The Homotopic Uniqueness of BS^3

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1 Introduction

Let p be a fixed prime number, F_p the field with p elements, and S^3 the unit sphere in R^4 considered as the multiplicative Lie group of norm 1 quaternions. The purpose of this paper is to prove the following theorem.

<u>*1.1 Theorem*</u>. *If X is any space with $H^*(X, F_p)$ isomorphic to $H^*(BS^3, F_p)$ as an algebra over the* mod p *Steenrod algebra, then the p-completion of X is homotopy equivalent to the p-completion of BS^3.*

<u>*1.2 Remark*</u>. It is easy to see that 1.1 implies that there is up to homotopy only one space B whose loop space is homotopy equivalent to the p-completion of S^3. To get such a strong uniqueness result it is definitely necessary to work one prime at a time: Rector [R] has produced an uncountable number of homotopically distinct spaces Y with loop space homotopy equivalent to S^3 itself.

Rector's deloopings $\{Y\}$ have the property that $Y_p \simeq (BS^3)^\wedge_p$ for all primes p. Theorem 1.1 implies that this condition is forced. Thus Rector's classification

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of the genus of BS^3 actually classifies all deloopings of S^3. McGibbon [Mc 1] proved in 1978 that any delooping of S^3 is stably equivalent at each odd prime to the standard BS^3. Rector [R] for odd primes and McGibbon [Mc 2] for $p = 2$ showed that the existence of a maximal torus in the sense of Rector distinguishes BS^3 from other members of its genus, and hence by 1.1, from other deloopings of S^3.

1.3 Remark. Theorem 1.1 is in some sense a delooping of the results of [M] and [D-M], although our techniques are somewhat different, especially in the complicated case p=2.

1.4 Organization of this paper. Section 2 contains an account of the main background material we will need from [La] and [D-Z]. Section 3 treats the odd primary case of 1.1, and section 5 the case p=2. The intervening section 4 describes a new way of homotopically constructing BS^3 at the prime 2. Section 6 is essentially an appendix which contains the proof of an auxiliary result needed in section 5.

1.5 Notation and terminology. Some of the methods in this paper are based on simplicial techniques, so we will occasionally use "space" to mean "simplicial set" and tacitly assume that any topological space involved in the argument has been replaced by its singular complex [Ma]. In particular, $X^\wedge_p$ (or $X^\wedge$, if p is understood) will denote the simplicial p-completion of the space X in the sense of [B-K,VII,5.1]. The space X is p-complete if the natural map $X \to X^\wedge_p$ is a weak equivalence. A p-completion $X^\wedge_p$ is itself p-complete iff the map $X \to X^\wedge_p$ induces an isomorphism on mod p homology; this map does give an isomorphism, for instance, if $H_1(X, F_p) = 0$ or if X is connected and $\pi_1(X)$ is a finite group [B-K]. Theorem 1.1 is equivalent to the claim that any p-complete space with the stated cohomology is homotopy equivalent to the p-completion of BS^3.

If X and Y are spaces, then $Hom(X, Y)$ denotes the full function complex of maps $X \to Y$; the subscripted variant $Hom(X, Y)_f$ stands for the component of $Hom(X, Y)$ containing a particular map f. As usual, $[X, Y]$ denotes the set of components of $Hom(X, Y)$, i.e., the set of homotopy classes of maps from X to Y.

If G is a (simplicial) group, then $EG \to BG$ is the functorial universal simplicial principal $G-$bundle [Ma,p83]. If G is abelian, then BG is also an abelian simplicial group and the classifying process can be iterated to form

$B^2G = B(BG), B^3G, \cdots$, etc. The symbol σ will denote the cyclic group Z/pZ of order p and κ the additive group of p-adic integers, so that, for example, $B\sigma$ is equivalent to an infinite lens space and $B^2\kappa$ to $(CP^\infty)^\wedge_p$.

Unless otherwise specified, all homology and cohomology is taken with simple F_p coefficients.

1.6 Remark. The goal of this paper is to present the complete picture for BS^3. Obviously the present techniques carry over to other classifying spaces and exotic loop spaces. The discussion of section 3 applies almost verbatim to other rank 1 loop spaces, and with a few changes to the general case with p prime to the order of the Weyl group. On the other hand, the work of sections 4,5 and 6 for $p = 2$ is special to S^3, and must be replaced with new constructions for the small prime cases. As yet no general methods for these primes which divide the order of the Weyl group are available.

2 Background

Fix the prime p, and let $\mathcal{K}$ denote the category of unstable algebras over the mod p Steenrod algebra $\mathcal{A}_p$ [La]. For any two spaces X and Y there is a natural map

$$[Y, X] \longrightarrow Hom_{\mathcal{K}}(H^*(X), H^*(Y))$$

2.1 Theorem [La]. *If X is a p-complete space with the property that $H^*(X)$ is finite in each dimension, then for each $n \geq 1$ the map*

$$[B\sigma^n, X] \longrightarrow Hom_{\mathcal{K}}(H^*(X), H^*(B\sigma^n))$$

is an isomorphism.

Lannes has studied the functor $T : \mathcal{K} \to \mathcal{K}$ which is left adjoint to the functor given by tensor product with $H^*(B\sigma^n)$. It is clear that the evaluation map

$$Hom(B\sigma^n, X) \times B\sigma^n \longrightarrow X$$

induces for any space X a cohomology map

$$T^n H^*(X) \longrightarrow H^*(Hom(B\sigma^n, X)).$$

2.2 Theorem [La] *If X is a p-complete space with the property that $H^*(X)$ is finite in each dimension, then the above map*

$$T^n H^*(X) \longrightarrow H^*(Hom(B\sigma^n, X))$$

is an isomorphism under either of the following two assumptions:

(1) $T^n H^*(X)$ *is zero in dimension 1, or*

(2) *there is a p-complete space* Z *and a map*

$$Z \longrightarrow Hom(B\sigma^n, X)$$

such that the induced cohomology map

$$T^n H^*(X) \longrightarrow H^*(Z)$$

is an isomorphism.

We will need a slight refinement of 2.2, which can be proved by the same argument. If X is any space and $g : H^*(X) \to H^*(B\sigma^n) \cong (H^*(B\sigma^n)) \otimes F_p$ is a map in $\mathcal{K}$, let $\tilde{g} : T^n H^*(X) \to F_p$ be its adjoint. If R^0 is the ring $(T^n H^*(X))^0$, then $\tilde{g}$ amounts to a ring homomorphism $R^0 \to F_p$; let $(T^n H^*(X))_g$ denote the tensor product $(T^n H^*(X)) \otimes_{R^0} F_p$, where the action of R^0 on F_p is via $\tilde{g}$. If $f : B\sigma^n \to X$ is an actual map, we will also write $(T^n H^*(X))_f$ for $(T^n H^*(X))_g$ where $g = f^* : H^*(X) \to H^*(B\sigma^n)$ is the induced cohomology morphism. It is easy to argue by functoriality that the natural map

$$T^n H^*(X) \longrightarrow H^*(Hom(B\sigma^n, X))$$

factors down to a map

$$(T^n H^*(X))_f \longrightarrow H^*(Hom(B\sigma^n, X)_f)$$

for each $f : B\sigma^n \to X$.

2.3 Theorem [La] *If X is a p-complete space with the property that $H^*(X)$ is finite in each dimension and $f : B\sigma^n \to X$ is a map, then the map*

$$(T^n H^*(X))_f \longrightarrow H^*(Hom(B\sigma^n, X)_f)$$

is an isomorphism under either of the following two assumptions:

(1) $(T^n H^*(X))_f$ *vanishes in dimension 1, or*

(2) *there is a connected p-complete space Z and a map*

$$Z \to Hom(B\sigma^n, X)_f$$

such that the corresponding cohomology map

$$(T^n H^*(X))_f \longrightarrow H^*Z$$

is an isomorphism.

There is one situation in which the algebraic functors of 2.1 and 2.3 can be computed. If π is a finite group and G is a compact Lie group, let $Rep(\pi, G)$ denote the set of G-conjugacy classes of homomorphisms $\pi \to G$. Passing to classifying spaces gives a map

$$Rep(\pi, G) \longrightarrow [B\pi, BG]$$

If $\varphi : \pi \to G$ is a particular homomorphism and $Z_G(\varphi) \subseteq G$ denotes the centralizer of the image of φ, then the obvious product map

$$Z_G(\varphi) \times \pi \overset{inc.\times\varphi}{\longrightarrow} G$$

also passes to the classifying space level and induces a map

$$BZ_G(\varphi) \longrightarrow Hom(B\pi, BG)_{B\varphi}$$

2.4 Theorem [La, Ad, M-W] *Suppose that G is a compact Lie group and that $n \geq 1$. Then*

(1) *the natural composite*

$$Rep(\sigma^n, G) \to [B\sigma^n, BG] \to Hom_{\mathcal{K}}(H^*(BG), H^*(B\sigma^n))$$

is an isomorphism of sets, and

(2) *for each homomorphism $\varphi : \sigma^n \to G$ the natural map*

$$BZ_G(\varphi) \to Hom(B\sigma^n, BG)_{B\varphi}$$

induces a composite map

$$T^n(H^*(BG)_{B\varphi}) \longrightarrow H^*(Hom(B\sigma^n, BG)_{B\varphi}) \longrightarrow H^*(BZ_G(\varphi))$$

which is an isomorphism in the category $\mathcal{K}$.

The following lemma allows 2.1 - 2.3 to be applied in some cases to spaces which are not p-complete.

2.5 Proposition If X is a connected space such that $\pi_1(X)$ is a finite p-group, then for any finite p-group π the natural map

$$Hom(B\pi, X) \longrightarrow Hom(B\pi, X_p^\wedge)$$

induces an isomorphism on mod p *cohomology.*

Proof This is a consequence of [B-K,proof of VII,5.1] and the obstruction theory argument of [D-Z].

We will need to apply 2.5 in one case in which the group π involved is not elementary. The necessary key piece of information for the application comes from the following theorem.

Theorem [D-Z]. *If G is a compact Lie group and π is a finite p-group, then the natural map*

$$Rep(\pi, G) \longrightarrow [B\pi, BG]$$

is an isomorphism of sets.

3 The odd primary case

In this section we will work under the assumption that p is a fixed odd prime and that X is a p-complete space with a fixed $\mathcal{K}$-isomorphism $H^*(X) \approx H^*(BS^3)$. Let b_2 denote a chosen generator of $H^2(B\sigma)$. If x is an element of $H^4(X)$, then $\mathcal{P}^2(x)$ is either $2x^{\frac{p+1}{2}}$ or $-2x^{\frac{p+1}{2}}$; let $x_4 \in H^4(X)$ denote a chosen generator of the first kind.

3.1 Proposition There exists up to homotopy a unique map $f : B\sigma \to X$ such that $f^*(x_4) = (b_2)^2$. The function space component $Hom(B\sigma, X)_f$ is homotopy equivalent to $B^2\kappa$.

Proof It is clear that in the category $\mathcal{K}$ there is a unique map $g : H^*(X) \approx H^*(BS^3) \to H^*(B\sigma)$ such that $g(x_4) = (b_2)^2$; by 2.1, g is f^* for a map $f : B\sigma \to X$ which is unique up to homotopy. By 2.4, f^* is the same as $(B\varphi)^*$ for some (non-trivial) representation $\varphi : \sigma \to S^3$. Since the centralizer of the image of φ in S^3 is a circle group, it follows from 2.4 that $(TH^*(X))_f$ is isomorphic to $H^*(B^2\kappa)$. It follows from 2.3 that the same holds true of $H^*(Hom(B\sigma, X)_f)$. The space $Hom(B\sigma, X)_f$ is p-complete in view of the fact that it is $H_*(-, F_p)-$local and has vanishing first homology group [Bo;12.6,proof of 4.3]], so the proof is finished by observing that $B^2\kappa$ is determined up to homotopy among p-complete spaces by its cohomology ring.

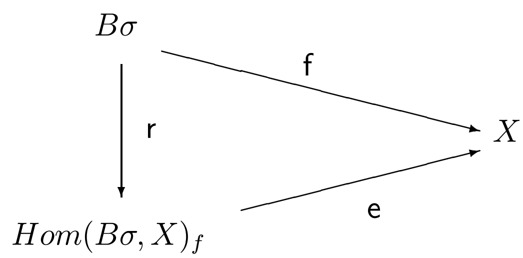
For the rest of this section, we will fix a particular f which satisfies the condition of 3.1. Let $e : Hom(B\sigma, X)_f \to X$ be the map obtained by evaluating a function at the basepoint of $B\sigma$.

3.2 Proposition The cohomology map

$$e^* : H^*(X) \longrightarrow H^*(Hom(B\sigma, X)_f)$$

is injective.

Proof Construct a commutative diagram



where the "right translation" map r is given by the formula $(r(g))(a) = f(a+g)$. Here $g, a \in B\sigma$ and $a + g$ denotes the sum of a and g with respect to the abelian group structure of $B\sigma$ (1.5). Commutativity follows from the fact that the basepoint of $B\sigma$ is the identity for the group structure. Since f^* is injective, the proof is complete.

Let $W = \{1, w\}$ be the group of order 2. Defining $w : \sigma \to \sigma$ to be the inverse map gives an action of W on σ which passes to a basepoint-preserving action of W on $B\sigma$ and induces a right-composition action of W on $Hom(B\sigma, X)$.

3.3 Proposition The above action of W on $Hom(B\sigma, X)$ carries the component $Hom(B\sigma, X)_f$ to itself.

Proof It is necessary to check that $f \cdot w$ is homotopic to f, where

$$w : B\sigma \to B\sigma$$

is given by the action of W. This follows from 3.1 and the fact that w^* is multiplication by (-1) on $H^2(B\sigma)$, so that $w^*((b_2)^2) = (-b_2)^2 = (b_2)^2$.

3.4 Proposition The basepoint evaluation map $e : Hom(B\sigma, X)_f \to X$ is equivariant with respect to the above action of W on $Hom(B\sigma, X)_f$ and the trivial action of W on X.

Proof This is a formal consequence of the fact that the action of W on $B\sigma$ is basepoint-preserving.

3.5 Proposition The automorphism w^* of $H^2(Hom(B\sigma, X))_f$ induced by the action of W on $Hom(B\sigma, X)_f$ is multiplication by (-1).

Proof Let $r : B\sigma \to Hom(B\sigma, X)_f$ be the map which appears in the proof of 3.2. Since $w^* : H^2(B\sigma) \to H^2(B\sigma)$ is multiplication by (-1) and r^* is injective (3.1, proof of 3.2), it is enough to show that the diagram

$$\begin{array}{ccc} B\sigma & \xrightarrow{r} & Hom(B\sigma, X)_f \\ w \downarrow & & \downarrow w \\ B\sigma & \xrightarrow{r} & Hom(B\sigma, X)_f \end{array}$$

commutes up to homotopy. Taking adjoints reduces this to the problem of showing that the two maps $f \cdot m \cdot (1 \times w)$ and $f \cdot m \cdot (w \times 1)$ are homotopic as maps $B\sigma \times B\sigma \to X$, where $m : B\sigma \times B\sigma \to B\sigma$ is the group multiplication. Since W acts on $B\sigma$ by group automorphisms, this follows from the fact that $f \cdot w$ is homotopic to f.

Proof of 1.1 (assuming p odd).
Let Y denote the quotient space $(Hom(B\sigma, X)_f \times EW)/W$. By 3.4, the basepoint evaluation map $e : Hom(B\sigma, X)_f \to X$ factors through a map $\bar{e} : Y \to X$. By 3.1, 3.2, 3.5 and a calculation with the Serre spectral sequence, the map $\bar{e}$ induces a mod p cohomology isomorphism $H^*(X) \to H^*(Y)$ and therefore a homotopy equivalence $Y_p^\wedge \to X_p^\wedge \sim X$. Since the order of W is prime to p and there is only one non-trivial homomorphism from W to the multiplicative group of p-adic integral units, it follows from 3.5 that up to

homotopy the space Y is just the bundle over BW associated to the inversion action of W on $B^2\kappa$. In particular, Y does not depend on X, so the above proof that X is equivalent to $Y_p^\wedge$ shows also that $(BS^3)_p^\wedge$ is equivalent to $Y_p^\wedge$.

4 Constructing BS^3 at the prime 2

In this section we will assume that p is 2. The goal of this section is to find an explicit way of passing from some sort of finite group data to the 2-completion of BS^3.

Let $\tilde{\mathcal{O}}_{48} \subseteq S^3$ be the binary octahedral group, i.e. the inverse image in $S^3 \approx Spin(3)$ of the group $\mathcal{O}_{24}$ in $SO(3)$ of orientation-preserving isometries of the cube. The 2-Sylow subgroup of $\tilde{\mathcal{O}}_{48}$ is the quaternion group Q_{16} of order 16. Let $NT \subseteq S^3$ be the normalizer of a maximal torus, and note that there is at least one injective homomorphism $Q_{16} \to NT$.

4.1 Theorem *Assume that $\varphi_1 : Q_{16} \to NT$ is an injective homomorphism and that $\varphi_2 : Q_{16} \to \tilde{\mathcal{O}}_{48}$ is the inclusion of a 2-Sylow subgroup. Let P be the homotopy pushout of the induced diagram*

$$B\tilde{\mathcal{O}}_{48} \overset{B\varphi_2}{\longleftarrow} BQ_{16} \overset{B\varphi_1}{\longrightarrow} BNT$$

Then the completion $P_2^\wedge$ of P is homotopy equivalent to $(BS^3)_2^\wedge$.

Proof Let $i : NT \to S^3$ be the inclusion. Up to conjugacy the group Q_{16} has two faithful representations in $S^3 \approx SU(2)$. These two representations have essentially the same image (since they differ by an outer automorphism of Q_{16}) so we can assume that the composite $i\varphi_1$ extends over φ_2 to the representation $\tilde{\mathcal{O}}_{48} \to S^3$ which was used above to define $\tilde{\mathcal{O}}_{48}$. This gives a commutative diagram of classifying spaces

$$\begin{array}{ccc} BQ_{16} & \overset{B\varphi_1}{\longrightarrow} & BNT \\ B\varphi_2 \downarrow & & \downarrow Bi \\ B\tilde{\mathcal{O}}_{48} & \longrightarrow & BS^3 \end{array}$$

and leads to a map $P \to BS^3$. To complete the proof it is enough to show that this map induces an isomorphism on mod 2 cohomology. Taking homotopy fibers over BS^3 reduces the problem to showing that the homotopy pushout of the diagram

$$S^3/\tilde{\mathcal{O}}_{48} \longleftarrow S^3/Q_{16} \longrightarrow S^3/NT$$

is mod 2 acyclic. This can be directly verified by a calculation with one-dimensional integral cohomology, since S^3/Q_{16} and $S^3/\tilde{\mathcal{O}}_{48}$ are orientable 3-manifolds while S^3/NT is RP^2.

Since the group S^3 is a double cover of $SO(3)$, the following corollary is just a restatement of 4.1. Recall that the 2-Sylow subgroup of the octahedral group $\mathcal{O}_{24}$ is the dihedral group D_8 of order 8, and that the normalizer of a maximal torus in S^3 is isomorphic to $O(2)$.

4.2 Corollary Assume that $\varphi_1 : D_8 \to O(2)$ *is an injective homomorphism and that* $\varphi_2 : D_8 \to \mathcal{O}_{24}$ *is the inclusion of a 2-Sylow subgroup. Let P be the homotopy pushout of the induced diagram*

$$B\mathcal{O}_{24} \xleftarrow{B\varphi_2} BD_8 \xrightarrow{B\varphi_1} BO(2)$$

then the completion $P_2^\wedge$ *is homotopy equivalent to* $(BSO(3))_2^\wedge$.

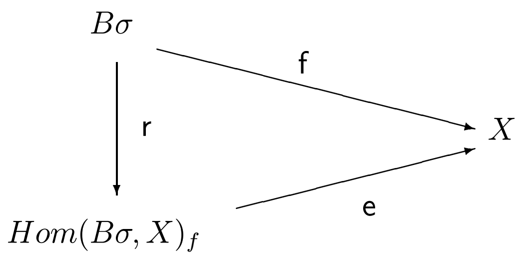
4.3 Remark The octahedral group $\mathcal{O}_{24}$ is isomorphic to the symmetric group on four letters, as well as to the semi-direct product constructed out of the natural action of the general linear group $GL_2(F_2)$ on $F_2 \times F_2$.

5 The case of the prime 2

In this section we will assume that p is equal to 2 and that X is a 2-complete space with the property that $H^*(X)$ is isomorphic in the category $\mathcal{K}$ to $H^*(BS^3)$. (Note that there is a unique way to choose the isomorphism). Let $x_4 \in H^4(X)$ and $b_1 \in H^1(B\sigma)$ label the generators of these groups.

5.1 Proposition There exists up to homotopy a unique map $f : B\sigma \to X$ *such that* $f^*(x_4) = (b_1)^4$. *The basepoint evaluation map* $e : Hom(B\sigma, X)_f \to X$ *is a homotopy equivalence.*

Proof By 2.1, there is a unique map f up to homotopy with the stated cohomological property and by 2.3 - 2.4 the cohomology ring $H^*(Hom(B\sigma, X)_f)$ is isomorphic to $H^*(X)$. As in the proof of 3.1, then, we will be done if we can show that the cohomology map e^* is bijective or even injective. Injectivity of e^* follows as in the proof of 3.2 from the existence of a commutative diagram



Fix a map f as in 5.1. The right translation action of $B\sigma$ on itself (1.5) induces an action of $B\sigma$ on the function space component $Hom(B\sigma, X)_f$. Let Y be the associated bundle over $B^2\sigma$, so that there is a principal fibration sequence

$$B\sigma \longrightarrow X \longrightarrow Y.$$

5.2 Proposition The space Y is 2-complete, and there is a unique $\mathcal{K}-$isomorphism $H^*(Y) \to H^*(BSO(3))$.

Proof The fact that Y is 2-complete follows from [B-K]. By calculation, the E_2- term of the Rothenberg-Steenrod spectral sequence [R-S]

$$E_2^{*,*} = Cotor_{H^*B\sigma}(H^*(X), F_2) \Longrightarrow H^*(Y)$$

is a polynomial algebra on classes y_2 and y_3 in positions (1,1) and (1,2) respectively. For positional reasons, then, the spectral sequence collapses and y_2, y_3 lift to unique classes (of the same name) in $H^*(Y)$ which generate $H^*(Y)$ as a polynomial algebra. The formula $Sq^1 y_2 = y_3$ holds already at E_2, and it is a priori clear that $Sq^2 y_2 = (y_2)^2$ and $Sq^3 y_3 = (y_3)^2$. The class $Sq^2 y_3$ does not vanish, because $Sq^1 Sq^2 y_3 = Sq^3 y_3 = (y_3)^2 \neq 0$; inasmuch as there is only one non-zero cohomology class in $H^5(Y)$, it follows that $Sq^2 y_3 = y_2 y_3$. These formulas completely determine the action of the Steenrod algebra on $H^*(Y)$ and lead immediately to the desired result.

Let $u_1, v_1 \in H^1(B\sigma^2)$ denote chosen generators.

5.3 Proposition *There exists up to homotopy a unique map* $g : B\sigma^2 \to Y$ *such that* $g^*(y_2) = u_1^2 + u_1v_1 + v_1^2$ and $g^*(y_3) = u_1v_1(u_1 + v_1)$.
The map $r : B\sigma^2 \to Hom(B\sigma^2, X)_g$ *constructed as in the proof of 3.2 is a homotopy equivalence.*

Proof The existence and essential uniqueness of g follows from 2.1 and 2.4; g corresponds cohomologically to the unique faithful representation $\varphi : \sigma^2 \to SO(3)$. To finish the proof, observe that for any space S and map $h^* : H^*(S) \to H^*(B\sigma^2)$, the Hopf algebra structure of $H^*(B\sigma^2)$ provides a map $T^2(H^*(S))_{h^*} \to H^*(B\sigma^2)$. By 2.4 this map is an isomorphism if $h = B\varphi : B\sigma^2 \to BSO(3)$; it follows that the map is also an isomorphism in the algebraically indistinguishable case $h = g$. The desired conclusion follows from 2.3, since the isomorphism $T^2(H^*(Y))_g \to H^*(B\sigma^2)$ is derived from an actual map $B\sigma^2 \to Hom(B\sigma^2, Y)_g$.

Fix a map as in 5.3. Let W denote the general linear group $GL_2(F_2)$; the group acts naturally on σ^2 and therefore also on $B\sigma^2$ as well as on $Hom(B\sigma^2, Y)$. The following proposition is proved by the arguments of propositions 3.3 - 3.5.

5.4 Proposition

(1) *The action of* W *on* $Hom(B\sigma^2, Y)$ *restricts to an action of* W *on* $Hom(B\sigma^2, Y)_g$.

(2) *The basepoint evaluation map*

$$e : Hom(B\sigma^2, Y)_g \longrightarrow Y$$

is W-equivariant with respect to the trivial action of W on Y.

(3) *The action of* W *on* $H_1(Hom(B\sigma^2, Y)_g)$ *is equivalent to the standard action of* W *on* $(F_2)^2$.

Proposition 5.4 immediately gives a map

$$\bar{e} : (EW \times Hom(B\sigma^2, Y)_g)/W \longrightarrow Y.$$

A short calculation shows that the cohomology group $H^2(GL_2(F_2), (F_2)^2)$ vanishes if $(F_2)^2$ is given the standard $GL_2(F_2)-$action; it follows, again by 5.4, that the fundamental group of the domain of $\bar{e}$ is the octahedral group $\mathcal{O}_{24}$ (4.3). Recall that the 2-Sylow subgroup of $\mathcal{O}_{24}$ is the dihedral group D_8 of order

8. Let $s : BD_8 \to B\mathcal{O}_{24}$ correspond to the inclusion of a 2-Sylow subgroup and $c : B\sigma \to BD_8$ to the inclusion of the center.

5.5 Proposition *There exists a map*

$$h : B\mathcal{O}_{24} \longrightarrow Y$$

such that $c^* s^* h^*(y_2) = (b_1)^2$.

Proof The map h is the map $\bar{e}$ described above. The formula for the restriction of y_2 to $B\sigma$ follows from 5.3 and the fact that, as in the proof of 3.2, $h \cdot s \cdot c$ can be factored as a composite

$$B\sigma \longrightarrow B\sigma^2 \xrightarrow{g} Y$$

in which the map $B\sigma \to B\sigma^2$ induces an inclusion $\sigma \to \sigma^2$.

Let $r : B\sigma \to Y$ denote the composite $h \cdot s \cdot c$ above.

For the moment, use the map c above to identify σ with the center of D_8. The multiplication homomorphism $D_8 \times \sigma \to D_8$ gives rise to a composite

$$BD_8 \times B\sigma \longrightarrow BD_8 \xrightarrow{s} B\mathcal{O}_{24} \xrightarrow{h} Y$$

The adjoint of this composite is a map

$$k : BD_8 \longrightarrow Hom(B\sigma, Y)_r$$

5.6 Proposition *The following diagram commutes:*

$$\begin{array}{ccc} BD_8 & \xrightarrow{k} & Hom(B\sigma, Y)_r \\ s \downarrow & & \downarrow e \\ B\mathcal{O}_{24} & \xrightarrow[h]{} & Y \end{array}$$

Here, as usual, the right vertical map e *is evaluation at the basepoint of* $B\sigma$.

5.7 Proposition *The function space component* $Hom(B\sigma, Y)_r$ *is homotopy equivalent to* $BO(2)^{\wedge}_2$.

This will be proved in section 6.

Proof of 1.1 (for the prime 2). We will show that the space Y constructed above is homotopy equivalent to $BSO(3)^\wedge_2$, and leave it to the reader to deduce that X is equivalent to $(BS^3)^\wedge_2$. Since $c^*k^*e^* = c^*s^*h^* = r^* \neq 0$ (5.5) the map k corresponds (2.4, 2.5) to an injective homomorphism $D_8 \to O(2)$. Let P be the homotopy pushout of the diagram

$$B\mathcal{O}_{24} \stackrel{s}{\longleftarrow} BD_8 \stackrel{\tilde{k}}{\longrightarrow} BO(2)$$

in which $\tilde{k}$ is a lift (2.5) of k to $BO(2)$. It follows from 4.2 that we will be done if we can show that the evident map $l : P \to Y$ induces an isomorphism on mod 2 cohomology. It is clear that $l^*(y_2) \neq 0$, since the restriction of y_2 to the center of D_8 is already non-zero; this implies that l^* is an isomorphism in dimension 1. A calculation with Sq^1 immediately shows that l^* is also an isomorphism in dimension 2. The fact that both of the cohomology rings involved are polynomial algebras on generators of dimension 1 and 2 now completes the proof.

6 A function space calculation

In this section we will prove proposition 5.7. Notation and terminology will be taken from section 5. Recall from section 5 the cohomologically non-trivial map $f : B\sigma \to X$ and the natural action of $B\sigma$ on $Hom(B\sigma, X)_f$. It is easy to produce a similarly non-trivial map $f' : B\sigma \to B^2\kappa$, as well as a corresponding action of $B\sigma$ on $Hom(B\sigma, B^2\kappa)_{f'}$. Basepoint evaluation gives homotopy equivalences

$$\begin{array}{ccc} Hom(B\sigma, X)_f & \longrightarrow & X \\ Hom(B\sigma, B^2\kappa)_{f'} & \longrightarrow & B^2\kappa \end{array}$$

Let Z denote the bundle over $B^2\sigma$ associated to the diagonal action of $B\sigma$ on $Hom(B\sigma, X)_f \times Hom(B\sigma, B^2\kappa)_{f'}$. Note that Z is just the analogue in this function space setting of the standard construction of $U(2)$ as the quotient of $SU(2) \times S^1$ by $Z/2Z$.

6.1 Proposition *The space Z is 2-complete, and there is a (unique) $\mathcal{K}$-isomorphism $H^*(Z) \to H^*(BU(2))$.*

Proof The fact that Z is 2-complete follows from [B-K]. It is easy to argue that the bundle over $B^2\sigma$ associated to the action of $B\sigma$ on $Hom(B\sigma, B^2\kappa)_{f'}$ has a total space homotopy equivalent to $B^2\kappa$ itself. As a consequence, the $B\sigma-$equivariant projection map

$$Hom(B\sigma, X)_f \times Hom(B\sigma, B^2\kappa)_{f'} \longrightarrow Hom(B\sigma, B^2\kappa)_{f'}$$

gives up to homotopy a fibration

$$X \longrightarrow Z \longrightarrow B^2\kappa.$$

The Serre spectral sequence of this fibration collapses and shows that $H^*(Z)$ is a polynomial algebra on classes z_2 and z_4 of dimensions 2 and 4 respectively. To show that $H^*(Z)$ is $\mathcal{K}$-isomorphic to $H^*(BU(2))$, it is necessary to check that $Sq^2(z_4)$ is either z_2z_4 or $z_2z_4 + (z_2)^2$ (these two possibilities differ by the substitution of the alternative polynomial generator $z_4 + (z_2)^2$ for z_4). This calculation can be made by examining the Eilenberg-Moore spectral sequence of the evident fibration

$$Z \longrightarrow Y \longrightarrow B^3\kappa.$$

This spectral sequence also collapses, and the residue class of z_4 modulo $(z_2)^2$is represented at E_1 by the bar construction cycle $\tilde{z}_4 = y_2 \otimes i_3 + 1 \otimes Sq^3 i_3$ (here $i_3 \in H^3(B^3\kappa)$ is the generator.) The fact that $Sq^1 i_3 = 0$ combines with the Cartan formula to give

$$Sq^2\tilde{z}_4 = (y_2)(\tilde{z}_4).$$

This formula implies the desired result, since the restriction of y_2 to Z is z_2.

6.2 Proposition *The map $r : B\sigma \to Y$ of 5.5-5.6 lifts to a map $\tilde{r} : B\sigma \to Z$ which is unique up to homotopy. The space $Hom(B\sigma, Z)_{\tilde{r}}$ is homotopy equivalent to $B^2\kappa^2$.*

Proof This follows from 2.1, 2.3 and 2.4. The map $\tilde{r}$ corresponds cohomologically to the map $B\sigma \to BU(2)$ derived from the regular representation of σ.

6.3 Proposition *The loop space $\Omega Hom(B\sigma, Y)_r$ is homotopy equivalent to the space $\sigma \times B\kappa$.*

Proof The fibration

$$Z \longrightarrow Y \longrightarrow B^3\kappa$$

gives rise to a function space fibration

$$Hom(B\sigma, Z)_{\tilde{r}} \longrightarrow Hom(B\sigma, Y)_r \longrightarrow Hom(B\sigma, B^3\kappa)$$

(the fiber is as described because the map $r : B\sigma \to Z$ composes to a null-homotopic map $B\sigma \to B^3\kappa$). The homotopy groups of $Hom(B\sigma, B^3\kappa)$ are given by the table

$$\pi_i Hom(B\sigma, B^3\kappa) = \begin{cases} \sigma & i = 1 \\ \kappa & i = 3 \\ * & \text{otherwise} \end{cases}$$

so the proof can be completed by showing that the boundary homomorphism

$$\kappa \approx \pi_3 Hom(B\sigma, B^3\kappa) \longrightarrow \pi_2 Hom(B\sigma, Z)_{\tilde{r}} \approx \kappa^2$$

remains non-zero after tensoring with F_2. Consider the diagram

$$\begin{array}{ccccc} \sigma \times B^2\kappa \sim & Hom(B\sigma, B^2\kappa) & \xrightarrow{a} & Hom(B\sigma, Z)_{\tilde{r}} & \sim B^2\kappa^2 \\ & e_1 \downarrow & & \downarrow e_2 & \\ & B^2\kappa & \xrightarrow{b} & Z & \end{array}$$

in which the vertical arrows are obtained by evaluation at the basepoint of $B\sigma$ and the horizontal arrows from the fiber inclusion in the shifted fibration

$$(*)\ B^2\kappa \xrightarrow{b} Z \longrightarrow Y$$

The map a induces a morphism on π_2 which is essentially the boundary homomorphism of interest, and the map e_1 restricts to a homotopy equivalence from either component of $\sigma \times B^2\kappa$ to $B^2\kappa$. A look at the Serre spectral sequence of (*) shows that $b^* : H^4(Z) \to H^4(B^2\kappa)$ is surjective, which implies that a^* induces epimorphisms $H^4(B^2k^2) \to H^4(\{\varepsilon\} \times B^2\kappa)$ for each ε in σ. This directly implies the desired result.

Proof of 5.7 By 6.3, the space $Hom(B\sigma, Y)_r$ is equivalent to the 2-completion of one of the following three spaces

$a)$ $B\sigma \times BS^1$
$b)$ $BO(2)$
$c)$ BNT

where $NT \subseteq S^3$ is the normalizer of a torus. Possibilities (a) and (c) are ruled out by combining 5.6, 2.1, 2.4 and 2.5, since there is no homomorphism $D_8 \to \sigma \times S^1$ or $D_8 \to NT$ which is non-trivial on the center of D_8.

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