

A PRIMER ON THE DICKSON INVARIANTS

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The ring of invariant forms of the full linear group $GL(V)$ of a finite dimensional vector space over the finite field F_q was computed early in the 20th century by L. E. Dickson [5], and was found to be a graded polynomial algebra on certain generators $\{c_{n,i}\}$. This ring of invariants, for $q = p$, has found use in algebraic topology in work of Milgram–Man [9], Singer [16,17], Adams–Wilkerson [1], Rector [13], Lam [6], Mui [11], and Smith–Switzer [18]. The aim of this exposition is to give a simple proof of the structure of the ring of invariants, and to compute the action of the Steenrod algebra on the generators of the invariants. The methods used are implicit in Adams–Wilkerson.¹

Dickson’s viewpoint was to vastly generalize the defining equation of \mathbb{F}_q ,

$$X^q = X.$$

This equation was replaced by the fundamental equation

$$f_n(X) = \prod_{v \in V} (X - v) = X^{q^n} + \sum_{i=0}^{n-1} (-1)^{n-i} c_{n,i} X^{q^i} = 0.$$

The polynomial $f_n(X)$ has the property that its roots form an \mathbb{F}_q -vector space. The sparse nature of the coefficients almost immediately gives the structure of the ring of invariants, and the Steenrod algebra action (for $q = p$) can be easily determined from Dickson’s fundamental equation. Other authors (Milgram–Mann, Singer, Smith–Switzer) have determined the action by other means, but it seems instructive to give a direct proof from Dickson’s original viewpoint.

I. The Fundamental Equation and the Ring of Invariants.

Let K be a field containing the \mathbb{F}_q -space V of dimension n . Here $q = p^s$ for p a rational integer prime and $s > 0$. We first prove the equivalence of the two forms of the fundamental equation.

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Proposition 1.1. *If $f_n(X)$ is a monic separable polynomial in $K[X]$ such that its roots are the elements of V , then*

$$f_n(X) = X^{q^n} + \sum_{i=0}^{n-1} (-1)^{n-i} c_{n,i} X^{q^i}$$

for $c_{n,i} \in K$.

Note: The choice of signs in $f_n(X)$ is best explained by the proof.

Proof. $f_n(X) = \prod_{v \in V} (X - v)$. Choose a basis for V over F_q , $\langle x_1, \dots, x_n \rangle$, and define V_{n-1} to be the subspace spanned by the first $n-1$ basis elements. The determinant

$$\Delta_n(X) = \det \begin{bmatrix} x_1 & \dots & X \\ x_1^q & \dots & X^q \\ \vdots & \ddots & \vdots \\ x_1^{q^{n-1}} & \dots & X^{q^{n-1}} \\ x_1^{q^n} & \dots & X^{q^n} \end{bmatrix}$$

is seen by column operations to have among its roots the elements of V . Hence $\Delta_n(X) = \Delta_{n-1}(x_n) f_n(X)$. It remains to verify that the constant $\Delta_{n-1}(x_n)$ is nonzero. For $n=1$, $\Delta_1(X) = x_1 f_1(X) \neq 0$, so the induction can be started. If the statement is true for vector spaces of dimension less than n , then

$$\Delta_{n-1}(X) = \Delta_{n-2}(x_{n-1}) f_{n-1}(X) \neq 0.$$

If V_{n-1} is used to define $\Delta_{n-1}(X)$, then since x_n is not in V_{n-1} , it is not a root and hence $\Delta_{n-1}(x_n) \neq 0$. It is also easily seen by this induction that $\Delta_{n-1}(x_n) = \prod_{u \in V^*} u$, where V^* is the subset of elements of V which are non-zero and for which the last non-zero coordinate is 1.

Now in the application to Dickson's theorem, we take K as the field of fractions of the symmetric algebra $S(V)$ over F_q . By a choice of basis, this can be identified with a polynomial ring on n -variables. It is convenient to grade $S(V)$ and K with V have grading 2 (resp. 1 for $p=2$). Then $GL(V)$ acts on $S(V)$, preserving the grading.

Theorem 1.2. (Dickson [5])

$$S(V)^{GL(V)} = F_q[c_{n,n-1}, \dots, c_{n,0}]$$

where

$$f_n(X) = \prod_{v \in V} (X - v) = x^{q^n} + \sum_{i=0}^{n-1} (-1)^{n-i} c_{n,i} X^{q^i}.$$

The $\{c_{n,i}\}$ have degrees $\{2(q^n - q^i)\}$, respectively, $\{q^n - q^i\}$ for $p=2$. The $\{c_{n,i}\}$ are the unique, up to scalar multiple, non-zero invariant elements in these degrees.

Proof. Clearly $\{c_{n,i}\} \subset S(V)^{GL(V)}$, and hence the algebra R^* generated by the $\{c_{n,i}\}$ is invariant. But $S(V)$ is integral over R^* , so the transcendence degree of

R^* over F_q is also n . There are exactly n of the $c_{n,i}$, so the elements of $\{c_{n,i}\}$ are algebraically independent. Thus

$$R^* = F_q[c_{n,0}, \dots, c_{n,n-1}] \subset S(V)^{GL(V)}.$$

Let L^* be the graded field of fractions of R^* , and K^* that of $S(V)$. Since K^* is the splitting field for $f_n(X)$ over L^* , the extension is Galois, with Galois group W . Then $GL(V) \subset W$, since L^* is $GL(V)$ invariant, and the action of $GL(V)$ on K^* is faithful. But $WV \subset V$, since the Galois group permutes the roots of $f_n(X)$. The action of W on K^* is L^* linear, and in particular, it is F_q linear. Since the action on V determines the action on K^* , $GL(V)$ is a subgroup of W and W is a subgroup of $GL(V)$. That is, $W = GL(V)$. Hence $K^{*GL(V)} = L^*$. Since R^* is a polynomial algebra, it is integrally closed. $S(V)^{GL(V)}$ is integral over R^* , and hence

$$F_q[c_{n,0}, \dots, c_{n,n-1}] = R^* = S(V)^{GL(V)}.$$

Finally, we provide some explicit formulae for the invariants.

Proposition 1.3. *Let $B = \langle x_1, \dots, x_n \rangle$ be an ordered basis for V over F_q .*

a) *If A_B is the $(n+1) \times n$ matrix with entries $\{x_j q^i : 0 \leq i \leq n, 1 \leq j \leq n\}$, and $A_B(i)$ is this matrix with the i -th row deleted, then*

$$c_{n,i} = \det(A_B(i)) / \Delta_{n-1}(x_n).$$

b) *Let V_{n-1} be span $\langle x_1, \dots, x_{n-1} \rangle$, and $\{c_{n-1,i}\}$ be the Dickson generators for the invariants of $GL(V_{n-1})$. Then*

$$c_{n,i} = c_{n-1,i-1}^q + c_{n-1,i} f_{n-1}(x_n)^{q-1},$$

where

$$f_{n-1}(x_n) = \prod_{v \in V_{n-1}} (x_n - v)$$

as in Proposition 1.1.

Proof. a) This follows immediately from $\Delta_n(X) = \Delta_{n-1}(x_n) f_n(X)$ and expansion by minors of $\Delta_n(X)$.

$$\begin{aligned} \text{b)} \quad f_n(X) &= \prod_{v \in V} (X - v) = \prod_{a \in \mathbb{F}_q} \prod_{v \in V_{n-1}} (X - ax_n - v) \\ &= \prod_{a \in \mathbb{F}_q} f_{n-1}(X - ax_n) = \prod_a \left(f_{n-1}(X) - a f_{n-1}(x_n) \right) \\ &= f_{n-1}(X)^q - f_{n-1}(X) f_{n-1}(x_n)^{q-1}. \end{aligned}$$

Hence by comparing coefficients, we have

$$c_{n,i} = c_{n-1,i-1}^q + c_{n-1,i} f_{n-1}(x_n)^{q-1}.$$

Corollary 1.4. *If $\varphi : V \rightarrow U$ is surjective, then*

$$\varphi_* : S(V)^{GL(V)} \rightarrow S(U)$$

has image exactly

$$\left(S(U)^{GL(U)} \right)^{q^{\dim V - \dim U}}.$$

[Rector, 13].

II. The Steenrod Algebra Action.

We now restrict our attention to $q = p$. Then $S(V)$ has a unique action of the Steenrod algebra \mathcal{A}_p compatible with the unstable axiom and the Cartan formula. Namely,

$$\begin{aligned}\beta v &= 0 \\ \mathcal{P}^0 v &= v \\ \mathcal{P}^1 v &= v^p \\ \mathcal{P}^{k+1} v &= 0 \quad \text{for } v \in V \quad \text{and } k \geq 1.\end{aligned}$$

The action of $GL(V)$ commutes with this \mathcal{A}_p -action, so the invariants inherit an unstable \mathcal{A}_p -action. This section gives explicit formulae for the action on the generators $\{c_{n,i}\}$ that are useful in applications. In this section, the indeterminate X is treated as a 2-dimensional (1 for $p = 2$) element with an unstable \mathcal{A}_p -action on the algebra it generates.

Proposition 2.1. *For*

$$f(X) = \prod_{v \in V} (X - v) = X^{q^n} + \sum_{i=0}^{n-1} (-1)^{n-i} c_{n,i} X^{q^i},$$

$f(X)$ divides $\mathcal{P}^k f(X)$ in $\mathbb{F}_q[c_{n,0}, \dots, c_{n,n-1}][X]$. Hence

$$\begin{aligned}\mathcal{P}^k f(X) &= -f(X) \mathcal{P}^{k-p^{n-1}} c_{n,n-i} \quad \text{for } k \neq p^n \quad \text{or } 0 \\ \mathcal{P}^{p^n} f(X) &= f(X)^p.\end{aligned}$$

Here $\mathcal{P}^j \equiv 0$ if $j < 0$, $c_{n,n} = 1$, and $c_{n,j} = 0$ if $j < 0$.

Example. The “total” Steenrod class for $p = 2$ is

$$\begin{aligned}\mathcal{SQ}_T(f(X)) &= \prod_v \mathcal{SQ}_T(X - v) = \prod_v ((X - v) + T(X - v)^2) = \\ &= \prod_v ((X - v)(1 + T(X - v))) = f(X) \prod_v T(T^{-1} + (X - v)) = \\ &= f(X) T^{2^n} \prod_v ((X + T^{-1}) - v) = f(X) (f(X + T^{-1})) T^{2^n} = \\ &= f(X) \left[f(X) T^{2^n} + c_{n,0} T^{2^n-1} + c_{n,1} T^{2^n-2} + \dots + c_{n,n-2} T^{2^n-2^{n-2}} + c_{n,n-1} T^{2^n-1} + 1 \right].\end{aligned}$$

The Steenrod algebra has Milnor’s [10] primitives $\{\mathcal{P}^{\Delta_i}\}$. These were denoted by $\{Q^i\}$ in Adams–Wilkerson [1].

Proposition 2.2. *Let \mathcal{P}^{Δ_i} be the primitive of dimension $2(p^i - 1)$ (respectively $2^i - 1$ for $p = 2$) in the Milnor basis of \mathcal{A}_p dual to ξ_i . Then*

$$\begin{aligned}\mathcal{P}^{\Delta_i} v &= v^{p^i} \\ \mathcal{P}^{\Delta_i} f(X) &= 0 \quad \text{for } i < n \\ \mathcal{P}^{\Delta_n} f(X) &= (-1)^n c_{n,0} f(X) \\ \mathcal{P}^{\Delta_i} f(X) &= \mathcal{P}^{p^{i-1}} \mathcal{P}^{\Delta_{i-1}} f(X), \quad \text{for } i > n.\end{aligned}$$

Given Propositions 2.1 and 2.2, one can quickly read off a recursion relation for the Steenrod algebra action.

Corollary 2.3.

$$\begin{aligned}
\text{a)} \quad & \mathcal{P}^k c_{n,i} = \mathcal{P}^{k-p^{i-1}} c_{n,i-1} - (\mathcal{P}^{k-p^{n-1}} c_{n,n-1}) c_{n,i}, \text{ if } k \neq 0 \text{ or } p^n \\
\text{b)} \quad & \mathcal{P}^{\Delta_j} c_{n,i} = 0 \text{ if } i \neq j, \quad 0 < j < n \\
& \mathcal{P}^{\Delta_j} c_{n,j} = (-1)^{j+1} c_{n,0} \text{ if } 0 < j < n \\
& \mathcal{P}^{\Delta_n} c_{n,i} = (-1)^n c_{n,0} c_{n,j} \\
& \mathcal{P}^{\Delta_j} c_{n,i} = \mathcal{P}^{p^{j-1}} \mathcal{P}^{\Delta_{j-1}} c_{n,i} \text{ for } j > n.
\end{aligned}$$

Corollary 2.4.

$$\begin{aligned}
\text{a)} \quad & \mathcal{P}^{p^k} c_{n,i} = 0 \text{ if } p^k < p^{n-1} \text{ and } k \neq i-1. \\
\text{b)} \quad & \mathcal{P}^{p^{i-1}} c_{n,i} = c_{n,i-1} \text{ for } i > 0. \\
\text{c)} \quad & \mathcal{P}^{p^{n-1}} c_{n,i} = -c_{n,i} c_{n,n-1}.
\end{aligned}$$

Proof of Proposition 2.1.

$$\begin{aligned}
\mathcal{P}^k(f(X)) &= \mathcal{P}^k \left(\prod_{v \in V} (X - v) \right) = \sum_I \mathcal{P}^1(X - v_{i_1}) \dots \mathcal{P}^1(X - v_{i_k}) \prod_{V - \{v_{i_1}, \dots, v_{i_k}\}} (X - v) \\
&= f(X) \sum_I (X - v_{i_1})^{p-1} \dots (X - v_{i_k})^{p-1} \text{ by the Cartan formula.}
\end{aligned}$$

Hence $f(X)$ divides $\mathcal{P}^k f(X)$ in $S(V)[X]$, and hence in $S(V)^{GL(V)}[X]$.

Also from the Cartan formula, if x_2 is 2-dimensional (respectively 1-dimensional for $p = 2$),

$$\begin{aligned}
\mathcal{P}^j x_2^{p^i} &= 0 \text{ if } j \neq p^i, 0 \\
\mathcal{P}^{p^i} x_2^{p^i} &= x_2^{p^{i+1}}.
\end{aligned}$$

Thus

$$\mathcal{P}^k f(X) = \sum_{i=0}^{n-1} (-1)^{n-i} (\mathcal{P}^{k-p^i} c_{n,i}) X^{p^{i+1}} + \sum_{i=0}^{n-1} (-1)^{n-i} (\mathcal{P}^k c_{n,i}) X^{p^i}$$

for $k \neq 0$ or p^n . This has degree at most p^n in X so

$$\mathcal{P}^k f(X) = -f(X) \mathcal{P}^{k-p^{n-1}} c_{n,n-1} \text{ for } k \neq 0 \text{ or } p^n.$$

Hence, comparing coefficients we have

$$\mathcal{P}^k c_{n,i} = \mathcal{P}^{k-p^{i-1}} c_{n,i-1} - c_{n,i} \mathcal{P}^{k-p^{n-1}} c_{n,n-1}$$

where \mathcal{P}^j and $c_{n,u}$ are interpreted as 0 for $j < 0$.

Proof of Proposition 2.2. The \mathcal{P}^{Δ_i} act as derivations, since they are primitive in \mathcal{A}_p . Furthermore, $\mathcal{P}^{\Delta_i} x_2 = x_2^{p^i}$. Hence

$$\mathcal{P}^{\Delta_j} f(X) = f(X) \sum_{v \in V} (X - v)^{p^j - 1}$$

in $S(V)[X]$. Thus $f(X)$ divides $\mathcal{P}^{\Delta_j} f(X)$ in $S(V)^{GL(V)}[X]$ also. By computation, using the derivation property,

$$\mathcal{P}^{\Delta_j} f(X) = \sum_{i=0}^{n-1} (-1)^{n-i} (\mathcal{P}^{\Delta_j} c_{n,i}) X^{p^i} + (-1)^n c_{n,0} X^{p^j}.$$

For $j < n$, this has degree less than p^n in X , and hence is zero. For $j = n$,

$$\mathcal{P}^{\Delta_n} f(X) = (-1)^n c_{n,0} f(X).$$

Comparing coefficients, we obtain

$$\begin{aligned} \mathcal{P}^{\Delta_j} c_{n,i} &= 0 \text{ for } i \neq j, \text{ and } j < n \\ \mathcal{P}^{\Delta_i} c_{n,i} &= (-1)^{i+1} c_{n,0} \end{aligned}$$

and

$$\mathcal{P}^{\Delta_n} c_{n,i} = (-1)^n c_{n,0} c_{n,i}.$$

In general,

$$\mathcal{P}^{\Delta_j} = \mathcal{P}^{p^{j-1}} \mathcal{P}^{\Delta_{j-1}} - \mathcal{P}^{\Delta_{j-1}} \mathcal{P}^{p^{j-1}},$$

while on an unstable class of dimension less than $2p^{j-1}$, the latter term on the right hand side is zero. Hence the last assertion is formal. The usual re-indexing changes for $p = 2$ are required.

Proof of the Corollaries. Corollary 2.3 has been proved in the course of the proof of Proposition 2.1 and 2.2. It remains to prove Corollary 2.4.

- a) If $k \neq i-1$ and $k < n-1$, then $m_j = p^k - p^{i-1} \dots - p^{i-j} \neq 0$ for $1 \leq j \leq i$.
Hence $\mathcal{P}^{p^k} c_{n,i} = \mathcal{P}^{m_i} c_{n,0} = 0$ from Corollary 2.3(a), since $m_i < p^{n-1}$.
- b) This follows directly from Corollary 2.3(a).
- c)

$$\mathcal{P}^{p^{n-1}} c_{n,i} = -c_{n,i} c_{n,n-1} + \mathcal{P}^{p^{n-1}-p^{i-1}} c_{n,i-1}.$$

As in a) this last term is zero, since $s = p^{n-1} - p^{i-1} \dots - 1$ is less than p^{n-1} .

III. Other Examples of Polynomial Invariants in Characteristic p .

If W is a generalized reflection group over the complex numbers such that the reflection representation can be defined over \mathbb{F}_q , where q is prime to the order of W , then one can show (e.g. Chevalley [2], Shephard–Todd [15], or Serre [14]) that the invariant subalgebra is again polynomial.

In this section, examples with the order of W not prime to p are discussed. The example of the Weyl group of the compact Lie group F_4 for $p = 3$ (Toda [20]), shows that the reflection condition is not sufficient to ensure polynomial invariants.

In the analysis of the Dickson invariants of §1, there were three main steps:

- 1) Make a good guess for a set of n algebra generators for the invariants!
- 2) Show that $S(V)$ is integral over the subalgebra R generated by the elements guessed in 1). If so, R must be polynomial.
- 3) Show that the degree of the underlying field extension from R to $S(V)$ is the order of the group W . Then the field of fractions of R must be the fixed subfield of W , and since R is integrally closed, and $S(V)$ is integral over R ,

$$R = S(V)^W.$$

In addition to the case $W = GL(V)$, three other cases are amenable to this strategy:

- a) W the symmetric group on some basis of V
- b) W the special linear group $SL(V)$
- c) W one of certain p -subgroups of $GL(V)$, including the case of the upper triangular matrices with diagonal 1's.

Theorem 3.1.

- a) $S(V)^{\Sigma_n} = \mathbb{F}_q[\sigma_1, \dots, \sigma_n]$ where Σ_n is the set of permutation matrices with respect to some fixed basis of V , and σ_i is the i -th elementary symmetric polynomial in the elements of this basis.
- b) $S(V)^{SL(V)} = \mathbb{F}_q[u, c_{n,1}, \dots, c_{n,n-1}]$ where $\{c_{n,i}\}$ are the Dickson invariants, and u is the product of a non-zero element v_L chosen from each line L in V .
- c) If W is a p -group contained in $GL(V)$ such that there is an ordered basis $\{x_i\}$ of V with $(Wx_i - x_i)$ contained in the span of $\langle x_1, \dots, x_{i-1} \rangle$ and $\prod_i \text{card}(Wx_i) = |W|$, then $S(V)^W$ is a polynomial algebra on generators

$$\left\{ y_i = \prod_{z \in \text{orbit } W(x_i)} z \right\}.$$

A useful lemma for computing the field extension degree in the graded case comes from Adams–Wilkerson [1, corr.].

Lemma 3.2. *If $R = \mathbb{F}_q[z_1, \dots, z_n]$ is contained in $S(V)$ such that $S(V)$ is integral over R , then the degree of the underlying field extension is $\prod (\deg(z_i)/2)$, respectively, $\prod \deg(z_i)$ for $p = 2$.*

Of course, the counting of degrees of generators will work in the $GL(V)$ case also, but it is unnecessary there.

Proof of Theorem 3.1.

a) This is left to the reader. It could be viewed as an easy non-inductive proof of the Fundamental Theorem of Symmetric Functions.

b) $c_{n,0} = u^{q-1}$, and hence $S(V)$ is integral over R since it is integral over a subalgebra of R . The product of the degrees of $\{c_{n,i}, i > 0\}$ and that of u is the order of $SL(V)$, so Theorem 3.1(b) is established if u is invariant. Obviously, $wu = \varphi(w)u$ for some map $\varphi : GL(V) \rightarrow \mathbb{F}_q^*$, the units, since the lines in V are permuted by the action of $GL(V)$ up to scalar multiples, φ is a homomorphism, and factors through $GL(V)_{ab}$. Hence $\varphi = (\det)^a$. In particular $\varphi|_{SL(V)} \equiv 1$, and u is invariant.

c) The degrees of the $\{y_i\}$ are correct by fiat, and the $\{y_i\}$ are obviously invariant. It remains only to show that $S(V)$ is integral over $R = \mathbb{F}_q[y_1, \dots, y_n]$. It is enough to show that $\{x_1, \dots, x_n\}$ are integral over R . Of course, $x_1 \in R$. Write W_i for the stabilizer of x_i in W , and set

$$y_i = \prod_{W/W_i} wx_i = \prod (x_i - (x_i - wx_i)).$$

Then x_i is integral over $\mathbb{F}_q[x_1, \dots, x_{i-1}, y_i]$. By the inductive hypothesis, $\{x_1, \dots, x_{i-1}\}$ are all integral over R , so x_i is integral over R . Note that this case includes the results of Mui [11] for the upper triangular groups (but does not treat his case of exterior generators).

Remarks.

- 1) The Steenrod operators on $S(V)^{SL(V)}$ are easily derived from the formulae of §2, using the fact that $u^{q-1} = c_{n,0}$.
- 2) The regular representation of the cyclic group of order p , $p > 2$ gives an example of a p -group with invariants not a polynomial algebra.

IV. Dickson Invariants, The Dyer-Lashof Algebra, and the Lambda Algebra.

I want to sketch an application of the Dickson invariants to a description of the Dyer–Lashof algebra and the lambda algebra. This description arises in the work of W. Singer [17] on the lambda algebra, but some parts of the description of the Dyer–Lashof algebra were known to Milgram, Madsen, and Priddy.

One essential ingredient is Milgram’s observation in [12, Quillen] that the characteristic classes of the regular representation of the \mathbb{F}_p –vector space V are exactly the Dickson invariants:

Proposition 4.1. (a) *Let $\rho_n : V \rightarrow 0(2^n)$ be the regular representation of the mod 2 vector space V . Then the Stiefel–Whitney classes of ρ_n are*

$$w_{2^n-2^i} = c_{n,i}(H^1(V, F_2)) \text{ and } 0 \text{ otherwise.}$$

b) *If $\rho_n : V \rightarrow U(p^n)$ is the regular representation of V , then the Chern classes of ρ_n are*

$$c_{p^n-p^i} = \pm c_{n,i}(H^2(V)) \text{ and } 0 \text{ otherwise.}$$

This follows easily from the decomposition of the regular representation into a sum of one–dimensional representations, together with the identification of the Dickson invariants as elementary symmetric functions.

In the following, p will be 2 unless otherwise noted. Details about the structure of the Dyer–Lashof algebra and its action can be found in Madsen [8] or the book of Cohen–Lada–May [3]. I want to only summarize the work needed to give the connection with the rings of invariants. I thank F. Cohen and W. Dwyer for providing me with the background material on the Dyer–Lashof algebra.

Now the infinite loop space QS^0 can be constructed from the spaces $B\Sigma_n$. In any case, there are maps $\{B\Sigma \rightarrow (QS^0)_n\}$. The Dyer–Lashof algebra R acts on the homology of QS^0 . From the structure of R , it follows that the sub–coalgebra $R[k]$ generated by monomials of length k is closed under the action of the Steenrod algebra defined by the Nishida relations. Madsen [8] calculated the linear dual of $R[k]$ and found it to be a rank k graded polynomial algebra on certain generators $\xi_{n,i}$ with an unstable action of the Steenrod algebra. It is easy to observe that $R[k]$ is isomorphic as an algebra over the Steenrod algebra to the Dickson invariants, and that the $\{c_{n,i}\}$ are just a re–indexing of Madsen’s generators. In fact, the duals can be identified by evaluating $R[k]$ on the homology class [1] in QS^0 and seeing that this is the image of the homology of

$$BV_k \rightarrow B\Sigma_{2^k} \rightarrow (QS^0)_{2^k}.$$

This follows from the fact

$$\text{image } H^*(B\Sigma_{2^k}) = (H^*(BV_k))^{GL(V)},$$

since the normalizer of V in Σ_{2^k} contains $GL(V)$.

Madsen also describes explicitly a “coproduct”

$$\tau_{k+j,k,j} : R[k+j]^* \rightarrow R[k]^* \otimes R[j]^*.$$

It is possible to give a description of this also in terms of invariant theory: it is just the “first-order approximation” to the natural inclusion

$$i_{k,j} : \text{invariants } (k+j) \rightarrow \text{invariants } (k) \otimes \text{invariants } (j),$$

i.e., restriction from $GL(V_{k+j})$ invariants to $GL(U_k) \times GL(W_j)$ invariants. One easily computes this on the $\{c_{n,i}\}$ by using properties of the polynomial $f_n(X)$ of §1., truncating the W -invariant terms at height 1, and extending multiplicatively. The formulae produced by this procedure agree with those of Madsen, so this has mnemonic value at the least. Presumably, the procedure could be justified by computing the deviation between the multiplication in R of elements x and y followed by evaluation on $[1]$, and the composition product $x[1]y[1]$.

Thus far, this has just reproduced descriptions known in part to the experts. But this does shed some light on W. Singer’s description of the dual to the lambda algebra. Curtis [4], for example, observed that the Dyer–Lashof algebra is a quotient DGA of the lambda algebra: namely the admissible monomials of negative excess generate the kernel. To describe the dual of Λ then, one can describe first the dual of R as the product of the $R[k]^*$ for all k , and then seek to adjoin enough generators to get up to Λ . Singer in effect shows that

$$\{c_{n,0}^{-s}c^I\}$$

can be adjoined to $R[n]^*$ in accordance with an excess rule on (I, s) so as to form $\Lambda[n]^*$.

Finally, Singer observes that the induced action of the Steenrod algebra on $\Lambda[n]^*$ is linear for the differential. On the other hand, Wellington [21] forces a formal action of the Steenrod algebra on the dual of $\Lambda[n]$ by the Nishida relations. This action is not linear over the differential, but there are interesting formulae relating it to the differential. Thus far, no applications have been suggested for either action, so the proper interpretation of these actions is still uncertain.

For p odd, some approximation of the above should be true. It is not true exactly, since the duals of the component coalgebras of the Dyer–Lashof algebra fail to be the entire ring of invariants

$$H^*(BV, F_p)^{GL(V)}.$$

However, if $c_{n,0}$ is inverted, this seems to be true. More details might appear in joint work with F. Cohen.

REFERENCES

1. J. F. Adams and C. W. Wilkerson, *Finite H -spaces and algebras over the Steenrod algebra*, Annals of Math. **111** (1980), 95–143; *Corrections*, Annals of Math. **113** (1981), 621–622.
2. C. Chevalley, *Invariants of finite groups generated by reflections*, Amer. J. Math. **77** (1955), 778–782.
3. F. R. Cohen, T. Lada, and J. P. May, *The Homology of Iterated Loop Spaces*, Lecture Notes in Math., Vol. 533, Springer-Verlag, New York, 1976.
4. E. B. Curtis, *The Dyer-Lashof algebra and the lambda-algebra*, Illinois Journal of Math. **18** (1975), 231–246.
5. L. E. Dickson, *A fundamental system of invariants of the general modular linear group with a solution of the form problem*, Trans. A.M.S. **12** (1911), 75–98.
6. S. P. Lam, *Unstable algebras over the Steenrod algebra and cohomology of classifying spaces*, Cambridge thesis (1982).
7. H. H. Li and W. M. Singer, *Resolutions of modules over the Steenrod Algebra and the classical theory of invariants*, preprint (1982).
8. I. Madsen, *On the action of the Dyer-Lashof algebra in $H^*(G)$* , Pac. J. Math. **60** (1975), 1, 235–275.
9. B. M. Mann and R. J. Milgram, *On the Chern classes of the regular representation of some finite groups*, Proc. of the Edinburgh Math. Soc. **25** (1982), 259–268.
10. J. Milnor, *The Steenrod algebra and its dual*, Annals of Math. **67:1** (1958), 150–171.
11. H. Mui, *Modular invariant theory and the cohomology algebras of symmetric groups*, Jour. Fac. Sci. U. of Tokyo **22** (1975), 319–369.
12. D. Quillen, *The Mod 2 cohomology rings of extra-special 2-groups and the Spinor Groups*, Math. Ann. **194** (1971), 197–212.
13. D. L. Rector, *Noetherian cohomology rings and finite loop spaces with torsion*, Journal of Pure and Applied Algebra **32** (1984), 191–217.
14. J.-P. Serre, *Groupes finis d'automorphismes d'anneaux locaux reguliers*, Colloque d'Algebra Paris, ENSJF **exp. 8** (1967), 11.
15. G. C. Shephard and J. A. Todd, *Finite unitary reflections generated by reflections*, Canadian Math. Jour. **6** (1954), 274–304.
16. W. M. Singer, *A new chain complex for the homology of the Steenrod algebra*, Proc. Cambridge Phil. Soc. **90** (1981), 279–292.
17. W. M. Singer, *Invariant theory and the lambda algebra*, Transactions of the A.M.S. **280** (1983), 673–693.
18. L. Smith and R. Switzer, *Realizability and nonrealizability of Dickson algebras as cohomology rings*, Proceedings of the A.M.S. **89** (1983), 303–313.
19. L. Smith and R. Switzer, *Polynomial algebras over the Steenrod algebra: variations on a theorem of Adams and Wilkerson*, Proc. Edinburgh Math. Soc. **27** (1984), 11–19.
20. H. Toda, *Cohomology mod 3 of the classifying space of the exceptional group F_4* , J. Math. Kyoto Univ. **13** (1973), 97–115.
21. R. J. Wellington, *The unstable Adams spectral sequence for free iterated loop spaces*, Memoirs of the A.M.S. No. 258, V. 36, March, 1982.
22. C. W. Wilkerson, *A primer on the Dickson invariants*, Contemporary Mathematics **19** (1983), 421–434.

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