DRAFT: THE IMAGE OF THE TRACE MAP FOR THE DICKSON INVARIANTS

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Abstract.

In [Wilkerson], [3] it was explicitly shown that the trace map for the extension of algebras

$$S[V]^{GL(V)} \to S[V],$$

is nonzero in certain degrees. In particular, for $p = 2$ and $\mathrm{rank}(V) = n$, it was calculated that

$$\mathrm{Trace}\big(\prod_{0\leq i<n} (x_{i+1})^{2^n-1-2^i}\big) = \big(\prod_{v\neq 0} v\big)^{n-1}.$$

That is, the $(n-1)$-th power of the top dimensional generator of the Dickson algebra is in the image of the trace map. The question left unsettled by [3] was the smallest dimension on which the trace was nonzero. This note answers that and more. For convenience we discuss only the $p = 2$ case.

Theorem I. *If the monomial* $\mathbf{X}^{\mathbf{k}} = \prod_{0<i\leq 0}(x_i)^{k_i}$ *has each* $k_i < 2^n$ *and some* k_j *has fewer than* $n-1$ *ones in its dyadic expansion, then*

$$\mathrm{Trace}(\mathbf{X}^{\mathbf{k}}) = 0.$$

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Corollary II. *The image of the trace is exactly the ideal in* $S[V]^{GL(V)}$ *generated by*

$$\prod_{v\neq 0} v^{n-1}.$$

The kernel of the trace is spanned as an $S[V]^{GL(V)}$ *module by the elements in* $S[V]$ *of degree less than*

$$\sum_{0\leq i<n}(2^n-2^i-1)=(n-1)(2^n-1).$$

The proof uses a few elementary facts about the trace:

1. If the trace into the subring of invariants of a proper subgroup is zero, then the trace into the subring of invariants of the entire group is zero and
2. If the trace of an element y in the same orbit as x is zero, then $\mathrm{Trace}(x)=0$.

There are two easily visualized useful subgroups of $GL(V)$:

1. The Borel subgroup B of upper triangular matrices. The action is arranged so that

   $$x_1\to x_1, x_2\to x_2+b_{12}x_1,\ldots,x_n\to x_n+\sum_{0<i\leq n-1} b_{in}x_i.$$

   And
2. The symmetric group Σ_n, embedded as the subgroup of monomial matrices. It permutes the set of monomials into itself.

Proof of Theorem I.

In the orbit of $\mathbf{X^k}$ under B, the general form of a single translate of $\mathbf{X^k}$ is

$$x_1^{k_1}(x_2+b_{12}x_1)^{k_2}\ldots(x_n+b_{1n}x_1\ldots+b_{n-1,n}x_{n-1})^{k_n}.$$

Form the symbolic expansion of this, regarding the b_{ij} as commutting formal variables satisfying $b_{ij}^2=b_{ij}$. Then it is expressible as

$$\sum_{\mathbf{J}}\mathbf{b^J}P_{\mathbf{J}}$$

where $P_{\mathbf{J}}\in S[V]$ and J is a multi-index with each term 0 or 1.

The orbit under B will be the the sum of this term over all evaluations of $(\mathbf{F}_p)^{\binom{n}{2}}$. If the P_J for the constant sequence $(1,1,\ldots,1)$ is zero in this expansion, then the trace of $\mathbf{X}^{\mathbf{k}}$ is zero, since any other monomial in the b_j's will be nonzero an even number of times on $(\mathbf{F}_p)^{\binom{n}{2}}$. Hence when summing over the B orbit, such terms will cancel. But we can break the analysis of such terms down to the individual

$$(x_i + \ldots b_{i-1,i}x_{i-1})^{k_i}$$

The worst case is for the expansion of

$$(x_n + b_{1n}x_1 \ldots + b_{n-1,n}x_{n-1})^{k_n}$$

By changing $\mathbf{X}^{\mathbf{k}}$ by a permutation, it can be assumed that any of the original k_i's is the exponent for x_n in $\mathbf{X}^{\mathbf{k}}$. So the question is reduced determining when does the product

$$b_{1n}b_{2n} \ldots b_{n-1,n}$$

appears with a nonzero coefficient in the expansion of

$$(x_n + b_{1n}x_1 \ldots + b_{n-1,n}x_{n-1})^{k}$$

. This is equivalent to having some multinomial coefficient

$$\binom{k}{i_1, i_2, \ldots i_{n-1}} = k/i!i_1! \ldots i_{n-1}!$$

where

$$i + i_1 + \ldots + i_{n-1} = k$$

be nonzero for some choice of positive i_j's.

By the usual algorithm for the calculation of binomial coefficients mod 2, this will require that for each position s and k_s the s-th coefficienct in the dyadic expansion of k, that

$$\binom{k_s}{(i_1)_s, (i_2)_s, \ldots (i_{n-1})_s} \neq 0.$$

But each i_j must have at least one 1 in its dyadic expansion, so at least $n-1$ of the k_s must be nonzero.

REFERENCES

1. J. F. Adams and C. W. Wilkerson, *Finite H-spaces and Algebras over the Steenrod Algebra*, Annals of Mathematics **111** (1980), 95–143.
2. Stewart Priddy and C.W. Wilkerson, *Hilbert's Theorem 90 and the Segal conjecture for elementary abelian p-groups*, Amer. J. Math **107(4)** (1985), 775–785.
3. C.W. Wilkerson, *A primer on the Dickson invariants*, Proc. of the Northwestern Homotopy Theory Conference, Contemp. Math. 19, (1983), A.M.S., 421–434.
4. C.W. Wilkerson, *Elements of trace 1 in* $S[V](L^{-1})$, URL http://hopf.math.purdue.edu/pub/Wilkerson-older/euler.dvi (1984), 1-3.

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