ has (in an appropriate sense) a unique maximal central subgroup, denoted $\mathbf{C}^{\mathrm{c}}(\mathcal{X})$. If $\mathcal{X}$ is connected, then $\mathbf{C}^{\mathrm{c}}(\mathcal{X})$ is a subgroup of the maximal torus of $\mathcal{X}$ and can be described explicitly in terms of the normalizer of the torus.*

The above maximal central subgroup $\mathbf{C}^{\mathrm{c}}(\mathcal{X})$ is called the *p-compact center* of $\mathcal{X}$. It is possible to form a quotient p-compact group $\mathcal{X}/\mathbf{C}^{\mathrm{c}}(\mathcal{X})$, and it turns out that if $\mathcal{X}$ is connected this quotient has a trivial p-compact center. The quotient $\mathcal{X}/\mathbf{C}^{\mathrm{c}}(\mathcal{X})$ is what we call the *adjoint form* of $\mathcal{X}$.

For a p-compact group $\mathcal{X}$, $\mathbf{C}^{\mathrm{c}}(\mathcal{X})$ is one candidate for the "center" of $\mathcal{X}$. There is another candidate, already mentioned above. This is the loop space $\mathbf{C}_{\mathcal{X}}(\mathrm{id}(\mathcal{X}))$, which is denoted $\mathbf{C}^{\mathrm{h}}(\mathcal{X})$ of $\mathcal{X}$ and called the *homotopy center* of $\mathcal{X}$. We will prove the following theorem (see [11, §5] for what is in retrospect a special case).

1.3 Theorem. *Let $\mathcal{X}$ be a p-compact group. Then the evident (§11) loop space homomorphism $\mathbf{C}^{\mathrm{c}}(\mathcal{X}) \to \mathbf{C}^{\mathrm{h}}(\mathcal{X})$ is an equivalence.*

This gives a generalization to non-simple groups of a fact which was proved for $G = \mathrm{SO}(3)$ in [9] and for general simple groups G in [15, Theorem 3].

1.4 Theorem. *Let G be a compact connected Lie group. Then the natural map*

$$\mathrm{B}\,\mathrm{Center}(G) \to \mathrm{Map}(\mathrm{B}G, \mathrm{B}G)_{\mathrm{id}}$$

induces an isomorphism on homology with any finite coefficients.

1.5 Organization of the paper. Section 2 gives some general properties of p-compact groups, including a recognition principle (2.15) for maximal tori, and §3 describes the sense in which certain p-compact groups have discrete approximations. Section 4 defines the notion of maximal rank subgroup and gives an "infinitesimal" criterion (4.7) for the inclusion of such a subgroup to be an isomorphism. Section 5 contains the proof of 1.1, while §6 contains a construction of the p-compact center and a study of the adjoint form. In §7 there is an explicit calculation of the center of a connected p-compact group in terms of the normalizer of a maximal torus; most of the interest here is at the prime 2, where the same complications arise for p-compact groups as for compact Lie groups. Section 7 also gives the Weyl group structure of the centralizer of a subgroup of the torus (7.6). Section 8 shows that the homology decomposition theorem of Jackowski and McClure [14] can be extended to p-compact groups, and §9 illustrates how to use this decomposition theorem to construct inductive proofs; the actual statement proved in §9 is a generalization of Miller's Theorem [17]. There are a few technical results in §10, including a key splitting theorem (10.7). Finally, §11 has the proof of 1.3 and §12 the proof of 1.4.

1.6 Notation and terminology. In this paper p denotes a fixed prime number, $\mathbf{F}_p$ the field with p elements, $\mathbf{Z}_p$ the ring of p-adic integers, and $\mathbf{Q}_p$ the field $\mathbf{Q} \otimes \mathbf{Z}_p$.

For a space $\mathcal{X}$ the symbol $\mathrm{H}^*\mathcal{X}$ denotes the mod p cohomology ring $\mathrm{H}^*(\mathcal{X}, \mathbf{F}_p)$ and $\mathrm{H}_{\mathbf{Q}_p}(\mathcal{X})$ the ring $Q \otimes \mathrm{H}^*(\mathcal{X}, \mathbf{Z}_p)$. The space $\mathcal{X}$ is *$\mathbf{F}_p$-finite* if $\mathrm{H}^*\mathcal{X}$ is finite dimensional, and $f : \mathcal{X} \to \mathcal{Y}$ is *an $\mathbf{F}_p$-equivalence* if it induces an isomorphism $\mathrm{H}^*\mathcal{Y} \cong \mathrm{H}^*\mathcal{X}$. If $\mathcal{X}$ is $\mathbf{F}_p$-finite then the *Euler characteristic* of $\mathcal{X}$ is the sum $\sum_i (-1)^i \operatorname{rk}_{\mathbf{F}_p} \mathrm{H}^i\mathcal{X}$; it turns out that this is the same as $\sum_i (-1)^i \operatorname{rk}_{\mathbf{Q}_p} \mathrm{H}^i_{\mathbf{Q}_p}(\mathcal{X})$. The ring $\mathrm{H}^*_{\mathbf{Q}_p}(\mathcal{X})$ serves as a replacement for rational cohomology in cases in which the actual rational cohomology ring may be too large to be useful [12, §4].

We assume that any space in this paper has the homotopy type of a CW-complex; if not, the space can be replaced by the geometric realization of its singular complex. The word "equivalence" stands for "homotopy equivalence". We will use $\mathrm{Comp}_{\mathbf{F}_p}(-)$ to denote the homotopical $\mathbf{F}_p$-completion functor constructed in [3] (see [12, §11] for a list of some of the properties of this functor); a space $\mathcal{X}$ is *$\mathbf{F}_p$-complete* if the natural map $\mathcal{X} \to \mathrm{Comp}_{\mathbf{F}_p}(\mathcal{X})$ is an equivalence.

1.7 Remark. Some of the results in this paper, including Theorem 1.1 and the first part of Theorem 1.2, have been obtained independently by Moller and Notbohm in [18].

§2. Loop spaces and p-compact groups

In this section we will summarize a few basic facts about loop spaces and p-compact groups. Note that any topological group G is a loop space in a natural way: the space $\mathrm{B}G$ is the usual classifying space of G. For example, any discrete group is a loop space. One way to motivate some of the definitions below is to look at what they specialize to in the case of discrete groups [12, §3].

2.1 Loop spaces. If $\mathcal{X}$ and $\mathcal{Y}$ are loop spaces, a *homomorphism* $f : \mathcal{X} \to \mathcal{Y}$ is by definition a pointed map $\mathrm{B}f : \mathrm{B}\mathcal{X} \to \mathrm{B}\mathcal{Y}$; the *space of homomorphisms* $\mathcal{X} \to \mathcal{Y}$ is the space $\mathrm{Map}_*(\mathrm{B}\mathcal{X}, \mathrm{B}\mathcal{Y})$ of pointed maps $\mathrm{B}\mathcal{X} \to \mathrm{B}\mathcal{Y}$. An *outer homomorphism* $\mathcal{X} \to \mathcal{Y}$ is a map $\mathrm{B}\mathcal{X} \to \mathrm{B}\mathcal{Y}$ which does not necessarily preserve the basepoints. Two homomorphisms f and g are *homotopic* if $\mathrm{B}f$ and $\mathrm{B}g$ are homotopic as pointed maps; the homomorphisms are *conjugate* if they are homotopic as outer homomorphisms, i.e., if $\mathrm{B}f$ and $\mathrm{B}g$ are freely homotopic. A homomorphism f is *trivial* if $\mathrm{B}f$ is the constant map, and *an equivalence* if $\mathrm{B}f$ is an equivalence. A *short exact sequence*

$$\{1\} \to \mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z} \to \{1\} \tag{2.2}$$

of loop spaces is by definition a fibration sequence $\mathrm{B}\mathcal{X} \xrightarrow{\mathrm{B}f} \mathrm{B}\mathcal{Y} \xrightarrow{\mathrm{B}g} \mathrm{B}\mathcal{Z}$.

If $f : \mathcal{X} \to \mathcal{Y}$ is a homomorphism of loop spaces, the *homogeneous space* $\mathcal{Y}/f(\mathcal{X})$ is defined to be the homotopy fibre of $\mathrm{B}f$ over the basepoint of $\mathrm{B}\mathcal{Y}$; this space is denoted $\mathcal{Y}/\mathcal{X}$ if f is understood. The *centralizer of* $f(\mathcal{X})$ *in* $\mathcal{Y}$, denoted $\mathbf{C}_{\mathcal{Y}}(f(\mathcal{X}))$ (or $\mathbf{C}_{\mathcal{Y}}(\mathcal{X})$ if f is understood) is the loop space of the mapping space component $\mathrm{Map}(\mathrm{B}\mathcal{X}, \mathrm{B}\mathcal{Y})_{\mathrm{B}f}$. (The map $\mathrm{B}f$ is used as a basepoint in forming this loop space.) Evaluation at the basepoint of $\mathrm{B}\mathcal{Y}$ gives a loop space homomorphism $\mathbf{C}_{\mathcal{Y}}(\mathcal{X}) \to \mathcal{Y}$, and the homomorphism $f : \mathcal{X} \to \mathcal{Y}$ is said to be *central* if the map $\mathbf{C}_{\mathcal{Y}}(\mathcal{X}) \to \mathcal{Y}$ is an equivalence.

2.3 p-compact groups. A *p-compact group* is a loop space $\mathcal{X}$ such that $\mathcal{X}$ is $\mathbf{F}_p$-finite and $\mathrm{B}\mathcal{X}$ is $\mathbf{F}_p$-complete, or equivalently [12, §2] a loop space $\mathcal{X}$ which satisfies the three conditions listed in §1. If $\mathcal{X}$ is a p-compact group and $H \subset \pi_0\mathcal{X}$ a subgroup, the inverse image $\mathcal{X}_H$ in $\mathcal{X}$ of H is a p-compact group with $\mathrm{B}(\mathcal{X}_H)$ an appropriate covering space of $\mathrm{B}\mathcal{X}$; for instance, the *identity component* $\mathcal{X}_1 = \mathcal{X}_{\{1\}}$ of $\mathcal{X}$ is a p-compact group. If $f : \mathcal{X} \to \mathcal{Y}$ is a homomorphism of p-compact groups, then f is said to be a *monomorphism* if $\mathcal{Y}/\mathcal{X}$ is $\mathbf{F}_p$-finite and an *epimorphism* if $\mathcal{Y}/\mathcal{X} = \mathrm{B}\mathcal{Z}$ for a p-compact group $\mathcal{Z}$. In the first case $\mathcal{X}$ is said to be a subgroup of $\mathcal{Y}$.

2.4 Proposition. [12, 2.3, 9.11] *For any p-compact group $\mathcal{X}$ the cohomology ring* $\mathrm{H}^*\mathrm{B}\mathcal{X}$ *is noetherian. If $f : \mathcal{X} \to \mathcal{Y}$ is a homomorphism of p-compact groups, then f is a monomorphism if and only if under* $(\mathrm{B}f)^*$ *the ring* $\mathrm{H}^*\mathrm{B}\mathcal{X}$ *is finitely generated as a module over* $\mathrm{H}^*\mathrm{B}\mathcal{Y}$.

Note that a composite of monomorphisms is a monomorphism. In a short exact sequence as in 2.2 of p-compact groups, the homomorphism f is a monomorphism and g is an epimorphism.

If G is a compact Lie group such that $\pi_0 G$ is a p-group, we will let $\hat{G}$ denote the p-compact group with $\mathrm{B}(\hat{G}) = \mathrm{Comp}_{\mathbf{F}_p}(\mathrm{B}G)$ [12, §11]; as a space $\hat{G}$ is just $\mathrm{Comp}_{\mathbf{F}_p} G$. *A p-compact torus* is a p-compact group of the form $\hat{T}$, where $T = \mathrm{SO}(2)^r$ is an ordinary torus; more generally, a *p-compact toral group* G is a p-compact group such that the identity component G_1 is a p-compact torus. We will work extensively with p-compact toral groups because they have good technical properties, some of which are listed in the next two propositions.

2.5 Proposition. [12, §6] *Suppose that $f : G \to \mathcal{X}$ is a homomorphism of p-compact groups, where G is a p-compact toral group. Then*

(1) *$\mathbf{C}_{\mathcal{X}}(G)$ is a p-compact group, and*
(2) *the map $\mathbf{C}_{\mathcal{X}}(G) \to \mathcal{X}$ is a monomorphism.*

2.6 Proposition. [12, 8.6] *Suppose that $f : G \to \mathcal{X}$ is a homomorphism of p-compact groups, where G is a p-compact toral group. Then f is central if and only if there exists a homomorphism $G \times \mathcal{X} \to \mathcal{X}$ which is homotopic to f on $G \times \{1\}$ and to $\mathrm{id}_{\mathcal{X}}$ on $\{1\} \times \mathcal{X}$.*

Remark. In saying that a homomorphism $h : G \times \mathcal{X} \to \mathcal{X}$ is "homotopic to f on $G \times \{1\}$", for instance, we mean that the restriction of $\mathrm{B}h$ to $\mathrm{B}G \cong \mathrm{B}G \times * \subset \mathrm{B}G \times \mathrm{B}\mathcal{X}$ is homotopic to $\mathrm{B}f$. Proposition 2.6 is true without the assumption that G is a p-compact toral group (10.2), but we do not know how to give a similar generalization of 2.5.

A loop space $\mathcal{X}$ is said to be *abelian* if the identity map of $\mathcal{X}$ is central. Any p-compact torus is abelian, as is more generally the product of a p-compact torus and a finite abelian p-group. (In 5.2 we will show that any abelian p-compact group is of this second form.)

2.7 Proposition. [12, 8.2] *Let $f : G \to \mathcal{X}$ be a homomorphism of p-compact groups, where G is an abelian p-compact toral group. Then f lifts naturally up to homotopy to a central homomorphism $G \to \mathbf{C}_{\mathcal{X}}(G)$.*

2.8 Proposition. [12, 8.3] *Suppose that $f : G \to \mathcal{X}$ is a central monomorphism of p-compact groups, where G is an abelian p-compact toral group. Then there is a p-compact group structure on the homogeneous space $\mathcal{X}/G$ and a natural short exact sequence*

$$\{1\} \to G \to \mathcal{X} \to \mathcal{X}/G \to \{1\}\,.$$

Maximal tori and Weyl groups. If $\mathcal{X}$ is a p-compact group, a homomorphism $f : T \to X$ with T a p-compact torus is said to be a *maximal torus* for $\mathcal{X}$ if the map $T \to \mathbf{C}_{\mathcal{X}}(T)$ of 2.7 gives an equivalence between T and the identity component $\mathbf{C}_{\mathcal{X}}(T)_1$ of its centralizer. Sometimes for brevity we will simply say that T is a maximal torus for $\mathcal{X}$. For a maximal torus $f : T \to \mathcal{X}$ the *Weyl space* $\mathbf{W}_{\mathcal{X},f}$ of T is defined to be the space of self-equivalences of $\mathrm{B}T$ over $\mathrm{B}\mathcal{X}$ (constructed after replacing $\mathrm{B}f : \mathrm{B}T \to \mathrm{B}\mathcal{X}$ by an equivalent fibration).

2.9 Theorem. [12, 9.5] *Any p-compact group $\mathcal{X}$ has a maximal torus $f : T \to \mathcal{X}$, unique up to conjugacy. If T is such a maximal torus, then the space $\mathbf{W}_{\mathcal{X},f}$ is homotopically discrete and $\pi_0\mathbf{W}_{\mathcal{X},f}$ is a finite group (under composition). The order of $\pi_0\mathbf{W}_{\mathcal{X},f}$ is equal to the Euler characteristic $\chi(\mathcal{X}/T)$.*

If $f : T \to \mathcal{X}$ is a maximal torus for $\mathcal{X}$, the *Weyl group* $W_{\mathcal{X},f}$ is defined to be the group $\pi_0\mathbf{W}_{\mathcal{X},f}$; this is sometimes denoted $W_{\mathcal{X},T}$ or $W_{\mathcal{X}}$ and called the Weyl group of $\mathcal{X}$. The *normalizer* $\mathbf{N}(T)$ of T is the loop space whose classifying space is the Borel construction of the action of $\mathbf{W}_{\mathcal{X},f}$ on $\mathrm{B}T$. It is clear that $\mathrm{B}\mathbf{N}(T)$ is up to homotopy a fibration over $\mathrm{B}W_{\mathcal{X}}$ with fibre $\mathrm{B}T$, so that $\mathbf{N}(T)$ is not a p-compact group in general unless $W_{\mathcal{X}}$ is a p-group. However, the loop space $\mathbf{N}_p(T)$ obtained as the union of components of $\mathbf{N}(T)$ corresponding to a Sylow p-subgroup of $W_{\mathcal{X}}$ *is* always a p-compact group. The p-compact group $\mathbf{N}_p(T)$ is called a "p-normalizer" of T. By construction, the homomorphism $T \to \mathcal{X}$ factors as a composite

$$T \to \mathbf{N}_p(T) \to \mathbf{N}(T) \to \mathcal{X}\,. \tag{2.9}$$

2.10 Proposition. [12, 9.9] *If $\mathcal{X}$ is a p-compact group with maximal torus T, then the above homomorphism $\mathbf{N}_p(T) \to X$ is a monomorphism, and the Euler characteristic $\chi(\mathcal{X}/\mathbf{N}_p(T))$ is relatively prime to p.*

2.11 Remark. The loop space homomorphism $\mathbf{N}(T) \to \mathcal{X}$ in 2.9 induces a group homomorphism $f : W_{\mathcal{X}} = \pi_0\mathbf{N}(T) \to \pi_0\mathcal{X}$. By definition the kernel of f is the Weyl group of the identity component $\mathcal{X}_1$ of $\mathcal{X}$. The map f is onto; this comes down to showing that if $\mathcal{U}$ is any component of $\mathcal{X}/T$ then the homotopy fixed point set $\mathcal{U}^{\mathrm{h}T}$ is nonempty [12, proof of 9.5], which is proved as in [12, proof of 8.11] by observing that $\chi(\mathcal{U}) \neq 0$. Since $\pi_0\mathcal{X}$ is a finite p-group, it follows that the map $\pi_0\mathbf{N}_p(T) \to \pi_0\mathcal{X}$ is also onto.

Recall that an element of finite order in $\mathrm{GL}(r, \mathbf{Q}_p)$ or $\mathrm{GL}(r, \mathbf{Z}_p)$ is said to be a *reflection* (or sometimes a *pseudoreflection*) if it pointwise fixes a codimension 1 subspace of $(\mathbf{Q}_p)^r$. The *rank* of a p-compact torus T is the number r such that $T \cong \hat{G}$ with $G = \mathrm{SO}(2)^r$; this equals the rank over $\mathbf{Z}_p$ of the free module $\mathrm{H}^2(\mathrm{B}T, \mathbf{Z}_p)$.

2.12 Proposition. [12, 9.7] *Let $\mathcal{X}$ be a connected p-compact group with maximal torus T, and let r be the rank of T. Then the natural action of the Weyl group $W_{\mathcal{X}}$ on $\mathrm{H}^2(\mathrm{B}T, \mathbf{Z}_p)$ is faithful and and represents $W_{\mathcal{X}}$ as a finite subgroup of $\mathrm{GL}(r, \mathbf{Z}_p)$ generated by reflections.*

2.13 Remark. The number r is 2.12 is called the *rank* of $\mathcal{X}$; it is also the number of generators in the polynomial algebra $\mathrm{H}^*_{\mathbf{Q}_p}\mathrm{B}\mathcal{X}$ [12, 9.7]. Since $\mathrm{H}^2(\mathrm{B}T, \mathbf{Z}_p)$ is the $\mathbf{Z}_p$-dual of $\pi_2\mathrm{B}T$, it follows that the natural action of $W_{\mathcal{X}}$ on $\pi_2\mathrm{B}T$ is also faithful and represents $W_{\mathcal{X}}$ as a group generated by reflections.

Euler characteristics. Here we extract two useful properties of Euler characteristics from [12].

2.14 Proposition. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a monomorphism of p-compact groups, G a p-compact toral group, and $g : G \to \mathcal{Y}$ a homomorphism of loop spaces. Suppose that one of the following conditions holds.*

1. *G is a p-compact toral group and $\chi(\mathcal{Y}/\mathcal{X})$ is not divisible by p.*
2. *G is a p-compact torus and $\chi(\mathcal{Y}/\mathcal{X})$ is not zero.*

Then g lifts up to conjugacy to a homomorphism $\tilde{g} : G \to \mathcal{X}$.

Sketch of proof. According to [12, 3.3] this is a question of proving that in the given cases the homotopy fixed point set $(\mathcal{Y}/\mathcal{X})^{\mathrm{h}G}$ is nonempty. Case (2) then follows from combining [12, 4.7, 5.7, 6.7] (see also the proof of [12, 8.11]). Case (1) is proved in essentially the same way, but using [12, 4.6] and the inductive argument in the proof of [12, 4.7] instead of using [12, 4.7] itself. □

2.15 Proposition. *Suppose $\mathcal{X}$ is a p-compact group, T is a p-compact torus, and $i : T \to \mathcal{X}$ is a monomorphism. Then i is a maximal torus for $\mathcal{X}$ if and only if $\chi(\mathcal{X}/T) \neq 0$.*

Proof. If i is a maximal torus, then $\chi(\mathcal{X}/T) \neq 0$ by 2.9. Suppose then that $\chi(\mathcal{X}/T) \neq 0$. Let $j : T' \to \mathcal{X}$ be a maximal torus for $\mathcal{X}$; note that j is also a monomorphism. According to 2.14, there are homomorphisms $f : T \to T'$ and $g : T' \to T$ such that $j \cdot f$ is conjugate to i and $i \cdot f'$ is conjugate to j, and a simple argument using 2.4 shows that the composites $f \cdot g$ and $g \cdot f$ are also monomorphisms. It is elementary to prove using 2.4 that any monomorphism $T \to T$ or $T' \to T'$ is an equivalence; for instance, this applies to $f \cdot g$ and $g \cdot f$. It follows that f and g are equivalences, and hence that T is conjugate in $\mathcal{X}$ to the maximal torus T'. □

2.16 Remark. Let G be a compact Lie group with $\pi_0 G$ a p-group, and $T \to G$ a Lie-theoretic maximal torus; it is a classical result that the Euler characteristic $\chi(G/T)$ is nonzero. An easy argument with the fibre lemma (cf. [12, proof of 5.7]) shows that G/T

is $\mathbf{F}_p$-good and that $\hat{G}/\hat{T}$ is equivalent to the $\mathbf{F}_p$-completion of G/T. It follows that $\chi(\hat{G}/\hat{T}) \neq 0$ and thus by 2.15 that $\hat{T}$ is a maximal torus for $\hat{G}$.

§3. Discrete approximations

Surprisingly, some questions about p-compact groups can be reduced to questions about discrete groups. This stems mostly from the fact that there is an effective way of constructing "discrete approximations" for p-compact toral groups.

Let $\mathbf{Z}/p^\infty$ denote the group $\mathbf{Z}[1/p]/\mathbf{Z} = \cup_n \mathbf{Z}/p^n$. By definition, a *$p$-discrete torus* is a discrete group A which is isomorphic to $(\mathbf{Z}/p^\infty)^r$ for some r. A *p-discrete toral group* is a discrete group G with a normal subgroup A such that A is a p-discrete torus and G/A is a finite p-group. The p-discrete torus A is then the maximal divisible subgroup of G. Any p-discrete toral group G can be expressed as an increasing union of finite p-groups [12, 6.19].

The *closure* $\bar{G}$ of a p-discrete toral group G is defined to be the loop space $\Omega\,\mathrm{Comp}_{\mathbf{F}_p}(\mathrm{B}G)$. The completion map $\mathrm{B}G \to \mathrm{Comp}_{\mathbf{F}_p}(\mathrm{B}G)$ is an isomorphism on mod p cohomology [12, 6.9] and gives a homomorphism $G \to \bar{G}$.

3.1 Proposition. *If G is a p-discrete toral group, then $\bar{G}$ is a p-compact toral group and the homogeneous space $\bar{G}/G$ is an Eilenberg-MacLane space of type $K(V,1)$ for some rational vector space V.*

Proof. All except the last statement is proven in [12, proof of 6.9]. By the arguments there, and the fact that the completion functor $\mathrm{Comp}_{\mathbf{F}_p}(-)$ preserves products [3, I, §7], it is enough to show that if $G = \mathbf{Z}/p^\infty$ then the homotopy fibre of the map $\epsilon : \mathrm{B}G \to \mathrm{B}\bar{G}$ is of the indicated type. There is a short exact sequence

$$\{1\} \to \mathbf{Z}_p \to \mathbf{Q}_p \to \mathbf{Z}/p^\infty \to \{1\}$$

of abelian groups, which gives rise to a fibration $f : K(\mathbf{Z}/p^\infty, 1) \to K(\mathbf{Z}_p, 2)$ with fibre $K(\mathbf{Q}_p, 1)$. The target of the map f is $\mathbf{F}_p$-complete, and f induces an isomorphism on mod p homology because the fibre of f has the mod p homology of a point. This implies that map f is in fact equivalent to the completion map ϵ and gives the required calculation of the homotopy fibre of ϵ. □

It is also possible to reverse the closure construction. A *discrete approximation* for a p-compact toral group G is defined to be a p-discrete toral group $\breve{G}$ together with a homomorphism $\alpha : \breve{G} \to G$ which induces an isomorphism $\mathrm{H}^*\mathrm{B}G \to \mathrm{H}^*\mathrm{B}\breve{G}$.

3.2 Proposition. *Any p-compact toral group G has a discrete approximation $\breve{G} \to G$. If G and H are p-compact toral groups with discrete approximations $\breve{G}$ and $\breve{H}$, then any homomorphism $f : G \to H$ lifts uniquely to a homomorphism $\breve{f} : \breve{G} \to \breve{H}$. The homomorphism f is central if and only if its lift $\breve{f}$ is central, which is the case if and only if $\breve{f}(\breve{G})$ is contained in the center of $\breve{H}$.*

Sketch of proof. The fact that discrete approximations exist is proven in [12, 6.8]. Suppose that $a : \breve{G} \to G$ is a discrete approximation. The map $\mathrm{B}a : \mathrm{B}\breve{G} \to \mathrm{B}G$ induces an isomorphism on mod p homology and the target of $\mathrm{B}a$ is $\mathbf{F}_p$-complete; this implies that $\mathrm{B}a$ is equivalent to the $\mathbf{F}_p$-completion map $\mathrm{B}\breve{G} \to \mathrm{Comp}_{\mathbf{F}_p}(\mathrm{B}\breve{G})$ and thus (3.1) that the homotopy fibre of $\mathrm{B}a$ is a space of type $K(V, 1)$ for a rational vector space V. The required lifting result now comes from obstruction theory. The final statement comes out of the observation that f is central if and only if there is a suitable homomorphism $G \times H \to H$ (2.6), whereas the map $\breve{f}$ of $K(\pi, 1)$'s is visibly central if and only if there is a corresponding homomorphism $\breve{G} \times \breve{H} \to \breve{H}$; one type of homomorphism can be obtained from the other by an appropriate closure/discrete approximation construction. □

3.3 Remark. Note that by 3.2 any two discrete approximations of a p-compact toral group G are canonically isomorphic (as groups). If $\breve{G} \to G$ is a discrete approximation, then by [12, 6.8] there is a short exact sequence

$$\{1\} \to (\pi_1 G) \otimes \mathbf{Z}/p^\infty \to \breve{G} \to \pi_0 G \to \{1\}\,.$$

The previous two propositions allow us to shift back and forth easily between p-compact toral groups and their discrete approximations. Because of the following proposition, which we think of as expressing the *density* in G of a discrete approximation, this shifting process is usually innocuous.

3.4 Proposition. *(Density)* [12, 6.7] *Let G be a p-compact toral group, $\alpha : \breve{G} \to G$ a discrete approximation, and $\mathcal{U}$ an $\mathbf{F}_p$-complete space (e.g., $\mathcal{U} = \mathrm{B}\mathcal{X}$ for a p-compact group $\mathcal{X}$). Then $\mathrm{B}\alpha$ induces equivalences $\mathrm{Map}(\mathrm{B}G, \mathcal{U}) \to \mathrm{Map}(\mathrm{B}\breve{G}, \mathcal{U})$ and $\mathrm{Map}_*(\mathrm{B}G, \mathcal{U}) \to \mathrm{Map}_*(\mathrm{B}\breve{G}, \mathcal{U})$.*

Remark. For example, if G is a p-discrete toral group and $f : G \to \mathcal{X}$ is a homomorphism with $\mathcal{X}$ a p-compact group, then "passing to the closure" (i.e. applying the functor $\mathrm{Comp}_{\mathbf{F}_p}(-)$ to $\mathrm{B}f : \mathrm{B}G \to \mathrm{B}\mathcal{X}$) gives a homomorphism $\bar{f} : \bar{G} \to \mathcal{X}$.

Kernels and exact sequences. One of the most useful features of a p-discrete toral group is that it has concrete elements, so that it is possible to describe kernels (or later on in §7 centers) as explicit subgroups.

Let G be a p-discrete toral group , $x \in G$ an element and $f : G \to \mathcal{X}$ a homomorphism of loop spaces. We will say that *$f(x)$ is trivial* if the restriction of f to the cyclic subgroup of G generated by x is trivial (2.1). The *kernel of f*, denoted $\ker(f)$ is the subset of G given by $\{x \mid f(x) \text{ is trivial}\}$. The kernel of f is said to be *trivial* if it contains only the identity element of G.

3.5 Proposition. [12, §7]. *Suppose that G is a p-discrete toral group, $\mathcal{X}$ a p-compact group, and $f : G \to \mathcal{X}$ a homomorphism. Then*

(1) $\ker(f)$ *is a normal subgroup of G,*
(2) *up to homotopy there is a unique homomorphism $f' : G/\ker(f) \to \mathcal{X}$ which extends f,*
(3) $\ker(f')$ *is trivial, and*
(4) *the induced homomorphism $\mathbf{C}_{\mathcal{X}}(G/\ker(f)) \to \mathbf{C}_{\mathcal{X}}(G)$ is an equivalence.*

The next proposition provides a simple way to recognize monomorphisms.

3.6 Proposition. [12, 7.3] *Suppose that G is a p-compact toral group, and $f : G \to \mathcal{X}$ is a homomorphism of p-compact groups. Let $\alpha : \breve{G} \to G$ be a discrete approximation. Then f is a monomorphism if and only if $\ker(f \cdot \alpha)$ is trivial.*

It is also useful to be able to recognize exact sequences.

3.7 Proposition. *Consider a commutative diagram of loop spaces and homomorphisms*

$$\begin{array}{ccccc} \breve{G} & \longrightarrow & \breve{H} & \longrightarrow & \breve{K} \\ \downarrow & & \downarrow & & \downarrow \\ G & \longrightarrow & H & \longrightarrow & K \end{array}$$

in which the lower row is a sequence of p-compact toral groups and the upper row is the corresponding (3.2) sequence of their discrete approximations. Then the lower sequence is exact if and only if the upper one is.

Proof. Suppose that the upper sequence is exact; in this case the problem is to show that the induced fibre sequence of classifying spaces remains a fibre sequence after $\mathbf{F}_p$-completion. By the fibre lemma [12, 11.7] this will be the case if the fundamental group of the base, namely $\breve{K}$, acts nilpotently on the mod p homology of the fibre. Since the mod p homology groups of the fibre are finite dimensional in each dimension [12, §12], the action of K on these homology groups must factor through the finite p-group obtained by dividing K by its maximal divisible subgroup. This implies that the action is nilpotent [12, 11.6].

Suppose on the other hand that the lower sequence is exact. The map $\breve{H} \to \breve{K}$ is onto by the description of discrete approximations given in 3.3: clearly $\pi_0 H \to \pi_0 K$ is onto, and, since the cokernel of $\pi_1 H \to \pi_1 K$ is a finite abelian p-group, $(\pi_1 H) \otimes \mathbf{Z}/p^\infty \to (\pi_1 K) \otimes \mathbf{Z}/p^\infty$ is onto. Let $\breve{G}'$ be the kernel of $\breve{H} \to \breve{K}$. Elementary algebra shows that $\breve{G}'$ is a p-discrete toral group, and by the uniqueness provision of 3.2 the map $\breve{G} \to \breve{H}$ lifts to a map $i : \breve{G} \to \breve{G}'$. Let G' be the closure of $\breve{G}$. The argument in the paragraph above shows that $G' \to H \to K$ is an exact sequence of p-compact toral groups, so by uniqueness of homotopy fibres the map $G \to G'$ induced by the closure of i is an equivalence, and hence (3.3) the map i is an isomorphism. □

3.8 Remark. One consequence of 3.7 is the following: if $G \to T \to K$ is a short exact sequence of p-compact toral groups in which T is a p-compact torus, then G is abelian and K is a p-compact torus. The proof of this amounts to inspecting the corresponding exact sequence of discrete approximations and noticing that any subgroup of $(\mathbf{Z}/p^\infty)^r$ is abelian, and any quotient group of $(\mathbf{Z}/p^\infty)^r$ is $(\mathbf{Z}/p^\infty)^s$ for some s.

3.9 One parameter subgroups. The p-discrete torus $\mathbf{Z}/p^\infty$ and its closure function like "one-parameter subgroups" of p-compact groups.

If $f : \mathcal{X} \to \mathcal{Y}$ is a homomorphism of loop spaces, we will write $f(\mathcal{X}) \subseteq \mathcal{Y}_1$ (and say that $f(\mathcal{X})$ lies in the identity component of $\mathcal{Y}$) if f lifts to a homomorphism $\mathcal{X} \to \mathcal{Y}_1$.

3.10 Proposition. *Suppose that $f : \mathbf{Z}/p^n \to \mathcal{X}$ is a homomorphism, where $\mathcal{X}$ is a p-compact group . Then $f(\mathbf{Z}/p^n) \subseteq \mathcal{X}_1$ if and only if f extends to a homomorphism $f' : \mathbf{Z}/p^\infty \to \mathcal{X}$.*

Proof. First suppose that the extension f' exists, and let A be the closure of $\mathbf{Z}/p^\infty$. Then A is connected (proof of 3.1) and the fact that $f(\mathbf{Z}/p^n) \subseteq \mathcal{X}_1$ follows from the fact that passing to closure gives an extension of f to a homomorphism $A \to \mathcal{X}$. If $f(\mathbf{Z}/p^n) \subseteq \mathcal{X}_1$ then the required extension f' exists by an inductive argument using [12, 5.5]. □

3.11 Proposition. *Suppose that $\mathcal{X}$ is a p-compact group. Then $\mathcal{X}$ is connected if and only if every homomorphism $f : \mathbf{Z}/p^n \to \mathcal{X}$ $(n \geq 1)$ extends to a homomorphism $f' : \mathbf{Z}/p^\infty \to \mathcal{X}$.*

Proof. If $\mathcal{X}$ is connected, the desired extensions exist by 3.10. Suppose conversely that $\mathcal{X}$ is not connected, and let $\breve{\mathbf{N}}_p(T)$ be a discrete approximation to a p-normalizer (2.10) of a maximal torus T for $\mathcal{X}$. By 2.11 the map $\breve{\mathbf{N}}_p(T) = \pi_0\breve{\mathbf{N}}_p(T) \to \pi_0\mathcal{X}$ is surjective. Since $\breve{\mathbf{N}}_p(T)$ is a union of finite p-groups, this implies that there exists a composite homomorphism $f : \mathbf{Z}/p^n \to \breve{\mathbf{N}}_p(T) \to \mathcal{X}$ such that $f(\mathbf{Z}/p^n) \nsubseteq \mathcal{X}_1$. □

3.12 A slight generalization. At some points in this paper it will be convenient for us to have a notion of discrete approximation somewhat more general than the one above.

Definition. A loop space G is an *extended p-compact torus* if $\pi_0 G$ is finite and the identity component G_1 is a p-compact torus. A discrete group G is an *extended p-discrete torus* if G has a normal subgroup T of finite index such that T is a p-discrete torus.

Note that if G is an extended p-discrete torus there is a *unique* normal p-discrete torus T in G such that G/T is finite; in fact T is the maximal divisible subgroup of G. We will denote this unique p-discrete torus by $G_{(1)}$. If $f : G \to \mathcal{X}$ is a homomorphism of loop spaces and $\pi_0\mathcal{X}$ is a finite group, then f induces a homomorphism $G/G_{(1)} \to \pi_0\mathcal{X}$ and the restriction of f to $G_{(1)}$ lifts uniquely to a homomorphism $G_{(1)} \to \mathcal{X}_1$.

Definition. A *discrete approximation* for an extended p-compact torus G is a homomorphism $f : \breve{G} \to G$, where $\breve{G}$ is an extended p-discrete torus and f induces

(1) an isomorphism $\breve{G}/\breve{G}_{(1)} \to \pi_0 G$, and
(2) an isomorphism $\mathrm{H}^*\mathrm{B}G_1 \to \mathrm{H}^*\mathrm{B}\breve{G}_{(1)}$.

It is not hard to see that the above notion specializes to the previous notion of discrete approximation (3.2) if G is a p-compact toral group (i.e., if $\pi_0 G$ is a p-group). The following proposition is proved in exactly the same way as 3.2 (cf. [12, 6.8]).

3.13 Proposition. *Any extended p-compact torus G has a discrete approximation $\breve{G} \to G$. If G and H are extended p-compact tori with discrete approximations $\breve{G}$ and $\breve{H}$, then any homomorphism $f : G \to H$ lifts uniquely to a homomorphism $\breve{f} : \breve{G} \to \breve{H}$.*

In particular, any two discrete approximations for an extended p-compact torus are canonically isomorphic (as ordinary discrete groups).

§4. Monomorphisms of maximal rank

In this section we will work out some points connected with maximal rank subgroups of a p-compact group. The conclusion is a theorem which can be interpreted as giving an "infinitesimal" criterion for a homomorphism of connected p-compact groups to be an equivalence (4.7).

4.1 Definition. A homomorphism $f : \mathcal{X} \to \mathcal{Y}$ of p-compact groups is said to be of *maximal rank* if f is a monomorphism and there is a maximal torus $i : T \to \mathcal{X}$ for $\mathcal{X}$ such that $f \cdot i$ is a maximal torus for $\mathcal{Y}$.

4.2 Remark. It follows from the uniqueness of maximal tori (2.9) that if $f : \mathcal{X} \to \mathcal{Y}$ is of maximal rank then *any* maximal torus for $\mathcal{X}$ gives a maximal torus for $\mathcal{Y}$. Similarly, if f is a monomorphism then f is of maximal rank if and only if some maximal torus $T \to \mathcal{Y}$ for $\mathcal{Y}$ lifts to a torus $T \to \mathcal{X}$; such a lift is necessarily a maximal torus for $\mathcal{X}$ (to see this, combine 2.15 with the multiplicativity of the Euler characteristic in the fibration $\mathcal{X}/T \to \mathcal{Y}/T \to \mathcal{X}/\mathcal{Y}$, e.g. 10.6).

There is one simple way to construct homomorphisms of maximal rank.

4.3 Propositions. *Let $\mathcal{X}$ be a p-compact group with maximal torus $T \to \mathcal{X}$, suppose that A is a p-compact toral group, and let $j : A \to T$ be a monomorphism. Then the natural monomorphism $\mathbf{C}_{\mathcal{X}}(A) \to \mathcal{X}$ is of maximal rank.*

Proof. There is a commutative diagram

$$\begin{array}{ccccc}
\mathbf{C}_T(T) & \longrightarrow & \mathbf{C}_T(A) & \longrightarrow & \mathbf{C}_{\mathcal{X}}(A) \\
\cong\downarrow & & \downarrow & & \downarrow \\
T & \xrightarrow{=} & T & \longrightarrow & \mathcal{X}
\end{array}$$

in which the left vertical arrow is an equivalence because T is abelian. This provides a lift of T up to conjugacy to $\mathbf{C}_{\mathcal{X}}(A)$. $\square$

4.4 Lemma. *Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is of maximal rank, $i : T \to \mathcal{X}$ is a maximal torus for $\mathcal{X}$, and $f \cdot i : T \to \mathcal{Y}$ the corresponding maximal torus for $\mathcal{Y}$. Then there is a natural monomorphism of Weyl groups $W_{\mathcal{X},i} \to W_{\mathcal{Y},fi}$.*

Remark. The above monomorphism $W_{\mathcal{X},i} \to W_{\mathcal{Y},fi}$ depends only up to inner automorphism on the choice of a torus $T \to \mathcal{X}$ for $\mathcal{X}$. Sometimes for brevity we will say that f induces a monomorphism $W_{\mathcal{X}} \to W_{\mathcal{Y}}$.

Proof of 4.4. By definition, the Weyl group $W_{\mathcal{X},i}$ is $\pi_0 \mathbf{W}_{\mathcal{X},i}$, where $\mathbf{W}_{\mathcal{X},i}$ is the space of self-maps over $\mathrm{B}\mathcal{X}$ of a fibration $u : U \to \mathrm{B}\mathcal{X}$ equivalent to $i : \mathrm{B}T \to \mathrm{B}\mathcal{X}$. Similarly, $W_{\mathcal{Y},fi} = \pi_0 \mathbf{W}_{\mathcal{Y},fi}$ where $\mathbf{W}_{\mathcal{Y},fi}$ is the space of self-maps over $\mathrm{B}\mathcal{Y}$ of a fibration $v : V \to \mathrm{B}\mathcal{Y}$ equivalent to $fi : \mathrm{B}T \to \mathrm{B}\mathcal{Y}$. We can assume without loss of generality that $f : \mathrm{B}\mathcal{X} \to \mathrm{B}\mathcal{Y}$ is a fibration and choose $v = f \cdot u$; in this case there is essentially an inclusion $\mathbf{W}_{\mathcal{X},i} \to \mathbf{W}_{\mathcal{Y},fi}$ of topological monoids and it is not hard to check that the induced component homomorphism $W_{\mathcal{X},i} \to W_{\mathcal{Y},fi}$ does not depend on the fibration choices involved. This gives the required map $W_{\mathcal{X},i} \to W_{\mathcal{Y},fi}$; it remains to show that the map is a monomorphism.

As in [12, 9.5] the group $W_{\mathcal{X},i}$ is isomorphic to the set of components of the homotopically discrete space $(\mathcal{X}/T)^{\mathrm{h}T}$, and $W_{\mathcal{Y},fi}$ to the set of components of $(\mathcal{Y}/T)^{\mathrm{h}T}$ (see [12, §10] for a discussion of these homotopy fixed point sets). The map between these groups can be produced on the level of sets by applying the functor $(-)^{\mathrm{h}T}$ to the map $\mathcal{X}/T \to \mathcal{Y}/T$ induced by f. Let C be the component of $(\mathcal{Y}/\mathcal{X})^{\mathrm{h}T}$ corresponding to $i : T \to \mathcal{X}$ [12, 3.3]. Applying $(-)^{\mathrm{h}T}$ to the fibration $\mathcal{X}/T \to \mathcal{Y}/T \to \mathcal{Y}/\mathcal{X}$ [12, 10.6] shows that the map $(\mathcal{X}/T)^{\mathrm{h}T} \to (\mathcal{Y}/T)^{\mathrm{h}T}$ is an injection on π_0 if and only if the natural action of $\pi_1 C$ on $\pi_0(\mathcal{X}/T)^{\mathrm{h}T}$ is trivial. To prove this action is trivial, we will prove that C is simply connected. Applying $(-)^{\mathrm{h}T}$ to the fibration $\mathcal{Y}/\mathcal{X} \to \mathrm{B}\mathcal{X} \to \mathrm{B}\mathcal{Y}$ (with T "acting" [12, 10.8] trivially on base and total space) shows that C is the basepoint component of $\mathbf{C}_{\mathcal{Y}}(T)/\mathbf{C}_{\mathcal{X}}(T)$. This last homogeneous space is in fact homotopically discrete because $\mathbf{C}_{\mathcal{X}}(T) \to \mathbf{C}_{\mathcal{Y}}(T)$ is a homomorphism of p-compact groups which is an equivalence on identity components, both identity components being equivalent to T (2.9). □

If $\mathcal{X}$ is a space let $\mathrm{cd}_{\mathbf{F}_p}\mathcal{X}$ denote the largest integer n such that $\mathrm{H}^n\mathcal{X} \neq 0$. Similarly, let $\mathrm{cd}_{\mathbf{Q}_p}\mathcal{X}$ denote the largest integer n such that $\mathrm{H}^n_{\mathbf{Q}_p}\mathcal{X} \neq 0$.

4.5 Lemma. *If $\mathcal{X}$ is a p-compact group then $\mathrm{cd}_{\mathbf{F}_p}\mathcal{X} = \mathrm{cd}_{\mathbf{Q}_p}\mathcal{X}$.*

Proof. This follows from the Bockstein spectral sequence arguments in [4]. The arguments depend only on the fact that $\mathcal{X}$ is an $\mathbf{F}_p$-finite H-space. □

4.6 Lemma. *Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a monomorphism of p-compact groups. Then*

(1) $\mathrm{cd}_{\mathbf{F}_p}\mathcal{Y} = \mathrm{cd}_{\mathbf{F}_p}\mathcal{X} + \mathrm{cd}_{\mathbf{F}_p}\mathcal{Y}/\mathcal{X}$,
(2) $\mathrm{cd}_{\mathbf{F}_p}\mathcal{Y} = \mathrm{cd}_{\mathbf{F}_p}\mathcal{X}$ *if and only if the map f gives an equivalence between $\mathcal{X}$ and a union of components of $\mathcal{Y}$, and*
(3) *If $\mathcal{X}$ is connected then* $\mathrm{cd}_{\mathbf{Q}_p}\mathcal{Y} = \mathrm{cd}_{\mathbf{Q}_p}\mathcal{X} + \mathrm{cd}_{\mathbf{Q}_p}\mathcal{Y}/\mathcal{X}$.

Proof. The statements involving $\mathrm{cd}_{\mathbf{F}_p}\mathcal{Y}$ are from [12, 6.14]. The equality in (3) follows along the lines of [12, proof of 6.14] from an examination of the Serre spectral sequence of the fibration $\mathcal{X} \to \mathcal{Y} \to \mathcal{Y}/\mathcal{X}$. Note that in this fibration the fundamental group of the base acts trivially on the (co)-homology of the fibre, since the fibration is pulled back from the path fibration over the simply connected space $\mathrm{B}\mathcal{X}$. □

4.7 Theorem. *A monomorphism $f : \mathcal{X} \to \mathcal{Y}$ between connected p-compact groups is an equivalence if and only if f is of maximal rank and induces (4.4) a Weyl group isomorphism $W_{\mathcal{X}} \to W_{\mathcal{Y}}$.*

Proof. It is only necessary to show that if f is of maximal rank and induces a Weyl group isomorphism then f is an equivalence. Let T be a maximal torus for $\mathcal{X}$, let W denote $W_{\mathcal{X},T} \cong W_{\mathcal{Y},T}$, and let R denote the ring of invariants $(\mathrm{H}^*_{\mathbf{Q}_p}\mathrm{B}T)^W$. Write S for the ring $\mathbf{Q}_p \otimes_R \mathrm{H}^*_{\mathbf{Q}_p}\mathrm{B}T$. According to [12, 9.7], $\mathrm{H}^*_{\mathbf{Q}_p}(\mathcal{X}/T)$ and $\mathrm{H}^*_{\mathbf{Q}_p}(\mathcal{Y}/T)$ are both isomorphic to S. By 4.6(3), $\mathrm{cd}_{\mathbf{Q}_p}\mathcal{X} = \mathrm{cd}_{\mathbf{Q}_p}\mathcal{Y}$ and hence $\mathrm{cd}_{\mathbf{F}_p}\mathcal{X} = \mathrm{cd}_{\mathbf{F}_p}\mathcal{Y}$ by 4.5. The result now follows from 4.6(2). □

§5. Abelian p-compact groups and central monomorphisms

The main goal of this section is to prove the following theorem.

5.1 Theorem. *If $f : \mathcal{X} \to \mathcal{Y}$ is a central monomorphism of p-compact groups, the $\mathcal{X}$ is equivalent to the product of a p-compact torus and a finite abelian p-group.*

This is analogous to the theorem that if $f : G \to H$ is a monomorphism of compact Lie groups with $f(G)$ contained in the center of H, then G is isomorphic to the product of a torus with a finite abelian group. Applying 5.1 to the identity map of a p-compact group $\mathcal{X}$ gives the following corollary (the "if" part of which is essentially obvious).

5.2 Corollary. *A p-compact group $\mathcal{X}$ is abelian if and only if $\mathcal{X}$ is equivalent to the product of a p-compact torus and a finite abelian p-group.*

Remark. A beautiful result of Bousfield [2, 6.9] implies that if $\mathcal{X}$ is a loop space which is abelian in the sense of §2 then $\mathrm{B}\mathcal{X}$ is equivalent to a product of Eilenberg-MacLane spaces. Given this result, it is possible to prove 5.2 by just enumerating the products of Eilenberg-MacLane spaces which can be classifying spaces of p-compact groups. We will use a different approach.

The proof of 5.1 depends on a few lemmas.

5.3 Lemma. *Suppose that $f : \mathcal{X} \to \mathcal{Z}$ and $g : \mathcal{Y} \to \mathcal{Z}$ are homomorphisms of loop spaces and that g is central. Then there is up to conjugacy a unique homomorphism $\mu(f,g) : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ which is conjugate to f on $\mathcal{X} \times \{1\}$ and to g on on $\{1\} \times \mathcal{Y}$. The homomorphism $\mu(f,g)$ is central if f is.*

Proof. This follows from the equivalence

$$\mathrm{Map}(\mathrm{B}\mathcal{X} \times \mathrm{B}\mathcal{Y}, \mathrm{B}\mathcal{Z}) \cong \mathrm{Map}(\mathrm{B}\mathcal{X}, \mathrm{Map}(\mathrm{B}\mathcal{Y}, \mathrm{B}\mathcal{Z}))\,.$$

The assumption that g is central gives $\mathrm{Map}(\mathrm{B}\mathcal{Y}, \mathrm{B}\mathcal{Z})_{\mathrm{B}g} \cong \mathrm{B}\mathcal{Z}$. □

5.4 Lemma. *Let G be a p-discrete toral group, $\mathcal{X}$ a p-compact group, and $f : G \to \mathcal{X}$ a homomorphism. Suppose that A is a cyclic p-group with generator a and $i, j : A \to G$ two homomorphisms such $i(a)$ commutes with $j(a)$, $f \cdot i$ is conjugate to $f \cdot j$, and $f \cdot i$ is central. Then $i(a)^{-1}j(a) \in \ker(f)$.*

Proof. Let $s : A \times A \to G$ be given by $s(x) = i(x)j(x)$. By the uniqueness provision of 5.3, the composite $f \cdot s$ is equal to the composite of $f \cdot i$ with the sum map $A \times A \to A$. This implies directly that $s(a^{-1}, a) \in \ker(f)$. □

5.5 Lemma. *Let $g : G \to H$ be a homomorphism of p-discrete toral groups and $h : H \to \mathcal{X}$ a homomorphism with $\ker(h) = \{1\}$, where $\mathcal{X}$ is a p-compact group. Suppose that $h \cdot g$ is central. Then g is central.*

Proof. Since $h \cdot g$ is central there is by 5.3 a homomorphism $G \times \mathcal{X} \to \mathcal{X}$ which is conjugate to $h \cdot g$ on $G \times \{1\}$ and to $\mathrm{id}_{\mathcal{X}}$ on $\{1\} \times \mathcal{X}$. This can be restricted to give a homomorphism $f : G \times H \to \mathcal{X}$ which is conjugate to $h \cdot g$ on $G \times \{1\}$ and to h on $\{1\} \times H$. By 5.4, for each $x \in G$ the element $(x, g(x)^{-1})$ in $G \times H$ lies in $\ker(f)$. Since $\ker(f)$ is a normal subgroup of $G \times H$ which intersects $\{1\} \times H$ only in the identity element, it follows from elementary algebra that $g(G)$ must lie in the center of H. □

5.6 Lemma. *Suppose that P is a p-compact toral group, $\mathcal{X}$ is a p-compact group, and $f : P \to \mathcal{X}$ is a central monomorphism. Then P is equivalent to the product of a p-compact torus and a finite abelian p-group.*

Proof. Let $\breve{P}$ be a discrete approximation (3.2) for P, and $\breve{f}$ the composite of f with the map $\breve{P} \to P$. Since $\ker(\breve{f})$ is trivial (3.6), it follows from 5.5 that $\breve{P}$ is abelian and hence for algebraic reasons isomorphic to the product of a finite abelian p-group and a group of the form $(\mathbf{Z}/p^\infty)^r$. This implies that P, which is equivalent to the closure (3.1) of $\breve{P}$, has the desired form. □

Proof of 5.1. Let $i : T \to \mathcal{X}$ be a maximal torus for $\mathcal{X}$ with discrete approximation $\breve{T} \to T$. Let $\breve{i}$ be the composite of i with $\breve{T} \to T$. The composites $f \cdot i$ and $f \cdot \breve{i}$ are central by 2.6. Suppose that the action of the Weyl group W of the identity component $\mathcal{X}_1$ of $\mathcal{X}$ on $\mathrm{H}^2(\mathrm{B}T; \mathbf{Z}_p)$ is nontrivial. In this case there must be a self homotopy equivalence $\mathrm{B}w : \mathrm{B}T \to \mathrm{B}T$, not homotopic to the identity, such that $\mathrm{B}i \cdot \mathrm{B}w$ is homotopic to $\mathrm{B}i$. Such a w lifts uniquely (3.2) to an automorphism $\breve{w} : \breve{T} \to \breve{T}$, not equal to the identity, such that $\breve{i} \cdot \breve{w}$ is conjugate to $\breve{i}$. Since $\breve{w}$ is not the identity, there must be some element $x \in \breve{T}$ such that $\breve{w}(x) \neq x$. By 5.4, then, $(x^{-1}\breve{w}(x)) \neq 1$ belongs to $\ker(\breve{f} \cdot i)$, which is impossible (3.6) because $f \cdot i$ is a monomorphism. This shows that W acts trivially on $\mathrm{H}^2(\mathrm{B}T; \mathbf{Z}_p)$ and hence, since this action is faithful [12, 9.7], that W itself is trivial. By 4.7 (applied to $T \to \mathcal{X}_1$), the p-compact group $\mathcal{X}_1$ is a torus. Since $\mathcal{X}$ is thus a p-compact toral group, the desired result is given by 5.6. □

§6. p-COMPACT CENTERS AND ADJOINT FORMS.

Let $f : C \to \mathcal{X}$ be a central monomorphism of p-compact groups. The map f is said to be a *p-compact center* for $\mathcal{X}$ (or more informally C is said to be a p-compact center for $\mathcal{X}$) if for any central monomorphism $g : A \to \mathcal{X}$ of p-compact groups there exists up to homotopy a unique homomorphism $h : A \to C$ such that $f \cdot h$ is homotopic to g. This signifies that up to homotopy f is a terminal object in the category of central monomorphisms $A \to \mathcal{X}$.

6.1 Theorem. *Any p-compact group $\mathcal{X}$ has a p-compact center $f : C \to \mathcal{X}$. Suppose that f is such a p-compact center and that $g : A \to \mathcal{X}$ is a central homomorphism in which A is a p-compact toral group; let $L(g, f)$ be the space of all homomorphisms $h : A \to C$ such that $f \cdot h = g$. Then $L(g, f)$ is contractible.*

Remark. Note above that if $A \to \mathcal{X}$ is a central *monomorphism*, then A is necessarily a p-compact toral group (5.1).

6.2 Remark. Let $(\mathrm{B}f)^{\mathrm{fib}} : (\mathrm{B}C)^{\mathrm{fib}} \to \mathrm{B}\mathcal{X}$ be a pointed fibration equivalent to $\mathrm{B}f$. By the above phrase *the space of all homomorphisms $h : A \to C$ such that $f \cdot h = g$* we mean the space of all basepoint preserving maps $\mathrm{B}h : \mathrm{B}A \to (\mathrm{B}C)^{\mathrm{fib}}$ such that $(\mathrm{B}f)^{\mathrm{fib}} \cdot \mathrm{B}h = \mathrm{B}g$.

Theorem 6.1 implies that p-compact centers are homotopically unique in a very strong sense. For this reason we will sometimes speak of *the* p-compact center of $\mathcal{X}$ and denote it by $\mathbf{C}^{\mathrm{c}}(\mathcal{X})$. According to 2.8 there is a natural p-compact group structure on $\mathcal{X}/\mathbf{C}^{\mathrm{c}}(\mathcal{X})$.

6.3 Theorem. *If $\mathcal{X}$ is a connected p-compact group then the p-compact center of $\mathcal{X}/\mathbf{C}^{\mathrm{c}}(\mathcal{X})$ is trivial.*

Remark. If $\mathcal{X}$ is a connected p-compact group, the quotient $\mathcal{X}/\mathbf{C}^{\mathrm{c}}(\mathcal{X})$ is called the *adjoint form* $\mathcal{X}_{\mathrm{ad}}$ of $\mathcal{X}$. Note that if $\mathcal{X}$ is not connected then $\mathcal{X}/\mathbf{C}^{\mathrm{c}}(\mathcal{X})$ may have a nontrivial p-compact center; consider, for instance, a nonabelian finite p-group or the Lie group O(2).

Let $\mathcal{X}$ be a p-compact group, C a p-discrete toral group, and $f : C \to \mathcal{X}$ a central homomorphism such that $\ker(f) = \{1\}$. We will say that f is a *p-discrete center* for $\mathcal{X}$ if the induced map $\bar{f} : \bar{C} \to \mathcal{X}$ is a p-compact center for $\mathcal{X}$. Such a p-discrete center is just a discrete approximation to $\mathbf{C}^{\mathrm{c}}(\mathcal{X})$ and has very similar uniqueness properties. In the course of proving 6.1 we will give the following description of the p-discrete center of a p-compact group $\mathcal{X}$; for connected $\mathcal{X}$ a more explicit description appears in §7.

6.4 Theorem. *Let $\mathcal{X}$ be a p-compact group, T a maximal torus for $\mathcal{X}$, $\mathbf{N}_p(T)$ the inverse image in the normalizer of T of a p-Sylow subgroup of the Weyl group of $\mathcal{X}$, and $\breve{\mathbf{N}}_p(T)$ a discrete approximation for $\mathbf{N}_p(T)$. Let $\breve{C} \subset \breve{\mathbf{N}}_p(T)$ be the set of elements $x \in \breve{\mathbf{N}}_p(T)$ such that $\langle x\rangle \to \mathcal{X}$ is central. Then $\breve{C}$ is a subgroup of $\breve{\mathbf{N}}_p(T)$ and $\breve{C} \to \mathcal{X}$ is a p-discrete center for $\mathcal{X}$.*

6.5 Lemma. *Let G be a p-compact toral group and $g : G \to \mathcal{X}$ a monomorphism of p-compact groups. Suppose that that H is a p-compact toral group, and that $h : H \to \mathcal{X}$ is a central homomorphism. Then if h lifts to any homomorphism $\tilde{h} : H \to G$ it lifts to exactly one, in the sense that the space (cf. 6.2) of all such lifts $\tilde{h}$ is contractible.*

Proof. Suppose that h lifts to a homomorphism $\tilde{h} : H \to G$. Let $i : \breve{H} \to H$ be a discrete approximation, and let $\breve{K} \subset \breve{H}$ be the kernel (3.5) of $h \cdot i$. It is clear from 3.6 that $\breve{K}$ is also the kernel of $\tilde{h} \cdot i$, since otherwise, after (3.2) lifting $\tilde{h} \cdot i$ to a map $\breve{H} \to \breve{G}$ for some discrete approximation $\breve{G}$ of G, we would find a nontrivial element in the kernel of the composite $\breve{G} \to G \to \mathcal{X}$. Let H' be the closure (3.1) of $\breve{H}/\breve{K}$. By naturality of the closure construction there is a homomorphism $H \to H'$. The map h extends (3.5, 3.4) to a central monomorphism $h' : H' \to \mathcal{X}$. Similarly, $\tilde{h}$ extends to a monomorphism $\tilde{h}' : H' \to G$ which is a lift of h'. An argument using 5.5 and discrete approximations shows that $\tilde{h}'$ is central.

Consider the diagram (2.8)

$$\begin{array}{ccccc} & & \mathrm{B}G & \longrightarrow & \mathrm{B}(G/\tilde{h}'(H')) \\ & & {\scriptstyle \mathrm{B}g}\downarrow & & \downarrow \\ \mathrm{B}H & \xrightarrow{\mathrm{B}h} & \mathrm{B}\mathcal{X} & \longrightarrow & \mathrm{B}(\mathcal{X}/h'(H')) \end{array}$$

in which the right hand square is a homotopy fibre square and the composite of the two lower arrows is null homotopic. Let $\mathcal{E}$ be the homotopy pullback of $\mathrm{B}G \to \mathrm{B}\mathcal{X}$ over $\mathrm{B}h$. It follows that there is a homotopy equivalence over $\mathrm{B}H$ between $\mathcal{E}$ and $\mathrm{B}H \times (\mathcal{X}/G)$, and so the space of lifts up to conjugacy of h into G (i.e. the homotopy fibre over $\mathrm{B}h$ of $\mathrm{Map}(\mathrm{B}H, \mathrm{B}G) \to \mathrm{Map}(\mathrm{B}H, \mathrm{B}\mathcal{X})$) is equivalent to $\mathrm{Map}(\mathrm{B}H, \mathcal{X}/G)$. This last space is equivalent by evaluation at the basepoint of $\mathrm{B}H$ to $\mathcal{X}/G$ itself [12, 6.1 and 5.3]. There is now a 3×3 square

$$\begin{array}{ccccc}
\mathcal{F} & \longrightarrow & \mathcal{X}/G & \xrightarrow{=} & \mathcal{X}/G \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Map}_*(\mathrm{B}H,\mathrm{B}G)_{\{\mathrm{B}h\}} & \longrightarrow & \mathrm{Map}(\mathrm{B}H,\mathrm{B}G)_{\{\mathrm{B}h\}} & \longrightarrow & \mathrm{B}G \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Map}_*(\mathrm{B}H,\mathrm{B}\mathcal{X})_{\mathrm{B}h} & \longrightarrow & \mathrm{Map}(\mathrm{B}H,\mathrm{B}\mathcal{X})_{\mathrm{B}h} & \longrightarrow & \mathrm{B}\mathcal{X}
\end{array}$$

in which the rows and columns are fibration sequences; a subscript "$\{\mathrm{B}h\}$" denotes restricting to the components of the indicated mapping space that cover $\mathrm{B}h$ up to homotopy. The space $\mathcal{F}$ is the desired space of lifts $\tilde{h}$, and is clearly contractible. □

6.6 Lemma. *Let $\mathcal{X}$ be a p-compact group, P a p-discrete toral group, and $f : P \to \mathcal{X}$ a monomorphism. Let $C \subset P$ be the set of all elements $x \in P$ such that $f(\langle x\rangle)$ is central. Then C is a subgroup of P and $f(C)$ is central.*

Proof. By 5.5, C is contained in the center of P. It is enough to show that if $\{C_\sigma\}_{\sigma\in S}$ is a countable collection of subgroups of the center of P such that $f(C_\sigma)$ is central for each $\sigma \in S$, then the subgroup of the center of P generated by the C_σ is central. For this, observe first that if C_1 and C_2 are two subgroups of the center of P such that $f(C_1)$ and $f(C_2)$ are central, then $C_1 + C_2$ is central; this is a consequence of 5.3 and 3.5. Secondly, apply a homotopy limit argument [12, 6.20] to show that if $C_0 \subset C_1 \subset C_2 \cdots$ is an increasing chain of subgroups of the center of P such that $f(C_i)$ is central for each i, then $f(\cup_i C_i)$ is central. □

Proof of 6.1 and 6.4. We will use the notation of 6.4. By 6.6 the group $\breve{C}$ is a subgroup of the center of $\breve{\mathbf{N}}_p(T)$ and $f(\breve{C})$ is central. Let $C \to \mathcal{X}$ be the closure of $\breve{C}$; by density the map $C \to \mathcal{X}$ is central and by 3.6 it is a monomorphism (cf. 2.10). Suppose that $f : A \to \mathcal{X}$ is a central homomorphism in which A is a p-compact toral group; for instance, f might be a central monomorphism (5.1). Since $\chi(\mathcal{X}/\mathbf{N}_p(T))$ is relatively prime to p (2.10), the map f lifts by 2.14 to a central (5.5) homomorphism $g : A \to \mathbf{N}_p(T)$. Let $\breve{A} \to A$ be a discrete approximation. By 3.2 the composite $\breve{A} \to A \to \mathbf{N}_p(T)$ lifts to a homomorphism $\breve{g} : \breve{A} \to \breve{\mathbf{N}}_p(T)$ such that, by 2.6 and the definition of $\breve{C}$, $f(\breve{A}) \subset \breve{C}$. Passing to closures gives a lift $A \to C$ of the homomorphism f. This proves in particular that C is a p-compact center for $\mathcal{X}$. The statement about $L(g, f)$ follows from 6.5. □

6.7 Remark. The subgroup $\breve{C}$ of $\breve{\mathbf{N}}_p(T)$ considered in the above proof is contained in the center of $\breve{\mathbf{N}}_p(T)$ (5.5). Let $\breve{T}$ be the maximal divisible subgroup of $\breve{\mathbf{N}}_p(T)$, so that $\breve{T}$ is a discrete approximation to the maximal torus T. If $\mathcal{X}$ is connected the quotient group $\breve{\mathbf{N}}_p(T)/\breve{T} \subset W_{\mathcal{X}}$ acts faithfully on $\pi_2 T$ (2.13) and hence faithfully by conjugation on $\breve{T} = \pi_2 T \otimes \mathbf{Z}/p^\infty$ (3.3). It follows that if $\mathcal{X}$ is connected the group $\breve{C}$ is contained in $\breve{T}$ and so, after passing to closures, that the map $\mathbf{C}^{\mathrm{c}}(\mathcal{X}) \to \mathcal{X}$ lifts to a unique (6.5) homomorphism $\mathbf{C}^{\mathrm{c}}(\mathcal{X}) \to T$.

6.8 Lemma. *Suppose that*

$$\{1\} \to A \xrightarrow{i} \mathcal{X} \to \mathcal{Y} \to \{1\}$$

is a short exact sequence of p-compact groups in which A is an abelian p-compact toral group and $\mathcal{Y}$ is connected. Then the monomorphism i is central.

Proof. Since A is abelian the identity component $\mathrm{Map}(\mathrm{B}A, \mathrm{B}A)_{\mathrm{id}}$ of the space of self homotopy equivalences of $\mathrm{B}A$ is equivalent to $\mathrm{B}A$. This implies that any fibration over a simply connected base with fibre $\mathrm{B}A$ is principal. In particular the fibration of classifying spaces corresponding to the above short exact sequence is principal, and the corresponding action map $\mathrm{B}A \times \mathrm{B}\mathcal{X} \to \mathrm{B}\mathcal{X}$ has the properties required to show that i is central (2.6). □

Proof of 6.3. Let T be a maximal torus for $\mathcal{X}$. As remarked above (6.7), the homomorphism $\mathbf{C}^{\mathrm{c}}(\mathcal{X}) \to \mathcal{X}$ lifts to a homomorphism $\mathbf{C}^{\mathrm{c}}(\mathcal{X}) \to T$; this lift is a central (5.5) monomorphism (3.6). Let $\mathcal{X}'$ denote $\mathcal{X}/\mathbf{C}^{\mathrm{c}}(\mathcal{X})$ and let T' denote $T/\mathbf{C}^{\mathrm{c}}(\mathcal{X})$ (see 2.8). By 3.8 the p-compact group T' is a p-compact torus. Consider the map of short exact sequences

$$\begin{array}{ccccccccc}
\{1\} & \longrightarrow & \mathbf{C}^{\mathrm{c}}(\mathcal{X}) & \longrightarrow & T & \longrightarrow & T' & \longrightarrow & \{1\} \\
 & & {\scriptstyle =}\downarrow & & \downarrow & & \downarrow & & \\
\{1\} & \longrightarrow & \mathbf{C}^{\mathrm{c}}(\mathcal{X}) & \longrightarrow & \mathcal{X} & \longrightarrow & \mathcal{X}' & \longrightarrow & \{1\}
\end{array} .$$

Taking vertical fibres in the induced diagram of classifying spaces shows that $\mathcal{X}/T \cong \mathcal{X}'/T'$, so that $T' \to \mathcal{X}'$ is a maximal torus for $\mathcal{X}'$ (2.15). Suppose that $C' \to \mathcal{X}'$ is a p-compact center and let $C \to \mathcal{X}$ be the "inverse image" of C' in $\mathcal{X}$, i.e., let $\mathrm{B}C$ be given by the homotopy fibre square

$$\begin{array}{ccc}
\mathrm{B}C & \longrightarrow & \mathrm{B}\mathcal{X} \\
\downarrow & & \downarrow \\
\mathrm{B}C' & \longrightarrow & \mathrm{B}\mathcal{X}'
\end{array} \tag{6.9}$$

It is clear from a homotopy group calculation that C is a p-compact toral group. Since $\mathcal{X}'$ is connected, the map $C' \to \mathcal{X}'$ lifts to a monomorphism $C' \to T'$ as above, and so the map $C \to \mathcal{X}$ lifts to a monomorphism $C \to T$ (the fact $C \to T$ is a monomorphism follows from the fact that T/C is equivalent to T'/C'). By 3.8, then, C is abelian. There is an evident short exact sequence $C \to \mathcal{X} \to \mathcal{X}'/C'$, and applying 6.8 to it shows that the map $C \to \mathcal{X}$ is central. As a consequence, the homomorphism $C \to \mathcal{X}$ lifts to a homomorphism $C \to \mathbf{C}^{\mathrm{c}}(\mathcal{X})$ and so the composite $C \to \mathcal{X} \to \mathcal{X}'$ is trivial. An argument with discrete approximations, for instance, now shows that $C' \to \mathcal{X}'$ is trivial and thus that C is trivial (3.7, 3.6). □

§7. Computing p-compact centers

In this section we will give an explicit way to compute the p-compact center of a p-compact group $\mathcal{X}$ in terms of the normalizer of a maximal torus for $\mathcal{X}$. Suppose that $\mathcal{X}$ is a p-compact group of rank r (2.13), T is a maximal torus for $\mathcal{X}$, and $W = W_{\mathcal{X}}$ is the Weyl group of $\mathcal{X}$. Let $\mathbf{N}(T)$ be the normalizer of T, $\breve{\mathbf{N}}(T) \to \mathbf{N}(T)$ a discrete approximation in the sense of §3.12, and $\breve{T}$ the maximal divisible subgroup $\breve{\mathbf{N}}(T)_{(1)}$ of $\breve{\mathbf{N}}(T)$, so that $\breve{T}$ is a discrete approximation in the usual sense for $\mathbf{N}(T)_1 = T$.

The Weyl group W acts on $\breve{T}$ by conjugation in the group extension

$$\{1\} \to \breve{T} \to \breve{\mathbf{N}}(T) \to W \to \{1\} \tag{7.1}$$

and on $\mathrm{B}T$ by monodromy in the fibration

$$\mathrm{B}T \to \mathrm{B}\mathbf{N}(T) \to \mathrm{B}W\,. \tag{7.2}$$

The fibration 7.2 is obtained from the exact sequence 7.1 by passing to classifying spaces and taking fibrewise $\mathbf{F}_p$-completion; under this construction the two actions correspond.

Let $s \in W$ be a reflection (2.12). The order $\mathrm{ord}(s)$ of s must divide the order of the multiplicative group of roots of unity in ${\mathbf{Q}_p}^*$, so that $\mathrm{ord}(s) = 2$ if $p = 2$ and $\mathrm{ord}(s)$ divides $(p-1)$ if p is odd. Since $\mathrm{H}^2(\mathrm{B}T, \mathbf{Z}_p)$ is the dual over $\mathbf{Z}_p$ of $\pi_2 \mathrm{B}T = (\mathbf{Z}_p)^r$, a nonidentity element $s \in W$ is a reflection if and only if the fixed point set of the action of s on $\pi_2\mathrm{B}T = \pi_1 T$ contains a subgroup isomorphic to $(\mathbf{Z}_p)^{r-1}$, or equivalently if and only if the action of s on $\breve{T} = \pi_1 T \otimes \mathbf{Z}/p^\infty$ (see 3.3) contains a subgroup isomorphic to $(\mathbf{Z}/p^\infty)^{r-1}$.

7.3 Definition. If $s \in W$ is a reflection,

(1) the *fixed point set* $F(s)$ of s is the fixed point set of the action of x on $\breve{T}$ by conjugation,
(2) the *singular hyperplane* $H(s)$ of s is the maximal divisible subgroup of $F(s)$ (so that $H(s) \cong (\mathbf{Z}/p^\infty)^{r-1}$),
(3) the *singular coset* $K(s)$ of s is the subset of $\breve{T}$ given by elements of the form $x^{\mathrm{ord}(s)}$, as x runs through elements of $\breve{\mathbf{N}}(T)$ which project to s in W, and
(4) the *singular set* $\sigma(s)$ of s is the union $\sigma(s) = H(s) \cup K(s)$.

7.4 Remark. The singular coset $K(s)$ is genuinely a coset of $H(s)$ in $\breve{T}$. On the one hand, suppose that x and x' are two elements of $\breve{\mathbf{N}}(T)$ which project to s. Then $x = x'a$ for some $a \in \breve{T}$, and it is easy to calculate directly that $x^{\mathrm{ord}(s)} = (x')^{\mathrm{ord}(s)}\nu(a)$, where ν is the endomorphism of $\breve{T}$ which sends a to the product of the elements $s^i a$, $1 \le i \le \mathrm{ord}(s)$. The image of ν is a divisible subgroup of $\breve{T}$ which is pointwise fixed by s and thus contained in $H(s)$. It follows that $x^{\mathrm{ord}(s)}$ and $(x')^{\mathrm{ord}(s)}$ are in the same coset of $H(s)$. Conversely, if $y = x^{\mathrm{ord}(s)} \in K(s)$ and $a \in H(s)$, it is possible (because $H(s)$ is divisible) to find $b \in H(s)$ with $b^{\mathrm{ord}(s)} = a$. Then $ay = (bx)^{\mathrm{ord}(s)}$.

Our main theorem in this section is the following one.

7.5 Theorem. *Let $\mathcal{X}$ be a connected p-compact group, T a maximal torus for $\mathcal{X}$, and $\breve{T} \to T$ a discrete approximation for T. Define $C \subset \breve{T}$ by*

$$C = \cap_s \sigma(s)$$

where the intersection is indexed by reflections $s \in W$. Then $C \to \mathcal{X}$ is a p-discrete center (6.4) for $\mathcal{X}$.

This is actually a corollary of a more general calculation. Suppose that $A \subset \breve{T}$ is a subgroup. Let $W_{\mathcal{X}}(A)$ denote the Weyl group of $\mathbf{C}_{\mathcal{X}}(A)$, and $W_{\mathcal{X}}(A)_1$ the Weyl group of the identity component $\mathbf{C}_{\mathcal{X}}(A)_1$ of $\mathbf{C}_{\mathcal{X}}(A)$. There are inclusions $W_{\mathcal{X}}(A)_1 \subset W_{\mathcal{X}}(A) \subset W$, where the last follows from §4.

7.6 Theorem. *Let $\mathcal{X}$ be a connected p-compact group with maximal torus T and Weyl group W. Suppose that $A \subset \breve{T}$ is a subgroup. Then*

1. *$W_{\mathcal{X}}(A)$ is the subgroup of W consisting of the elements which, under the conjugation action of W on $\breve{T}$, pointwise fix the subgroup A, and*
2. *$W_{\mathcal{X}}(A)_1$ is the subgroup of $W_{\mathcal{X}}(A)$ generated by those elements $s \in W_{\mathcal{X}}(A)$ such that $s \in W$ is a reflection and $A \subset \sigma(s)$.*

7.7 Remark. The above theorems raise some natural questions about the structure of $\sigma(s)$. By definition there are inclusions

$$H(s) \subset \sigma(s) = H(s) \cup K(s) \subset F(s)\,,$$

and $K(s)$ is a coset of $H(s)$. If p is odd, then s acts on $\breve{T}/H(s) \cong \mathbf{Z}/p^\infty$ with only the identity as a fixed point (this follows from the fact that at an odd prime a p-adic root of unity has a nontrivial reduction mod p), and so the above chain of inclusions collapses to $H(s) = \sigma(s) = F(s)$. If $p = 2$ the situation is more complex. In this case the reflection s is of order 2 and acts on $\breve{T}/H(s) \cong \mathbf{Z}/2^\infty$ by inversion; the fixed subgroup here is of order 2, and this potentially allows $H(s)$ to be of index 2 in $F(s)$. There are three possibilities (we list with each one an example of a compact Lie group G such that the possibility is realized in $\hat{G}$):

1. $H(s) \subsetneq \sigma(s) = F(s)$: $SU(2)$.
2. $H(s) = \sigma(s) \subsetneq F(s)$: $SO(3)$.
3. $H(s) = \sigma(s) = F(s)$: $U(2)$

Note that in all cases $\sigma(s)$ is a subgroup of $\breve{T}$.

Remark. The above description of $\sigma(s)$ shows that if p is odd, then

1. the p-discrete center C of $\mathcal{X}$ (7.5) is the center of $\breve{\mathbf{N}}(T)$ (i.e. the subgroup of $\breve{T}$ which is fixed by the conjugation action of W), and
2. the group $W_{\mathcal{X}}(A)_1$ of 7.6 is the subgroup of $W_{\mathcal{X}}(A)$ generated by the reflections contained in $W_{\mathcal{X}}(A)$.

If $p = 2$ then the p-discrete center of $\mathcal{X}$ is contained in the center of $\breve{\mathbf{N}}(T)$ but is not necessarily equal to it, unless of course the center of $\breve{\mathbf{N}}(T)$ is trivial. This last possibility actually comes up if $\mathcal{X}$ is the 2-compact group DI(4) from [11]; in this case $\mathcal{X}$ is of rank 3 and it is possible to verify that the action of the Weyl group W on the elements of exponent 2 in $\breve{T}$ gives an epimorphism $W \to \mathrm{GL}(3, \mathbf{F}_2)$ (see [11, §4] for instance). This implies that the center of $\breve{\mathbf{N}}(T)$ is trivial, hence that the p-discrete center C of $\mathcal{X}$ is trivial and that the p-compact center $\mathbf{C}^{\mathrm{c}}(\mathcal{X})$ is trivial.

Proof of 7.5 (given 7.6). Let A be a subgroup of $\breve{T}$. By 7.6 and 4.7 the map $A \to \mathcal{X}$ is central if and only if $A \subset \sigma(s)$ for each reflection s in the Weyl group W of $\mathcal{X}$ (recall that W is generated by such reflections). The result follows from the description of the p-discrete center of $\mathcal{X}$ provided by 6.4, and the fact (6.7) that since $\mathcal{X}$ is connected the p-discrete center is contained in $\breve{T}$. □

7.8 Lemma. *Suppose that $\mathcal{X}$ is a connected p-compact group, A is a p-discrete torus and $i : A \to \mathcal{X}$ is a homomorphism. Then $\mathbf{C}_{\mathcal{X}}(i(A))$ is connected.*

Proof. It is enough (3.11) to show that any homomorphism $\mathbf{Z}/p^n \to \mathbf{C}_{\mathcal{X}}(A)$ $(n \geq 0)$ extends to a homomorphism $\mathbf{Z}/p^\infty \to \mathbf{C}_{\mathcal{X}}(A)$ or equivalently that any homomorphism $\mathbf{Z}/p^n \times A \to \mathcal{X}$ extends to a homomorphism $\mathbf{Z}/p^\infty \times A \to \mathcal{X}$. This is easily seen to be true if $\mathcal{X}$ is a p-compact torus. For the general case, then, it is enough to choose a maximal torus $T \to \mathcal{X}$ and show that any homomorphism $f : \mathbf{Z}/p^n \times A \to \mathcal{X}$ lifts up to conjugacy to a homomorphism $\tilde{f} : \mathbf{Z}/p^n \times A \to T$. To simplify notation, let C denote $\mathbf{Z}/p^n$. Such a lift $\tilde{f}$ exists if and only if the homotopy fixed point set

$$(\mathcal{X}/T)^{\mathrm{h}(C\times A)} \cong ((\mathcal{X}/T)^{\mathrm{h}C})^{\mathrm{h}A}$$

is nonempty (see [12, 3.3, 10.5]). The "left translation action" of C via f on $\mathcal{X}/T$ (see [12, 3.3]) extends to an action of the connected loop space $\mathcal{X}$ and so is homotopically trivial, i.e., each element of C acts by a self-map of $\mathcal{X}/T$ which is homotopic to the identity. This implies that the Lefschetz number of the action of C on $\mathcal{X}/T$ is equal to the Euler characteristic of $\mathcal{X}/T$ and so in particular is nonzero (2.9). This Lefschetz number is equal to the Euler characteristic $(\mathcal{X}/T)^{\mathrm{h}C}$ [12, 4.5, 5.7], and so $\chi((\mathcal{X}/T)^{\mathrm{h}C})$ is nonzero. By [12, 4.7, 5.7], the space $((\mathcal{X}/T)^{\mathrm{h}C})^{\mathrm{h}A}$ is not empty. □

7.9 Proof of 7.6(1). Let $W' \subset W$ be the group of elements which, under the conjugation action of W on $\breve{T}$, pointwise fix the subgroup A, and let C denote the centralizer of A in $\breve{\mathbf{N}}(T)$. By calculation there is an exact sequence (of discrete groups)

$$\{1\} \to \breve{T} \to C \to W' \to \{1\}\,.$$

The homomorphism $\breve{\mathbf{N}}(T) \to \mathcal{X}$ induces a homomorphism $C \to \mathbf{C}_{\mathcal{X}}(A)$ which immediately gives the inclusion $W' \subset W_{\mathcal{X}}(A)$. Conversely, if $W_{\mathcal{X}}(A)$ is not contained in W', then there are elements $a \in A$ and $w \in W_{\mathcal{X}}(A)$ such that $w \cdot a \neq a$. Let $C \subset A$ be the cyclic subgroup generated by a, let $i : C \to \breve{T}$ be the inclusion homomorphism, and $j : C \to \breve{T}$ the composite of i with the conjugation action of w. Clearly i and j become conjugate when composed with the homomorphism $\breve{T} \to \mathbf{C}_{\mathcal{X}}(A)$, and so it follows from 5.4 that $a(w \cdot a)^{-1}$ belongs to the kernel of $\breve{T} \to \mathbf{C}_{\mathcal{X}}(A)$. This is impossible, since the composite $\breve{T} \to \mathbf{C}_{\mathcal{X}}(A) \to \mathcal{X}$ has a trivial kernel. □

7.10 Lemma. *Let $\mathcal{X}$ be a p-compact group, A an abelian p-discrete toral group, and $f : A \to \mathcal{X}$ a homomorphism. Let U and V be two subgroups of A and $U \cdot V \subset A$ the subgroup generated by both. Then there is a natural map*

$$\mathbf{C}_{\mathcal{X}}(U \cdot V) \to \mathbf{C}_{\mathbf{C}_{\mathcal{X}}(V)}(U) \tag{7.11}$$

which is an equivalence.

Proof. Multiplication gives a surjective homomorphism $U \times V \to U \cdot V$, an induced map $\mathrm{B}U \times \mathrm{B}V \to \mathrm{B}(U \cdot V)$, and hence a map

$$\mathrm{Map}(\mathrm{B}(U \cdot V), \mathrm{B}\mathcal{X}) \to \mathrm{Map}(\mathrm{B}U \times \mathrm{B}V, \mathrm{B}\mathcal{X}) = \mathrm{Map}(\mathrm{B}U, \mathrm{Map}(\mathrm{B}V, \mathrm{B}\mathcal{X}))\,.$$

Restricting to appropriate components provides the required natural map 7.11. Since all of the elements in the kernel of the map $U \times V \to U \cdot V$ are also in the kernel of the composite $U \times V \to U \cdot V \to \mathcal{X}$, the fact that the map 7.11 is an equivalence follows from 3.5. □

7.12 Lemma. *In the situation of 7.6, let $s \in W$ be a reflection. Then $s \in W_{\mathcal{X}}(A)_1$ if and only if $s \in W_{\mathbf{C}_{\mathcal{X}}(H(s))}(A)_1$.*

Proof. The maximal rank map $\mathbf{C}_{\mathbf{C}_{\mathcal{X}}(H(s))}(A) \to \mathbf{C}_{\mathcal{X}}(A)$ clearly carries $W_{\mathbf{C}_{\mathcal{X}}(H(s))}(A)_1$ into $W_{\mathcal{X}}(A)_1$; this gives one of the implications. Suppose then that $s \in W_{\mathcal{X}}(A)_1$. Let $\mathcal{Z}$ be the centralizer of $H(s)$ in $\mathbf{C}_{\mathcal{X}}(A)_1$. Then $s \in W_{\mathcal{Z}_1}$ because $s \in W_{\mathcal{Z}}$ by 7.9 and $W_{\mathcal{Z}_1} = W_{\mathcal{Z}}$ because $\mathcal{Z}$ is connected (7.8). However, by 7.10 the p-compact group $\mathcal{Z}_1$ equals the identity component of $\mathbf{C}_{\mathbf{C}_{\mathcal{X}}(H(s))}(A)$, and so it follows that $s \in W_{\mathbf{C}_{\mathcal{X}}(H(s))}(A)_1$. □

Proof of 7.6(2). By [12, 9.7] the subgroup $W_{\mathcal{X}}(A)_1$ of W is generated by the reflections it contains, so proving (2) comes down to showing that a reflection $s \in W$ belongs to $W_{\mathcal{X}}(A)_1$ if and only if $A \subset \sigma(s)$.

Suppose first that $A \subset \sigma(s)$. To show that $s \in W_{\mathcal{X}}(A)_1$ is enough to show that $s \in W_{\mathcal{X}}(\sigma(s))_1$, and it is this second inclusion that we will prove. By 7.12, we can assume that $H(s) \to \mathcal{X}$ is central. If $H(s) = \sigma(s)$, then $W_{\mathcal{X}}(\sigma(s))_1 = W_{\mathcal{X}}(H(s))_1 = W_{\mathcal{X}_1} = W_{\mathcal{X}}$, so $s \in W_{\mathcal{X}}(\sigma(s))_1$ for trivial reasons and we are done. If $H(s) \neq \sigma(s)$, then $p = 2$ and $\sigma(s) = H(s) \times \mathbf{Z}/2$ (see 7.7). Choose $t \in K(s) \subset \sigma(s)$ to generate the second factor of $\sigma(s) = H(s) \times \mathbf{Z}/2$, and let $x \in \breve{\mathbf{N}}(T)$ be an element, with image $s \in W_{\mathcal{X}}$, such that $x^2 = t$. There is a commutative diagram of loop spaces and homomorphisms

$$\begin{array}{ccccc}
\mathbf{Z}/2 & \longrightarrow & \mathbf{Z}/4 & \longrightarrow & \mathbf{Z}/2^{\infty} \\
{\scriptstyle t}\big\downarrow & & {\scriptstyle x}\big\downarrow & & {\scriptstyle f}\big\downarrow \\
\breve{T} & \longrightarrow & \breve{\mathbf{N}}(T) & \longrightarrow & \mathcal{X}
\end{array}$$

in which the map f is obtained from 3.10 by using the connectivity of $\mathcal{X}$. Taking centralizers of $\langle t \rangle$ everywhere in this diagram and using 7.10 for the equivalence $\mathbf{C}_{\mathcal{X}}(\langle t \rangle) \cong \mathbf{C}_{\mathcal{X}}(\sigma(s))$ gives another commutative diagram

$$\begin{array}{ccccc}
\mathbf{Z}/2 & \longrightarrow & \mathbf{Z}/4 & \longrightarrow & \mathbf{Z}/2^{\infty} \\
{\scriptstyle t}\big\downarrow & & {\scriptstyle x}\big\downarrow & & {\scriptstyle f}\big\downarrow \\
\breve{T} & \longrightarrow & \mathbf{C}_{\breve{\mathbf{N}}(T)}(\langle t \rangle) & \longrightarrow & \mathbf{C}_{\mathcal{X}}(\sigma(s))
\end{array}$$

This shows that x, which represents $s \in W_{\mathcal{X}}(\sigma(s))$, lies in the identity component of $\mathbf{C}_{\mathcal{X}}(\sigma(s))$ (3.10).

Conversely, suppose that A is not contained in $\sigma(s)$. If A is not contained in $F(s)$ then $s \notin W_{\mathcal{X}}(A)$ by 7.6(1), and the result we want, that $s \notin W_{\mathcal{X}}(A)_1$, certainly follows. If $A \subseteq \breve{F}(s)$ but $A \nsubseteq \sigma(s)$, then $p = 2$, $H(s) = \sigma(s)$, and $F(s) \cong H(s) \times \mathbf{Z}/2$ (see 7.7). By 7.12, in showing $s \notin W_{\mathcal{X}}(A)_1$, we can replace $\mathcal{X}$ by $\mathbf{C}_{\mathcal{X}}(H(s))$ or, equivalently, assume that $H(s) \to \mathcal{X}$ is central. By 7.10, then, $\mathbf{C}_{\mathcal{X}}(A)$ is equivalent to $\mathbf{C}_{\mathcal{X}}(F(s))$. Choose $x \in \breve{\mathbf{N}}(T)$ so that the image of x in W is s and such that $x^2 = 1$; this is possible because $K(s) = H(s)$ in the current situation and so $1 \in K(s)$. If $s \in W_{\mathcal{X}}(A)_1$ then x lies in the identity component of $\mathbf{C}_{\mathcal{X}}(F(s))$ and so the homomorphism $f : \mathbf{Z}/2 \to \mathbf{C}_{\mathcal{X}}(F(s))$ representing x lifts to a homomorphism $Z/2 \to \mathbf{C}_{\mathcal{X}}(F(s))_1$. Since the Euler characteristic $\chi(\mathbf{C}_{\mathcal{X}}(F(s))_1/T)$ is non-zero (2.9), f lifts further (2.14) to a homomorphism $\mathbf{Z}/2 \to T$ and even (3.2) to an actual group homomorphism $\mathbf{Z}/2 \to \breve{T}$. This implies that $f(\mathbf{Z}/2)$ is central in $\mathbf{C}_{\mathcal{X}}(F(s))$, since every element of order 2 in $\breve{T}$ is contained in $F(s)$ and so is central in $\mathbf{C}_{\mathcal{X}}(F(s))$. However it is impossible for f to be central in this way, since x acts nontrivially by conjugation on $\breve{T}$ and so x is not in the center of the evident subgroup of $\breve{\mathbf{N}}(T)$ which is a discrete approximation to the 2-normalizer of the torus in $\mathbf{C}_{\mathcal{X}}(F(s))$ (see 5.5). This contradiction establishes that A is contained in $\sigma(s)$. □

§8. A homology decomposition theorem

Suppose that $\mathcal{X}$ is a p-compact group. Let $\mathbf{A}_{\mathcal{X}}$ denote the category whose objects are the pairs (V, f), where V is a nontrivial elementary abelian p-group and $f : V \to X$ is a conjugacy class of monomorphisms. A morphism $(V, f) \to (V', f')$ in this category is a injection $i : V \to V'$ such that $f' \cdot i$ is conjugate to f. There is a functor $\alpha_{\mathcal{X}} : \mathbf{A}_{\mathcal{X}}^{\mathrm{op}} \to$ (Spaces) given by

$$\alpha_{\mathcal{X}}(V, f) = \mathrm{Map}(\mathrm{B}V, \mathrm{B}\mathcal{X})_{\mathrm{B}f}\,.$$

By definition, $\alpha_{\mathcal{X}}(V, f)$ is equivalent to $\mathrm{B}\mathbf{C}_{\mathcal{X}}(g(V))$ for any $g : V \to \mathcal{X}$ contained in the conjugacy class f.

Evaluation at the basepoint of $\mathrm{B}V$ gives a map

$$\alpha_{\mathcal{X}}(V, f) \to \mathrm{B}\mathcal{X}$$

which is natural in (V, f) [10, §6]; together these induce a map

$$a_{\mathcal{X}} : \operatorname{hocolim} \alpha_{\mathcal{X}} \to \mathrm{B}\mathcal{X}\,.$$

In this section we will prove the following theorem.

8.1 Theorem. *For any p-compact group $\mathcal{X}$ the map $a_{\mathcal{X}} : \operatorname{hocolim} \alpha_{\mathcal{X}} \to \mathrm{B}\mathcal{X}$ induces an isomorphism on mod p cohomology.*

Remark. This is called a *homology decomposition theorem* because it shows how to express $\mathrm{B}\mathcal{X}$ up to mod p homology in terms of classifying spaces of other p-compact groups; if $\mathcal{X}$ has trivial p-compact center, these other p-compact groups will be strictly smaller than $\mathcal{X}$ (cf. proof of 9.10). A decomposition theorem like this was first proved for classifying spaces of compact Lie groups in [14]. We will follow the more algebraic treatment from [10].

Proof of 8.1. Let R denote the object $\mathrm{H}^*\mathrm{B}\mathcal{X}$ of the category $\mathcal{K}$ of unstable algebras over the mod p Steenrod algebra, and let $\mathbf{A}_R$ be the category whose objects are the pairs (V,g), where V is a nontrivial elementary abelian p-group and $g : R \to \mathrm{H}^*\mathrm{B}V$ is a map in $\mathcal{K}$ which makes $\mathrm{H}^*\mathrm{B}V$ into a finitely generated R-module. A morphism $(V,g) \to (V',g')$ in this category is an injection $i : V \to V'$ such that $(\mathrm{B}i)^* \cdot g' = g$. Let $T^V : \mathcal{K} \to \mathcal{K}$ be the functor which is left adjoint to tensor product with $\mathrm{H}^*\mathrm{B}V$; this functor is studied by Lannes in [16] . There is a functor $\alpha_R : \mathbf{A}_R \to \mathcal{K}$ which assigns to (V,g) the object $T(V,R)_g$, also denoted $T^V_g(R)$, which is the summand or "component" of $T^V(R)$ corresponding to the map g [10, §3]. Let $T \to \mathcal{X}$ be a maximal torus, and let $S = \mathrm{H}^*\mathrm{B}\mathbf{N}_p(T)$. The monomorphism (2.10) $\mathbf{N}_p(T) \to \mathcal{X}$ induces a map $i : R \to S$. Since $\mathcal{X}/\mathbf{N}_p(T)$ is $\mathbf{F}_p$-finite and has Euler characteristic prime to p, the Becker-Gottlieb transfer construction (as generalized in [5]) provides a left inverse $t : S \to R$ for i, such that t is both a map of R-modules and a map of modules over the mod p Steenrod algebra. Moreover, the object $S \in \mathcal{K}$ has a nontrivial "center" in the sense of [10, §4] because any discrete approximation $\breve{\mathbf{N}}_p(T)$ to $\mathbf{N}_p(T)$ has a nontrivial group theoretic center [10, 1.4]. By [10, 1.2], then, the natural map $R \to \lim \alpha_R$ is an isomorphism, and the higher limits $\lim^i \alpha_R$ vanish for $i > 0$. (Note that these higher limits are calculated in the category of graded mod p vector spaces.)

For any elementary abelian p-group V there is a map Φ from the set of conjugacy classes of homomorphisms $f : V \to \mathcal{X}$ to the set of $\mathcal{K}$-maps $g : \mathrm{H}^*\mathrm{B}V \to R$; the map Φ assigns to f the cohomology map $(\mathrm{B}f)^*$ induced by $\mathrm{B}f$. According to [16, 3.1.4] the map Φ is a bijection. Since a homomorphism $f : V \to \mathcal{X}$ is a monomorphism if and only if $\Phi(f)$ makes $\mathrm{H}^*\mathrm{B}V$ into a finitely generated module over R (2.4), it follows that Φ gives an isomorphism of categories $\Phi : \mathbf{A}_\mathcal{X} \to \mathbf{A}_R$.

If V is an elementary abelian p-group and $f : V \to \mathcal{X}$ is a homomorphism, then the mapping space component $\mathrm{Map}(\mathrm{B}V, \mathrm{B}\mathcal{X})_{\mathrm{B}f}$ is the classifying space of a p-compact group and so in particular it is $\mathbf{F}_p$-complete and has finite dimensional mod p cohomology groups. By [16] then, the natural map $\lambda_f : T(V,R)_{\Phi(f)} \to \mathrm{H}^*\,\mathrm{Map}(\mathrm{B}V,\mathrm{B}\mathcal{X})_{\mathrm{B}f}$ is an isomorphism; here λ_f is adjoint to the map on cohomology induced by the evaluation map $\mathrm{B}V \times \mathrm{Map}(\mathrm{B}V,\mathrm{B}\mathcal{X})_{\mathrm{B}f} \to \mathrm{B}\mathcal{X}$. These isomorphisms combine to give a natural equivalence from the functor α_R to the composite $\mathrm{H}^*(\alpha_\mathcal{X}) \cdot \Phi$. It follows then from the remarks above about α_R that $\lim^i \mathrm{H}^*(\alpha_\mathcal{X}) = 0$ for $i > 0$ and that the natural map $\lim \mathrm{H}^*(\alpha_\mathcal{X}) \to \mathrm{H}^*(\mathrm{B}\mathcal{X})$ is an isomorphism. The theorem can now be derived by a straightforward application of the spectral sequence of [3, XII, 5.8]; recall that this is a first quadrant spectral converging to $\mathrm{H}^*(\mathrm{hocolim}\,\alpha_\mathcal{X})$ which at E^2 contains the groups $\lim^i \mathrm{H}^*(\alpha_\mathcal{X})$. □

The above proof of 8.1 reveals a few properties of $\mathbf{A}_\mathcal{X}$ which we will use later on.

8.2 Proposition. *If $\mathcal{X}$ is a p-compact group, then the nerve (or underlying space* [3, XI, §2]*)* $\mathrm{Nerve}(\mathbf{A}_\mathcal{X})$ *of the category $\mathbf{A}_\mathcal{X}$ has the mod p (co-)homology of a point.*

Proof. An inspection of [3, XI, §6] shows that $\mathrm{H}^i\,\mathrm{Nerve}(\mathbf{A}_\mathcal{X})$ is isomorphic to $\lim^i \beta$, where β is the constant functor on $\mathbf{A}^{\mathrm{op}}_\mathcal{X}$ with value $\mathbf{F}_p$. Since β is naturally equivalent to the functor $\mathrm{H}^0(\alpha_\mathcal{X})$, this proposition follows from the vanishing result referred to in the proof of 8.1. □

8.3 Proposition. *If $\mathcal{X}$ is a p-compact group, then $\mathbf{A}_{\mathcal{X}}$ is equivalent to a category with a finite number of objects and a finite number of morphisms between any two objects.*

Proof. By definition the set of morphisms between any two objects of $\mathbf{A}_{\mathcal{X}}$ is finite, so it is enough to show that up to isomorphism $\mathbf{A}_{\mathcal{X}}$ has only a finite number of objects. As in the proof of 8.1, such objects correspond to pairs (V, g), where V is a nontrivial elementary abelian p-group and $g : \mathrm{H}^*\mathrm{B}\mathcal{X} \to \mathrm{H}^*\mathrm{B}V$ makes $\mathrm{H}^*\mathrm{B}V$ a finitely generated module over $\mathrm{H}^*\mathrm{B}\mathcal{X}$. Since $\mathrm{H}^*\mathrm{B}\mathcal{X}$ is a finitely generated, i.e. noetherian, algebra over $\mathbf{F}_p$, an elementary Poincaré series argument shows that any such pair (V, g) must have the Krull dimension of $\mathrm{H}^*\mathrm{B}V$ (which equals $\mathrm{rk}_{\mathbf{F}_p} V$) less than or equal to the Krull dimension of $\mathrm{H}^*\mathrm{B}\mathcal{X}$; in particular, up to isomorphism there are only a finite number of possible V. For any given V, a $\mathcal{K}$-map $g : \mathrm{H}^*\mathrm{B}\mathcal{X} \to \mathrm{H}^*\mathrm{B}V$ is determined by the images under g of some finite set of algebra generators for $\mathrm{H}^*\mathrm{B}\mathcal{X}$; since $\mathrm{H}^*\mathrm{B}V$ is finite in each dimension, there are only a finite number of choices for these images. □

§9. An inductive principle and the "Sullivan Conjecture"

In this section we will describe an inductive principle which can sometimes be used in conjunction with 8.1 to prove statements about p-compact groups. We then show that the principle leads to a proof for p-compact groups of an analogue of Miller's Theorem (the "Sullivan Conjecture") [17].

9.1 Definition. A class $\mathbf{Cl}$ of p-compact groups is said to be *saturated* if it satisfies the following five conditions.

(1) $\mathbf{Cl}$ is closed under equivalences, in the sense that if $f : \mathcal{X} \to \mathcal{Y}$ is a homomorphism of p-compact groups which is an equivalence, then $\mathcal{X} \in \mathbf{Cl}$ if and only if $\mathcal{Y} \in \mathbf{Cl}$.
(2) The trivial p-compact group belongs to $\mathbf{Cl}$.
(3) If the identity component $\mathcal{X}_1$ of $\mathcal{X}$ is in $\mathbf{Cl}$, then $\mathcal{X} \in \mathbf{Cl}$.
(4) If $\mathcal{X}$ is connected and $\mathcal{X}/\mathbf{C}^{\mathrm{c}}(\mathcal{X}) \in \mathbf{Cl}$, then $\mathcal{X} \in \mathbf{Cl}$.
(5) If $\mathcal{X}$ is connected, $\mathbf{C}^{\mathrm{c}}(\mathcal{X})$ is trivial, and $\mathcal{Y} \in \mathbf{Cl}$ for all p-compact groups $\mathcal{Y}$ such that $\mathrm{cd}_{\mathbf{F}_p} \mathcal{Y} < \mathrm{cd}_{\mathbf{F}_p} \mathcal{X}$, then $\mathcal{X} \in \mathbf{Cl}$.

9.2 Theorem. *Any saturated class* $\mathbf{Cl}$ *contains all p-compact groups.*

Proof. We will work by induction on $\mathrm{cd}_{\mathbf{F}_p} \mathcal{X}$ to show that $\mathcal{X} \in \mathbf{Cl}$. If $\mathrm{cd}_{\mathbf{F}_p} \mathcal{X} = 0$ then $\mathcal{X}_1$ is contractible and $\mathcal{X} \in \mathbf{Cl}$ by the first three conditions. Suppose that $\mathrm{cd}_{\mathbf{F}_p} \mathcal{X} > 0$ and that $\mathcal{Y} \in \mathbf{Cl}$ for all $\mathcal{Y}$ with $\mathrm{cd}_{\mathbf{F}_p} \mathcal{Y} < \mathrm{cd}_{\mathbf{F}_p} \mathcal{X}$. Let C be the p-compact center of $\mathcal{X}_1$. Then $\mathcal{X}_1/C \in \mathbf{Cl}$ by (5), $\mathcal{X}_1 \in \mathbf{Cl}$ by (4), and hence $\mathcal{X} \in \mathbf{Cl}$ by (3). □

9.3 Theorem. *(Sullivan conjecture for p-compact groups) Let $\mathcal{X}$ be a p-compact group and $\mathcal{U}$ a space which is $\mathbf{F}_p$-complete and $\mathbf{F}_p$-finite. Then inclusion of constant maps gives an equivalence $\mathcal{U} \to \mathrm{Map}(\mathrm{B}\mathcal{X}, \mathcal{U})$.*

Before proving 9.3 we will list a few lemmas. If $\mathcal{X}$ and $\mathcal{U}$ are spaces, write "$\mathcal{X} \perp \mathcal{U}$" if inclusion of constant maps gives an equivalence $\mathcal{U} \to \mathrm{Map}(\mathcal{X}, \mathcal{U})$. The first three lemmas are relatively elementary.

9.4 Lemma. [2, §2-3] *Let $\mathcal{F} \to \mathcal{E} \to \mathcal{B}$ be a fibration. If $\mathcal{F} \perp \mathcal{U}$ and $\mathcal{B} \perp \mathcal{U}$ then $\mathcal{E} \perp \mathcal{U}$.*

9.5 Lemma. *Let C be a category and $F : C \to$ (Spaces) a functor. If* $\mathrm{Nerve}(C) \perp \mathcal{U}$ *and* $F(c) \perp \mathcal{U}$ *for each object c of C, then* $\operatorname{hocolim} F \perp \mathcal{U}$.

Proof. See [3, XII, §4] and observe that the homotopy limit of a constant functor on C with value $\mathcal{U}$ is the space $\mathrm{Map}(\mathrm{Nerve}(C), \mathcal{U})$. □

9.6 Lemma. [12, 11.13] *If $\mathcal{U}$ is an $\mathbf{F}_p$-complete space and $\mathcal{X} \to \mathcal{Y}$ is an $\mathbf{F}_p$-equivalence, then $\mathcal{X} \perp \mathcal{U}$ if and only if $\mathcal{Y} \perp \mathcal{U}$.*

9.7 Lemma. *If $\mathcal{X}$ is a p-compact toral group and $\mathcal{U}$ is a space which is $\mathbf{F}_p$-complete and $\mathbf{F}_p$-finite, then* $\mathrm{B}\mathcal{X} \perp \mathcal{U}$.

Proof. By density (3.4) we can assume instead that $\mathcal{X}$ is a p-discrete toral group. In this case $\mathcal{X}$ is an increasing union of finite p-groups [12, 6.19], and $\mathrm{B}\mathcal{X}$ is equivalent to the corresponding sequential homotopy colimit of classifying spaces of finite p-groups [3, XII, 3.6]. Since $\mathrm{B}G \perp \mathcal{U}$ for any finite p-group G by Miller's Theorem, the lemma follows from 9.5. □

Let **Cl** be the class of all p-compact groups $\mathcal{X}$ such that $\mathrm{B}\mathcal{X} \perp \mathcal{U}$ for each space $\mathcal{U}$ which is $\mathbf{F}_p$-complete and $\mathbf{F}_p$-finite. It is clear that **Cl** satisfies (1) and (2) of 9.1. In order to complete the proof of 9.3, i.e., to show that **Cl** is the class of all p-compact groups, it is enough by 9.2 to prove the following three propositions. In each of them $\mathcal{X}$ is a p-compact group and $\mathcal{U}$ is a space which is $\mathbf{F}_p$-complete and $\mathbf{F}_p$-finite.

9.8 Proposition. *If* $\mathrm{B}(\mathcal{X}_1) \perp \mathcal{U}$, *then* $\mathrm{B}\mathcal{X} \perp \mathcal{U}$.

Proof. There is a fibration sequence $\mathrm{B}\mathcal{X}_1 \to \mathrm{B}\mathcal{X} \to \mathrm{B}\pi_0\mathcal{X}$. Since $\mathrm{B}\pi_0\mathcal{X} \perp \mathcal{U}$ by [17], the proposition follows from 9.4. □

9.9 Proposition. *Suppose that $\mathcal{X}$ is connected and that $\mathcal{Z} = \mathcal{X}/\mathbf{C}^{\mathrm{c}}(\mathcal{X})$. If* $\mathrm{B}\mathcal{Z} \perp \mathcal{Y}$, *then* $\mathrm{B}\mathcal{X} \perp \mathcal{Y}$.

Proof. There is a fibration sequence $\mathrm{B}\mathbf{C}^{\mathrm{c}}(\mathcal{X}) \to \mathrm{B}\mathcal{X} \to \mathrm{B}\mathcal{Y}$. Since $\mathrm{B}\mathbf{C}^{\mathrm{c}}(\mathcal{X}) \perp \mathcal{Y}$ by 9.7, the proposition follows from 9.4. □

9.10 Proposition. *If $\mathcal{X}$ is connected, $\mathbf{C}^{\mathrm{c}}(\mathcal{X})$ is trivial, and* $\mathrm{B}\mathcal{Y} \perp \mathcal{U}$ *for all $\mathcal{Y}$ such that* $\mathrm{cd}_{\mathbf{F}_p} \mathcal{Y} < \mathrm{cd}_{\mathbf{F}_p} \mathcal{X}$, *then* $\mathrm{B}\mathcal{X} \perp \mathcal{U}$.

Proof. Let $f : V \to \mathcal{X}$ be a monomorphism with V a nontrivial elementary abelian p-group. Since $\mathbf{C}^{\mathrm{c}}(\mathcal{X})$ is trivial the homomorphism f cannot be central, and so (by 4.6) $\mathrm{cd}_{\mathbf{F}_p} \mathbf{C}_{\mathcal{X}}(f(V)) < \mathrm{cd}_{\mathbf{F}_p} \mathcal{X}$. It follows that $\alpha_{\mathcal{X}}(V, f) \perp \mathcal{U}$ for each object (V, f) of $\mathbf{A}_{\mathcal{X}}$, and so $\operatorname{hocolim} \alpha_{\mathcal{X}} \perp \mathcal{U}$ by 8.2 and 9.5. By 9.6, $\mathrm{B}\mathcal{X} \perp \mathcal{U}$. □

§10. A few technical results

In this section we begin by proving a collection of results which are more or less immediate corollaries of 9.3. Some of these are interesting because they are generalizations for arbitrary p-compact groups of results that were previously known only for p-compact toral groups. We finish by giving a splitting result (10.7) for extensions of a p-compact torus by a connected p-compact group in adjoint form; this will be used in the proof of 11.7.

10.1 Proposition. *(cf. [12, 6.1 and 5.3]) If $f : \mathcal{X} \to \mathcal{Y}$ is a trivial homomorphism between p-compact groups, then f is central.*

Proof. Let $f : \mathrm{B}\mathcal{X} \to \mathrm{B}\mathcal{Y}$ be the constant map. The proposition follows from the fact that the loop space $\Omega\,\mathrm{Map}(\mathrm{B}\mathcal{X}, \mathrm{B}\mathcal{Y})_f$ is equivalent to $\mathrm{Map}(\mathrm{B}\mathcal{X}, \mathcal{Y})$, which, by 9.3, is in turn equivalent to $\mathcal{Y}$. □

10.2 Proposition. *(cf. 2.6) A homomorphism $f : \mathcal{X} \to \mathcal{Y}$ between p-compact groups is central if and only if there is a homomorphism $\mu : \mathcal{X} \times \mathcal{Y} \to \mathcal{Y}$ which restricts to f on $\mathcal{X} \times \{1\}$ and to the identity on $\{1\} \times \mathcal{Y}$.*

Proof. This is the same as the proof of [12, 8.6]. If f is central then an evaluation map gives the required homomorphism μ. If μ exists, then it induces an equivalence between $\mathbf{C}_{\mathcal{Y}}(f(\mathcal{X}))$ and $\mathbf{C}_{\mathcal{Y}}(g(\mathcal{X}))$, where $g : \mathcal{X} \to \mathcal{Y}$ is the trivial homomorphism. This second centralizer is equivalent to $\mathcal{Y}$ by 10.1. □

10.3 Lemma. *(cf. 3.5) Let*

$$\{1\} \to \mathcal{X} \xrightarrow{f} \mathcal{Y} \to \mathcal{Z} \to \{1\}$$

be a short exact sequence of p-compact groups and $g : \mathcal{Y} \to \mathcal{U}$ a homomorphism such that $g \cdot f$ is a trivial homomorphism. Then up to homotopy there is a unique homomorphism $g' : \mathcal{Z} \to \mathcal{U}$ which extends g, and the natural map $\mathbf{C}_{\mathcal{U}}(g'(\mathcal{Z})) \to \mathbf{C}_{\mathcal{U}}(g(\mathcal{Y}))$ is an equivalence.

Proof. This is the same as the proof of [12, 7.5]. Let $h : \mathcal{X} \to \mathcal{U}$ be the trivial homomorphism. The idea is to express the space of maps $\mathrm{B}\mathcal{Y} \to \mathrm{B}\mathcal{U}$ which up to homotopy are trivial on $\mathrm{B}\mathcal{X}$ as the space of sections of a fibration over $\mathrm{B}\mathcal{Z}$ with fibre $\mathrm{Map}(\mathrm{B}\mathcal{X}, \mathrm{B}\mathcal{U})_{\mathrm{B}h}$, and then to use 10.1 to identify this fibration as the product $\mathrm{B}\mathcal{Z} \times \mathrm{B}\mathcal{U} \to \mathrm{B}\mathcal{Z}$. □

10.4 Proposition. *Suppose that*

$$\{1\} \to \mathcal{X} \to \mathcal{Y} \xrightarrow{f} \mathcal{Z} \to \{1\}$$

is a short exact sequence of p-compact groups, and that $g : \mathcal{U} \to \mathcal{Y}$ is a central homomorphism of p-compact groups. Then the composite $f \cdot g : \mathcal{U} \to \mathcal{Z}$ is also central.

Proof. By 10.2 it is enough to construct an appropriate homomorphism $\mu : \mathcal{U} \times \mathcal{Z} \to \mathcal{Z}$. Consider the corresponding homomorphism $\mathcal{U} \times \mathcal{Y} \to \mathcal{Y}$ which expresses the fact that g is central. The adjoint of the associated map of classifying spaces is a map $\mathrm{B}\mathcal{U} \to \mathrm{Map}(\mathrm{B}\mathcal{Y}, \mathrm{B}\mathcal{Y})_{\mathrm{id}}$ which induces (by composing points in the target with $\mathrm{B}f$) a map $\mathrm{B}\mathcal{U} \to \mathrm{Map}(\mathrm{B}\mathcal{Y}, \mathrm{B}\mathcal{Z})_{\mathrm{B}f}$. By 10.3, however, $\mathrm{Map}(\mathrm{B}\mathcal{Y}, \mathrm{B}\mathcal{Z})_{\mathrm{B}f}$ is equivalent via precomposition with $\mathrm{B}f$ to $\mathrm{Map}(\mathrm{B}\mathcal{Z}, \mathrm{B}\mathcal{Z})_{\mathrm{id}}$, and it is easy to check that the adjoint of the resulting map $\mathrm{B}\mathcal{U} \to \mathrm{Map}(\mathrm{B}\mathcal{Z}, \mathrm{B}\mathcal{Z})_{\mathrm{id}}$ gives the required homomorphism μ. □

10.5 Proposition. *Let*

$$\{1\} \to \mathcal{X} \xrightarrow{f} \mathcal{Y} \to \mathcal{Z} \to \{1\}$$

be a short exact sequence of p-compact groups, $\mathcal{U}$ a p-compact group, and $g : \mathcal{U} \to \mathcal{X}$ a homomorphism. Then

(1) *if $f \cdot g$ is central then g is central, and*
(2) *if g is central and $\mathcal{Z}$ is connected then $f \cdot g$ is central*

Proof. Let $h : \mathcal{U} \to \mathcal{Z}$ denote the trivial homomorphism. Evaluation at the basepoint of $\mathrm{B}\mathcal{U}$ gives a map of fibration sequences

$$\begin{array}{ccccc} \mathcal{F} & \longrightarrow & \mathrm{Map}(\mathrm{B}\mathcal{U}, \mathrm{B}\mathcal{Y})_{\mathrm{B}(f\cdot g)} & \longrightarrow & \mathrm{Map}(\mathrm{B}\mathcal{U}, \mathrm{B}\mathcal{Z})_{\mathrm{B}h} \\ {\scriptstyle a}\big\downarrow & & {\scriptstyle b}\big\downarrow & & {\scriptstyle c}\big\downarrow \\ \mathrm{B}\mathcal{X} & \longrightarrow & \mathrm{B}\mathcal{Y} & \longrightarrow & \mathrm{B}\mathcal{Z} \end{array}$$

where $\mathcal{F}$ denotes the space of maps from $\mathrm{B}\mathcal{U}$ to $\mathrm{B}\mathcal{X}$ which after composition with $\mathrm{B}f$ are homotopic to $\mathrm{B}(f \cdot g)$. Note that if $\mathcal{F}$ is connected then $\mathcal{F} = \mathrm{Map}(\mathrm{B}\mathcal{U}, \mathrm{B}\mathcal{X})_{\mathrm{B}g}$. By 10.1 the map c is an equivalence. It follows that if b is an equivalence if and only if a is an equivalence. If $f \cdot g$ is central then b is an equivalence, so that a is an equivalence, $\mathcal{F}$ is connected, and g is central. This gives (1). If $\mathcal{Z}$ is connected then $\mathcal{F}$ is connected by the long exact homotopy sequence of the upper fibration; thus if g is central, the map a is an equivalence, so b is an equivalence too and $f \cdot g$ is central. This gives (2). □

10.6 Lemma. *Suppose that $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ is a fibration sequence of $\mathbf{F}_p$-finite spaces, such that $\mathcal{Z}$ is connected and the action of $\pi_1\mathcal{Z}$ on $\mathrm{H}_*\mathcal{X}$ is nilpotent. Then $\chi(\mathcal{Y}) = \chi(\mathcal{X})\chi(\mathcal{Z})$.*

Proof. It follows from [12, 4.13] that the desired multiplicativity holds in E^2 of the fibration Serre spectral sequence, and a standard inductive argument shows that this persists to E^∞. □

Remark. Lemma 10.6 does not necessarily hold for a general fibration sequence of $\mathbf{F}_p$-finite spaces. For example, let p be odd and consider the fibration

$$S^2 \to \mathbf{R}P^2 \to \mathbf{R}P^\infty \quad .$$

The mod p cohomology Euler characteristics of base and total space are 1, while that of the fibre is 2.

10.7 Proposition. *Suppose that*

$$\{1\} \to \mathcal{X} \to \mathcal{Y} \to A \to \{1\}$$

is a short exact sequence of connected p-compact groups, where $\mathcal{X}$ has a trivial p-compact center and A is a p-compact torus. Then the sequence is equivalent to the product sequence

$$\{1\} \to \mathcal{X} \to \mathcal{X} \times A \to A \to \{1\}.$$

Proof. Let T be a maximal torus for $\mathcal{Y}$ and $\breve{T} \to T$ a discrete approximation. Let f be the composite $\breve{T} \to Y \to A$, and denote $\ker(f)$ by $\breve{K}$ and $\breve{T}/\breve{K}$ by $\breve{Q}$. Let K and Q denote the closures of $\breve{K}$ and $\breve{Q}$ respectively. Observe that $\breve{Q}$, as a quotient of $\breve{T}$, is a p-discrete torus and hence that Q is a p-compact torus. There is a commutative diagram of loop spaces and homomorphisms (3.5)

$$\begin{array}{ccccc} \breve{K} & \longrightarrow & \breve{T} & \longrightarrow & \breve{Q} \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Y} & \longrightarrow & A \end{array}$$

which on passing to closures gives a map of short exact sequences

$$
\begin{array}{ccccccccc}
\{1\} & \longrightarrow & K & \longrightarrow & T & \longrightarrow & Q & \longrightarrow & \{1\} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\{1\} & \longrightarrow & \mathcal{X} & \longrightarrow & \mathcal{Y} & \longrightarrow & A & \longrightarrow & \{1\}
\end{array}
\tag{10.8}
$$

By construction the map $\breve{Q} \to A$ has trivial kernel, and so $Q \to A$ is a monomorphism (3.6). The composite $\breve{K} \to \mathcal{X} \to \mathcal{Y}$ has trivial kernel (since $T \to \mathcal{Y}$ is a monomorphism) and so $\breve{K} \to \mathcal{X}$ has trivial kernel and $K \to \mathcal{X}$ is a monomorphism. Taking vertical homotopy fibres in the diagram of classifying spaces induced by 10.8 gives a fibre sequence

$$\mathcal{X}/K \to \mathcal{Y}/T \to A/Q$$

in which all three spaces are $\mathbf{F}_p$-finite and in which it is straightforward to verify that the fundamental group of the base, which is a quotient of $\pi_1 A$, acts nilpotently on the homology of the fibre. By multiplicativity of Euler characteristic (10.6), $\chi(\mathcal{Y}/T) = \chi(\mathcal{X}/K)\chi(A/Q)$. Since $\chi(\mathcal{Y}/T) \neq 0$, it follows that $\chi(\mathcal{X}/K) \neq 0 \neq \chi(A/Q)$ and hence (2.15) that Q is a maximal torus for A and (use in addition 10.6) that the identity component K_1 is a maximal torus for $\mathcal{X}$. Since A is itself a p-compact torus, the uniqueness property of maximal tori implies that the map $Q \to A$ is an equivalence. As in 3.8, K is a product of a p-compact torus and a finite abelian p-group; since $\mathcal{X}$ is connected and a maximal torus in a connected p-compact group is self-centralizing [12, 9.1], we conclude that the finite abelian p-group factor is trivial, K is connected, and K itself is a maximal torus for $\mathcal{X}$.

Let $C \to \breve{T}$ be the p-discrete center of $\mathcal{Y}$ (6.7). By 7.5, $C = \cap_s \sigma(s)$, where s runs through all reflections in the Weyl group W of $\mathcal{Y}$. By 10.5 the intersection $C \cap \breve{K}$ is trivial, since $\mathcal{X}$ has a trivial p-discrete center. Let $\breve{T}^W$ be the fixed point set of the conjugation action of W on $\breve{T}$, and $\breve{T}_{\mathrm{nrm}}$ the image of the norm map $\breve{T} \to \breve{T}$ which sends an element of $\breve{T}$ to the sum of its W-conjugates. Then $\breve{T}_{\mathrm{nrm}} \subset \breve{T}^W$, $C \subset \breve{T}^W$, the quotient $\breve{T}^W/\breve{T}_{\mathrm{nrm}}$ is finite because it is annihilated by $|W|$, and $\breve{T}^W/C$ is finite by 7.7. The quotient $\breve{T} \to \breve{Q}$ is clearly equivariant with respect to the conjugation action of W on $\breve{T}$ and the trivial action of W on $\breve{Q}$. Under this trivial action of W on $\breve{Q}$, the norm map $\breve{Q} \to \breve{Q}$ is just multiplication by $|W|$ and so is surjective, because $\breve{Q}$ is divisible. As a consequence, the composite $\breve{T}_{\mathrm{nrm}} \to \breve{T} \to \breve{Q}$ is surjective. Since $\breve{Q}$ has no proper subgroups of finite index, it follows that the maps $\breve{T}^W \to \breve{Q}$ and $C \to \breve{Q}$ are surjective too, this last map in fact being an isomorphism because as mentioned above the intersection $C \cap \breve{K}$ is trivial. Passing to closures shows that the homomorphism $\mathcal{Y} \to A$ induces an equivalence $\mathbf{C}^{\mathrm{c}}(\mathcal{Y}) \to A$. The composite homomorphism $\mathbf{C}^{\mathrm{c}}(\mathcal{Y}) \times \mathcal{X} \to \mathbf{C}^{\mathrm{c}}(\mathcal{Y}) \times \mathcal{Y} \to \mathcal{Y}$ (where the second map comes from 2.6) is the required equivalence of $\mathcal{Y}$ with $\mathbf{C}^{\mathrm{c}}(\mathcal{Y}) \times \mathcal{X} \cong A \times \mathcal{X}$. It is easy to check that this equivalence is compatible with the original short exact sequence. □

§11. Homotopy centers

In this section we will prove 1.3. Let $\mathcal{X}$ be a p-compact group. The multiplication homomorphism (2.6):

$$\mu_c : \mathbf{C}^{\mathrm{c}}(\mathcal{X}) \times \mathcal{X} \to \mathcal{X}$$

gives a classifying space map $B\mu_c : B\mathbf{C}^c(\mathcal{X}) \times B\mathcal{X} \to B\mathcal{X}$. The adjoint of $B\mu_c$ is a map $B\mathbf{C}^c(\mathcal{X}) \to \mathrm{Map}(B\mathcal{X}, B\mathcal{X})_{\mathrm{id}}$, or equivalently a loop space homomorphism $j_{\mathcal{X}} : \mathbf{C}^c(\mathcal{X}) \to \mathbf{C}^h(\mathcal{X})$. This is the homomorphism which will turn out to be an equivalence.

11.1 Remark. Let $\mathcal{X}$ be a p-compact group. The construction in [12, proof of 8.3] shows that the fibration sequence $B\mathbf{C}^c(\mathcal{X}) \to B\mathcal{X} \to B\mathcal{X}_{\mathrm{ad}}$ extends one step further to the right to give a fibration sequence

$$B\mathcal{X} \to \mathcal{B}\mathcal{X}_{\mathrm{ad}} \to B^2\mathbf{C}^c(\mathcal{X})\,. \tag{11.2}$$

It follows directly from Theorem 1.3 that 11.2 is equivalent to the universal fibration

$$B\mathcal{X} \to \mathcal{E} \to B(\mathrm{Aut}(B\mathcal{X})_{\mathrm{id}})$$

with fibre $B\mathcal{X}$ over a simply connected space; in particular, the total space of this universal bundle is $B\mathcal{X}_{\mathrm{ad}}$.

The next proposition shows that in order to prove that $j_{\mathcal{X}}$ is an equivalence it is enough to prove that $\mathbf{C}^h(\mathcal{X})$ is a p-compact group.

11.3 Proposition. *Suppose that $\mathcal{X}$ is a p-compact group and that $\mathbf{C}^h(\mathcal{X})$ is a p-compact group. Then the map $j_{\mathcal{X}} : \mathbf{C}^c(\mathcal{X}) \to \mathbf{C}^h(\mathcal{X})$ is an equivalence.*

Proof. The composition operation on $\mathrm{Map}(B\mathcal{X}, B\mathcal{X})_{\mathrm{id}}$ gives a homomorphism $\mathbf{C}^h(\mathcal{X}) \times \mathbf{C}^h(\mathcal{X}) \to \mathbf{C}^h(\mathcal{X})$ which shows (10.2) that $\mathbf{C}^h(\mathcal{X})$ is abelian and hence (5.2) that $\mathbf{C}^h(\mathcal{X})$ is a p-compact toral group. Consider the evaluation map

$$B\mu_h : B\mathbf{C}^h(\mathcal{X}) \times B\mathcal{X} = \mathrm{Map}(B\mathcal{X}, B\mathcal{X})_{\mathrm{id}} \times B\mathcal{X} \to B\mathcal{X}\,.$$

This induces a homomorphism $\mu_h : \mathbf{C}^h(\mathcal{X}) \times \mathcal{X} \to \mathcal{X}$ whose restriction to $\mathbf{C}^h(\mathcal{X}) \times \{1\}$ is a central homomorphism $\mathbf{C}^h(\mathcal{X}) \to \mathcal{X}$ (2.6). By 6.1 this central homomorphism lifts uniquely to a homomorphism $k_{\mathcal{X}} : \mathbf{C}^h(\mathcal{X}) \to \mathbf{C}^c(\mathcal{X})$. The homomorphism diagrams

$$\begin{array}{ccc} \mathbf{C}^h(\mathcal{X}) \times \mathcal{X} & \xrightarrow{\mu_h} & \mathcal{X} \\ {\scriptstyle k_{\mathcal{X}} \times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathbf{C}^c(\mathcal{X}) \times \mathcal{X} & \xrightarrow{\mu_c} & \mathcal{X} \end{array} \qquad \begin{array}{ccc} \mathbf{C}^c(\mathcal{X}) \times \mathcal{X} & \xrightarrow{\mu_c} & \mathcal{X} \\ {\scriptstyle j_{\mathcal{X}} \times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathbf{C}^h(\mathcal{X}) \times \mathcal{X} & \xrightarrow{\mu_h} & \mathcal{X} \end{array}$$

commute up to homotopy, the first by the uniqueness provision of 5.3 and the second by construction. It follows directly that $j_{\mathcal{X}} \cdot k_{\mathcal{X}} : \mathbf{C}^h(\mathcal{X}) \to \mathbf{C}^h(\mathcal{X})$ is homotopic to the identity, and from the uniqueness provision of 6.1 that $k_{\mathcal{X}} \cdot j_{\mathcal{X}}$ is also homotopic to the identity. $\square$

The above proof also gives a slightly more specialized result which we will use later on.

11.4 Proposition. *Suppose that $\mathcal{X}$ is a p-compact group, and that the identity component $\mathbf{C}^{\mathrm{h}}(\mathcal{X})_1$ is a p-compact torus. Then the restriction of $j_{\mathcal{X}} : \mathbf{C}^{\mathrm{c}}(\mathcal{X}) \to \mathbf{C}^{\mathrm{h}}(\mathcal{X})$ to identity components gives an equivalence $\mathbf{C}^{\mathrm{c}}(\mathcal{X})_1 \to \mathbf{C}^{\mathrm{h}}(\mathcal{X})_1$.*

Let **Cl** be the class consisting of all p-compact groups $\mathcal{X}$ such that $\mathbf{C}^{\mathrm{h}}(\mathcal{X})$ is a p-compact group. By 11.3, Theorem 1.3 is equivalent to the statement that **Cl** contains all p-compact groups. It is clear that **Cl** satisfies (1) and (2) of 9.1. By 9.2, then, in order to prove 1.3 it is enough to prove the following three propositions. In each one, $\mathcal{X}$ is a p-compact group.

11.5 Proposition. *If $\mathbf{C}^{\mathrm{h}}(\mathcal{X}_1)$ is a p-compact group, then so is $\mathbf{C}^{\mathrm{h}}(\mathcal{X})$.*

11.6 Proposition. *Suppose that $\mathcal{X}$ is connected, and let $C = \mathbf{C}^{\mathrm{c}}(\mathcal{X})$. If $\mathbf{C}^{\mathrm{h}}(\mathcal{X}/C)$ is a p-compact group, then so is $\mathbf{C}^{\mathrm{h}}(\mathcal{X})$.*

11.7 Proposition. *Suppose that $\mathcal{X}$ is connected, and that $\mathbf{C}^{\mathrm{c}}(\mathcal{X})$ is trivial. If $\mathbf{C}^{\mathrm{h}}(\mathcal{Y})$ is a p-compact group for each p-compact group $\mathcal{Y}$ with $\mathrm{cd}_{\mathbf{F}_p}\mathcal{Y} < \mathrm{cd}_{\mathbf{F}_p}\mathcal{X}$, then $\mathbf{C}^{\mathrm{h}}(\mathcal{X})$ is a p-compact group.*

The proofs of 11.5 and 11.6 amount to fairly standard homotopy theoretic calculations; in order to organize these calculations we set up some notation.

If $\mathcal{U}$ is a space, let $\mathrm{Aut}(\mathcal{U})$ denote the space of (unpointed) self homotopy equivalences of $\mathcal{U}$; this is a loop space, and $\mathrm{B}\,\mathrm{Aut}(\mathcal{U})$ is the classifying space for fibrations with fibre equivalent to $\mathcal{U}$. If $f : \mathcal{E} \to \mathcal{B}$ is a fibration, let $\mathrm{Aut}(f)$ denote the space of commutative diagrams

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{E} \\ {\scriptstyle f}\downarrow & & {\scriptstyle f}\downarrow \\ \mathcal{B} & \longrightarrow & \mathcal{B} \end{array} \tag{11.8}$$

in which both of the horizontal arrows are equivalences. Let $\mathrm{Aut}_{\mathcal{B}}(f)$ denote the subspace of $\mathrm{Aut}(f)$ consisting of diagrams as above in which the map $\mathcal{B} \to \mathcal{B}$ is the identity. The spaces $\mathrm{Aut}(f)$ and $\mathrm{Aut}_{\mathcal{B}}(f)$ are loop spaces such that $\mathrm{B}\,\mathrm{Aut}(f)$ and $\mathrm{B}\,\mathrm{Aut}_{\mathcal{B}}(f)$ classify certain types of iterated fibrations [7].

If $\mathcal{X}$ and $\mathcal{Y}$ are spaces let $[\mathcal{X}, \mathcal{Y}]$ denote $\pi_0\,\mathrm{Map}(\mathcal{X}, \mathcal{Y})$. Suppose that $\mathcal{B}$ is a connected space and that $f : \mathcal{E} \to \mathcal{B}$ is a fibration with fibre $\mathcal{F}$. We will denote by $c(f)$ the element of $[\mathcal{B}, \mathrm{B}\,\mathrm{Aut}(\mathcal{F})]$ consisting of classifying maps for f, and by $C(f)$ the subset of $[\mathcal{B}, \mathrm{B}\,\mathrm{Aut}(\mathcal{F})]$ obtained by taking the orbit of $c(f)$ under the precomposition action of $\pi_0\,\mathrm{Aut}(\mathcal{B})$ on $[\mathcal{B}, \mathrm{B}\,\mathrm{Aut}(\mathcal{F})]$.

11.9 Proposition. [7] *Let $f : \mathcal{E} \to \mathcal{B}$ be a fibration with fibre $\mathcal{F}$ over the connected space $\mathcal{B}$. Then there is a fibration sequence*

$$\mathrm{Map}(\mathcal{B}, \mathrm{B}\,\mathrm{Aut}(\mathcal{F}))_{C(f)} \to \mathrm{B}\,\mathrm{Aut}(f) \to \mathrm{B}\,\mathrm{Aut}(\mathcal{B})\,.$$

Remark. The above fibration sequence is associated to the composition action of $\mathrm{Aut}(\mathcal{B})$ on $\mathrm{Map}(\mathcal{B}, \mathrm{B}\,\mathrm{Aut}(\mathcal{F})_{C(f)}$.

If $\mathcal{X}$ and $\mathcal{U}$ are spaces with $\mathcal{U}$ connected, let $\mathrm{Map}(\mathcal{X}, \mathcal{U})_0$ denote the subspace of $\mathrm{Map}(\mathcal{X}, \mathcal{U})$ consisting of maps which are homotopic to constant maps. Write $\mathcal{X} \perp_0 \mathcal{U}$ if inclusion of constant maps gives an equivalence $\mathcal{U} \to \mathrm{Map}(\mathcal{X}, \mathcal{U})_0$ (cf. §9).

11.10 Proposition. *Suppose that $\mathcal{B}$ is a connected space, that $f : \mathcal{E} \to \mathcal{B}$ is a fibration with fibre $\mathcal{F}$, and that $\mathcal{F} \perp_0 \mathcal{B}$. Then the forgetful map $u : \mathrm{Aut}(f)_{\mathrm{id}} \to \mathrm{Aut}(\mathcal{E})_{\mathrm{id}}$ (cf. 11.8) is an equivalence.*

Proof. Let $\mathrm{Aut}_{\mathcal{B}}(f)_{\{\mathrm{id}_{\mathcal{E}}\}}$ denote the subspace of $\mathrm{Aut}_{\mathcal{B}}(f)$ consisting of maps of $\mathcal{E}$ over $\mathcal{B}$ which as maps of $\mathcal{E}$ are homotopic to the identity, and let $\mathrm{Aut}(f)_{\{\mathrm{id}_{\mathcal{E}},\mathrm{id}_{\mathcal{B}}\}}$ denote the subspace of $\mathrm{Aut}(f)$ consisting of commutative diagrams 11.8 in which the upper arrow is homotopic to the identity map of $\mathcal{E}$ and the lower one to the identity map of $\mathcal{B}$. Consider the map of fibration sequences

$$\begin{array}{ccccc}
\mathrm{Aut}_{\mathcal{B}}(f)_{\{\mathrm{id}_{\mathcal{E}}\}} & \longrightarrow & \mathrm{Aut}(f)_{\{\mathrm{id}_{\mathcal{E}},\mathrm{id}_{\mathcal{B}}\}} & \xrightarrow{u'} & \mathrm{Aut}(\mathcal{B})_{\mathrm{id}} \\
{\scriptstyle =}\downarrow & & {\scriptstyle u}\downarrow & & {\scriptstyle v}\downarrow \\
\mathrm{Aut}_{\mathcal{B}}(f)_{\{\mathrm{id}_{\mathcal{E}}\}} & \longrightarrow & \mathrm{Aut}(\mathcal{E})_{\mathrm{id}} & \xrightarrow{w} & \mathrm{Map}(\mathcal{E},\mathcal{B})_f
\end{array}$$

in which u' like u is a forgetful map and the maps v and w are given by composition with f (on the left and right, respectively). Since $\mathcal{F} \perp_0 \mathcal{B}$, the argument used in the proof of 10.3 shows that v is an equivalence. The fibration sequence then shows that u is an equivalence and also that the domain of u, being connected, is the identity component $\mathrm{Aut}(f)_{\mathrm{id}}$ of $\mathrm{Aut}(f)$. □

The next two lemmas are proved by direct homotopy group calculation, and depend on the following observation.

11.11 Remark. A connected space $\mathcal{U}$ is the classifying space $\mathrm{B}G$ of a p-compact toral group G if and only if

(1) $\pi_1\mathcal{U}$ is a finite p-group,
(2) $\pi_2\mathcal{U}$ is a finitely generated free module over $\mathbf{Z}_p$ (since $\mathbf{Z}_p$ is a principal ideal domain, this is equivalent to the condition that $\pi_2\mathcal{U}$ be a finitely generated torsion free module over $\mathbf{Z}_p$) and
(3) $\pi_i(\mathcal{U}) = \{0\}$ for $i > 2$.

If $\mathcal{U}$ is both a loop space *and* the classifying space of a p-compact toral group then $\mathcal{U}$ is the classifying space of an abelian p-compact toral group (2.6).

11.12 Lemma. *Suppose that $\mathcal{Y}$ is a connected space with $\mathrm{H}^i\mathcal{Y}$ finite for $i = 1, 2$ and $\mathrm{H}^1(\mathcal{Y}, \mathbf{Z}_p) = 0$. Let $\mathcal{U}$ be connected space such that $\Omega\mathcal{U}$ is the classifying space of a p-compact toral group and let $f : \mathcal{Y} \to \mathcal{U}$ be a map. Then each component of $\Omega\,\mathrm{Map}(\mathcal{Y},\mathcal{U})_f$ is the classifying space of a p-compact toral group.*

11.13 Remark. The above lemma applies if $\mathcal{Y} = \mathrm{B}\mathcal{X}$ for a p-compact group $\mathcal{X}$.

11.14 Lemma. *If G is a p-compact toral group, then $\mathrm{Aut}(\mathrm{B}G)_{\mathrm{id}} = \mathrm{B}\mathbf{C}^{\mathrm{h}}(G)$ is the classifying space of a p-compact toral group.*

Proof of 11.5. By 11.3, $\mathbf{C}^{\mathrm{h}}(\mathcal{X}_1)$ is a p-compact toral group. The fibration

$$\mathrm{B}(\mathcal{X}_1) \to \mathrm{B}\mathcal{X} \xrightarrow{f} \mathrm{B}\pi_0\mathcal{X}$$

satisfies the conditions of 11.10, so the map

$$\operatorname{Aut}(f)_{\mathrm{id}} \to \operatorname{Aut}(\mathrm{B}\mathcal{X})_{\mathrm{id}} = \mathrm{B}\mathbf{C}^{\mathrm{h}}(\mathcal{X})$$

is an equivalence. Looping down the fibration sequence from 11.9 then gives a fibration sequence for $\mathrm{B}\mathbf{C}^{\mathrm{h}}(\mathcal{X})$ in which the fibre is a disjoint union of classifying spaces of p-compact toral groups (11.12) and the base is $\operatorname{Aut}(\mathrm{B}\pi_0\mathcal{X})_{\mathrm{id}} \cong \mathrm{B}C$, where C is the center of $\pi_0\mathcal{X}$. It follows from the long exact homotopy sequence of a fibration (cf. 11.11) that $\mathrm{B}\mathbf{C}^{\mathrm{h}}(\mathcal{X})$ is the classifying space of a p-compact toral group. □

Proof of 11.6. The fibration

$$\mathrm{B}\mathbf{C}^{\mathrm{c}}(\mathcal{X}) \to \mathrm{B}\mathcal{X} \xrightarrow{f} \mathrm{B}(\mathcal{X}/\mathbf{C}^{\mathrm{c}}(\mathcal{X}))$$

satisfies the conditions of 11.10 (see 10.1), so the map

$$\operatorname{Aut}(f)_{\mathrm{id}} \to \operatorname{Aut}(\mathrm{B}\mathcal{X})_{\mathrm{id}} = \mathrm{B}\mathbf{C}^{\mathrm{h}}(\mathcal{X})$$

is an equivalence. Looping down the fibration sequence from 11.9 then gives a fibration sequence for $\mathrm{B}\mathbf{C}^{\mathrm{h}}(\mathcal{X})$ in which the base is contractible (6.3, 11.3) and the fibre is the classifying space of a p-compact toral group (11.12, 11.13). This gives the desired result. □

To prove the final proposition we will need a concept from [6]. A map $f : \mathcal{X} \to \mathcal{Y}$ of spaces is said to be *centric* if composition with f induces an equivalence $\operatorname{Aut}(\mathcal{X})_{\mathrm{id}} \to \operatorname{Map}(\mathcal{X}, \mathcal{Y})_f$.

11.15 Lemma. *Let $\mathcal{X}$ be a p-compact group, V an elementary abelian group, $f : V \to \mathcal{X}$ a monomorphism, and $i : \mathbf{C}_{\mathcal{X}}(V) \to \mathcal{X}$ the induced monomorphism. Then the map $\mathrm{B}i : \mathrm{B}\mathbf{C}_{\mathcal{X}}(V) \to \mathrm{B}\mathcal{X}$ is centric.*

Proof. This is the same as the proof of [6, 4.2]. The rough idea is that, since V is central in $\mathbf{C}_{\mathcal{X}}(V)$, the centralizer of $\mathbf{C}_{\mathcal{X}}(V)$ in $\mathcal{X}$ is the same as the centralizer of $\mathbf{C}_{\mathcal{X}}(V)$ in the centralizer of V in $\mathcal{X}$. Encoding this in mapping space terms gives the stated equivalence. □

The following is clear from the complex given in [3, XI, §6] for computing higher derived functors of the inverse limit.

11.16 Lemma. *Let $\mathbf{A}$ be a category, R a noetherian ring and $\alpha : \mathbf{A} \to$ (R–modules) a functor. Suppose that $\mathbf{A}$ is equivalent to a category with a finite number of objects and a finite number of morphisms between any two objects, and that $\alpha(x)$ is a finitely generated R-module for any object x of $\mathbf{A}$. Then $\lim^i \alpha$ is a finitely generated R-module for each $i \geq 0$.*

Proof of 11.7. The $\mathbf{F}_p$-equivalence $a_{\mathcal{X}}$ of 8.1 gives an equivalence [12, 11.3]

$$a_{\mathcal{X}}^{\#} : \operatorname{Map}(\mathrm{B}\mathcal{X}, \mathrm{B}\mathcal{X}) \xrightarrow{\cong} \operatorname{Map}(\operatorname{hocolim} \alpha_{\mathcal{X}}, \mathrm{B}\mathcal{X})\,.$$

As explained in [3, XII, §4] the target of this equivalence is the same as $\operatorname{holim}\beta_{\mathcal{X}}$, where $\beta_{\mathcal{X}} : \mathbf{A}_{\mathcal{X}} \to$ (Spaces) sends (V, f) to $\operatorname{Map}(\alpha_{\mathcal{X}}(V, f), \mathrm{B}\mathcal{X})$. For each object (V, f) of $\mathbf{A}_{\mathcal{X}}$ let

$$e_{V,f} : \alpha_{\mathcal{X}}(V, f) = \operatorname{Map}(\mathrm{B}V, \mathrm{B}\mathcal{X})_{\mathrm{B}f} \to \mathrm{B}\mathcal{X}$$

be given by evaluation at the basepoint of $\mathrm{B}V$. The definition of $a_{\mathcal{X}}$ implies that $a_{\mathcal{X}}^{\#}$ carries the component $\operatorname{Aut}(\mathrm{B}\mathcal{X})_{\mathrm{id}}$ of $\operatorname{Map}(\mathrm{B}\mathcal{X}, \mathrm{B}\mathcal{X})$ to the basepoint component of $\operatorname{holim}\beta'_{\mathcal{X}}$, where for each object (V, f) of $\mathbf{A}_{\mathcal{X}}$, $\beta'_{\mathcal{X}}(V, F)$ is the pointed space given by

$$\beta'_{\mathcal{X}}(V, F) = \operatorname{Map}(\alpha_{\mathcal{X}}(V, f), \mathrm{B}\mathcal{X})_{e_{V,f}} .$$

Since $\mathbf{C}^{\mathrm{c}}(\mathcal{X})$ is trivial, each space $\alpha_{\mathcal{X}}(V, f)$ is the classifying space of some p-compact group $\mathcal{Y}$ with $\operatorname{cd}_{\mathbf{F}_p} \mathcal{Y} < \operatorname{cd}_{\mathbf{F}_p} \mathcal{X}$ (cf. proof of 9.10). Consequently, by 11.3 and the given hypotheses, $\operatorname{Aut}(\alpha_{\mathcal{X}}(V, f))_{\mathrm{id}}$ is the classifying space of a p-compact toral group, in fact, of an abelian p-compact toral group (11.11). By 11.15, each space $\beta'_{\mathcal{X}}(V, f)$ is also the classifying space of an abelian p-compact toral group. There is a second quadrant homotopy spectral sequence [3, XI, §7] [1]

$$E^2_{i,j} = \lim{}^{-i} \pi_j \beta'_{\mathcal{X}} \Rightarrow \pi_{j-i} \operatorname{holim}\beta'_{\mathcal{X}} = \pi_{j-i} \operatorname{Aut}(\mathrm{B}\mathcal{X})_{\mathrm{id}} = \pi_{j-i}\mathrm{B}\mathbf{C}^{\mathrm{h}}(\mathcal{X})$$

The E^2-term of the spectral sequence has these properties:

(1) $E^2_{i,j} = 0$ for $j \geq 3$, because $\pi_j\beta'_{\mathcal{X}}(V, f) = 0$ for all (V, f) and all $j \geq 3$.
(2) $E^2_{0,2} = \lim \pi_2\beta'_{\mathcal{X}} \subset \prod_{(V,f)} \pi_2\beta'_{\mathcal{X}}(V, f)$ is a torsion-free module over $\mathbf{Z}_p$, because $\pi_2\beta'_{\mathcal{X}}(V, f)$ is a torsion-free module over $\mathbf{Z}_p$ for each (V, f).
(3) $E^2_{i,j}$ is a finitely generated module over $\mathbf{Z}_p$ for $j = 1, 2$ and all $i \leq 0$, by 8.3 and 11.16.

It follows from these observations and the structure of the spectral sequence that $\pi_i \operatorname{Aut}(\mathrm{B}\mathcal{X})_{\mathrm{id}} = 0$ for $i \geq 3$, that $\pi_2 \operatorname{Aut}(\mathrm{B}\mathcal{X})_{\mathrm{id}}$ is a finitely generated torsion-free (equivalently, free) module over $\mathbf{Z}_p$, and that $\pi_1 \operatorname{Aut}(\mathrm{B}\mathcal{X})_{\mathrm{id}}$ is a finitely generated module over $\mathbf{Z}_p$. (Note that this fundamental group is necessarily abelian, because $\operatorname{Aut}(\mathrm{B}\mathcal{X})$ is a loop space.) We will be done if we can show that $\pi_1 \operatorname{Aut}(\mathrm{B}\mathcal{X})_{\mathrm{id}}$ is finite, since then $\operatorname{Aut}(\mathrm{B}\mathcal{X})_{\mathrm{id}} = \mathrm{B}\mathbf{C}^{\mathrm{h}}(\mathcal{X})$ will be the classifying space of a p-compact toral group. It is clear in any case that the identity component $\mathbf{C}^{\mathrm{h}}(\mathcal{X})_1$ is a p-compact torus, since the nonzero homotopy of this component consists at most of a single finitely generated free module over $\mathbf{Z}_p$ in dimension 1. Since $\mathbf{C}^{\mathrm{c}}(\mathcal{X})$ is trivial by assumption, it follows from 11.4 that $\mathbf{C}^{\mathrm{h}}(\mathcal{X})_1$ is contractible, and hence that $\operatorname{Aut}(\mathrm{B}\mathcal{X})_{\mathrm{id}}$ is of type $K(M, 1)$ for some finitely generated module M over $\mathbf{Z}_p$. Therefore $\mathrm{B}(\operatorname{Aut}(\mathrm{B}\mathcal{X})_{\mathrm{id}})$ is of type $K(M, 2)$. We will prove directly that $M = 0$, which is certainly enough to show that M is finite. Pick a homomorphism $\mathbf{Z}_p \to M$ and consider the fibration sequence

$$\mathrm{B}\mathcal{X} \to \mathcal{E} \to K(\mathbf{Z}_p, 2)$$

obtained from the induced map $K(\mathbf{Z}_p, 2) \to K(M, 2)$ by pulling back the universal fibration over $K(M, 2) = \mathrm{B}(\operatorname{Aut}(\mathrm{B}\mathcal{X})_{\mathrm{id}})$. This gives a short exact sequence of p-compact groups which by 10.7 is a product sequence. This implies that the above fibration is a product fibration and, by the meaning of universality, that the homomorphism $\mathbf{Z}_p \to M$ is trivial. Since this is true for any homomorphism $\mathbf{Z}_p \to M$, M itself is trivial. □

§12. Relationship to Lie groups

In this section we will prove 1.4; this gives one explicit connection between the theory described in this paper and the homotopy theory of compact Lie groups.

Let G be a compact Lie group and $\mathbf{C}^{\mathrm{a}}(G)$ the algebraic center of G, so that $\mathbf{C}^{\mathrm{a}}(G)$ is a compact abelian Lie group. The multiplication homomorphism $\mathbf{C}^{\mathrm{a}}(G) \times G \to G$ induces a map on classifying spaces whose adjoint is a map $e_1 : \mathrm{B}\mathbf{C}^{\mathrm{a}}(G) \to \mathrm{Map}(\mathrm{B}G, \mathrm{B}G)_{\mathrm{id}}$ (cf. [15, §4]). This is the natural map referred to in the statement of 1.4.

Recall from §2 that if G is compact Lie group with $\pi_0 G$ a p-group, $\hat{G}$ denotes the p-compact group obtained from G by $\mathbf{F}_p$-completion. There is a natural homomorphism $\delta : G \to \hat{G}$.

12.1 Lemma. *Let G be a connected compact Lie group, A a p-discrete toral group, and $f : A \to G$ a homomorphism. Then the maps*

$$\mathrm{Map}(\mathrm{B}G, \mathrm{B}G)_{\mathrm{id}} \xrightarrow{h} \mathrm{Map}(\mathrm{B}G, \mathrm{B}\hat{G})_{B\delta}$$

$$\mathrm{Map}(\mathrm{B}A, \mathrm{B}G)_{\mathrm{B}f} \xrightarrow{k} \mathrm{Map}(\mathrm{B}A, \mathrm{B}\hat{G})_{B(\delta\cdot f)}$$

induced by composition with δ are $\mathbf{F}_p$-equivalences.

Proof. We will only treat the map h; the case of the map k is similar but simpler. Let $\mathrm{Comp}'_{\mathbf{F}_p}(\mathrm{B}G)$ denote the product $\prod_{q \neq p} \mathrm{Comp}_{\mathbf{F}_q}(\mathrm{B}G)$. By [15, 3.1], the completion map $\epsilon : \mathrm{B}G \to \mathrm{Comp}'_{\mathbf{F}_p}(\mathrm{B}G)$ combines with $\mathrm{B}\delta$ to give an $\mathbf{F}_p$-equivalence

$$\mathrm{Map}(\mathrm{B}G, \mathrm{B}G)_{\mathrm{id}} \to \mathrm{Map}(\mathrm{B}G, \mathrm{B}\hat{G})_{B\delta} \times \mathrm{Map}(\mathrm{B}G, \mathrm{Comp}'_{\mathbf{F}_p}(\mathrm{B}G))_{\epsilon}$$

To finish the proof it is enough to show that the space

$$\mathrm{Map}(\mathrm{B}G, \mathrm{Comp}'_{\mathbf{F}_p}(\mathrm{B}G))_{\epsilon} \cong \prod_{q \neq p} \mathrm{Map}(\mathrm{B}G, \mathrm{Comp}_{\mathbf{F}_q}(\mathrm{B}G))_{\epsilon_q} \tag{12.2}$$

has the mod p homology of a point, where $\epsilon_q : \mathrm{B}G \to \mathrm{Comp}_{\mathbf{F}_q}(\mathrm{B}G)$ is the $\mathbf{F}_q$-completion map. However, as described in [12, 11.10] each completion map ϵ_q induces an equivalence

$$\mathrm{Map}(\mathrm{Comp}_{\mathbf{F}_q}(\mathrm{B}G), \mathrm{Comp}_{\mathbf{F}_q}(\mathrm{B}G))_{\mathrm{id}} \to \mathrm{Map}(\mathrm{B}G, \mathrm{Comp}_{\mathbf{F}_q}(\mathrm{B}G))_{\epsilon_q}$$

whose domain, by 1.3, is the classifying space of an abelian q-compact group. It follows that the space in 12.2 is a nilpotent (even simple) space with uniquely p-divisible homotopy groups, and thus has the desired acyclicity property. □

12.3 Lemma. *Suppose that G is a connected compact Lie group and K a subgroup of G such that $\pi_0 K$ is a p-group and the inclusion $K \to G$ induces an isomorphism $\mathrm{H}^*\mathrm{B}G \to \mathrm{H}^*\mathrm{B}K$. Then $K = G$.*

Proof. It follows from application of the Eilenberg-Moore spectral sequence or the fibre lemma [12, 11.7] to appropriate path fibrations that the map $K \to G$ is an $\mathbf{F}_p$-equivalence. Since K and G are both orientable manifolds, this implies that K and G have the same dimension; the fact that G is connected now easily gives $K = G$. □

12.4 Lemma. [15, A.4] *Suppose that G is a connected compact Lie group, K a subgroup of G which is a finite p-group, and C the centralizer of K in G. Then $\pi_0 C$ is a finite p-group.*

12.5 Theorem. *Suppose that G is a connected compact Lie group and $i : K \subset G$ a subgroup of G which is a finite p-group. Then K lies in the center of G if and only if the composite $K \to G \to \hat{G}$ is central.*

Proof. If K lies in the center of G, then applying the $\mathbf{F}_p$-completion functor to the map of classifying spaces induced by the multiplication map $K \times G \to G$ gives the homomorphism $K \times \hat{G} \to \hat{G}$ required for showing that $K \to \hat{G}$ is central (2.6). If $K \to \hat{G}$ is central, consider the square

$$\begin{array}{ccc} \mathrm{Map}(\mathrm{B}K, \mathrm{B}G)_{\mathrm{B}i} & \longrightarrow & \mathrm{Map}(\mathrm{B}K, \mathrm{B}\hat{G})_{\mathrm{B}(\delta\cdot i)} \\ \downarrow & & \downarrow \\ \mathrm{B}G & \longrightarrow & \mathrm{B}\hat{G} \end{array}$$

where the vertical maps are induced by evaluation at the basepoint of BK. Here the two horizontal maps are $\mathbf{F}_p$-equivalences (the upper one by 12.3) and the right hand map is an equivalence by assumption. It follows that the map $\mathrm{Map}(\mathrm{B}K, \mathrm{B}G)_{\mathrm{B}i} \to \mathrm{B}G$ is an isomorphism on homology. Let H be the centralizer of K in G. The main result of [13] implies that the natural map $\mathrm{B}H \to \mathrm{Map}(\mathrm{B}K, \mathrm{B}G)_{\mathrm{B}i}$ is an $\mathbf{F}_p$-equivalence, and it follows that the inclusion $H \to G$ induces an $\mathbf{F}_p$-equivalence $\mathrm{B}H \to \mathrm{B}G$. By 12.4 and 12.5, then, $H = G$. □

12.6 Lemma. *Let G be a connected compact Lie group and C the subgroup of $\mathbf{C}^{\mathrm{a}}(G)$ given by elements whose order is a power of p. (We are treating C as a discrete group). Then the composite homomorphism $C \to G \to \hat{G}$ is a p-discrete center for $\hat{G}$.*

Proof. Let $T \to G$ be a Lie-theoretic maximal torus for G; as in 2.16, the induced homomorphism $\hat{T} \to \hat{G}$ is a maximal torus for $\hat{G}$. Let $\breve{T}$ be the subgroup of T given by elements with order a power of p. The composite of the inclusion $\breve{T} \to T$ with the completion map $\delta : T \to \hat{T}$ is a discrete approximation for $\hat{T}$. By ordinary Lie group theory, $\mathbf{C}^{\mathrm{a}}(G)$ is contained in T and so C is the set of elements $x \in \breve{T}$ such that x lies in the center of G. By 12.5 this is the same as the set of elements $x \in \breve{T}$ such that the composite $\langle x \rangle \to \breve{T} \to \hat{G}$ is central; by 6.4 and 6.7, this set is the p-discrete center of $\hat{G}$. □

Proof of 1.4. It is enough to show that for each prime p the map $e_1 : \mathrm{B}\mathbf{C}^{\mathrm{a}}(G) \to \mathrm{Map}(\mathrm{B}G, \mathrm{B}G)_{\mathrm{id}}$ is an $\mathbf{F}_p$-equivalence. Fix p as usual, and let $C \subset \mathbf{C}^{\mathrm{a}}(G)$ be the group of elements of order a power of p. There are maps

$$\mathrm{B}C \xrightarrow{u} \mathrm{B}\mathbf{C}^{\mathrm{a}}(G) \xrightarrow{e_1} \mathrm{Map}(\mathrm{B}G, \mathrm{B}G)_{\mathrm{id}} \xrightarrow{h} \mathrm{Map}(\mathrm{B}G, \mathrm{B}\hat{G})_{B\delta} \xleftarrow{\cong} \mathrm{Map}(\mathrm{B}\hat{G}, \mathrm{B}\hat{G})_{\mathrm{id}}$$

where u is induced by the inclusion $C \to \mathbf{C}^{\mathrm{a}}(G)$, h is given by composition with $B\delta$, and the last equivalence is also induced by composition (on the other side) with $B\delta$. A straightforward argument using 1.3 and 12.6 shows that the composite $\mathrm{B}C \to \mathrm{Map}(\mathrm{B}G, \mathrm{B}\hat{G})_{B\delta}$ is an $\mathbf{F}_p$-equivalence. By inspection the map u is an $\mathbf{F}_p$-equivalence (recall that $\mathbf{C}^{\mathrm{a}}(G)$ is the product of a torus and a finite abelian group), and by 12.1 the map h is an $\mathbf{F}_p$-equivalence too. It follows that e_1 is an $\mathbf{F}_p$-equivalence. □

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