Sample Problems for MIDTERM EXAM I,
Ma 553

1. (a) List representatives for each conjugacy class in the symmetric group S_4 and state the number of elements in each conjugacy class.

   (b) List representatives for each conjugacy class in the alternating group A_4 and state the number of elements in each conjugacy class.

   (c) Determine the number of elements of order 2 in the symmetric group S_5.

   (d) Determine the number of elements of order 2 in the symmetric group A_5.

2. Let G be a group having order $2k$, where k is an odd integer. Prove that G has a subgroup of order k.

3. (a) What is the order of the group $GL_3(F_3)$ of 3×3 invertible matrices with entries in F_3 ?

   (b) What is the order of the group $GL_3(F_3)$ of 3×3 invertible upper triangular matrices with entries in F_3 ?

   (c) Does there exist a nonabelian group of order 27 ? Justify your answer.

4. Let G be a group of order p^4 ,where p is prime, and suppose H is a subgroup of G with $|H| = p^2$. Prove or disprove that there must exist a subgroup K of G such that $H \subseteq K$ and $|K| = p^3$.

5. Let G be a finite group, $p > 0$ a prime number, and H a normal subgroup of G. Prove the following assertions.

   (a) Any Sylow p-subgroup of H is the intersection $P \cap H$ of a Sylow p-subgroup P of G and subgroup H.

   (b) Any Sylow p-subgroup of G/H is the is the quotient PH/H , where P is a Sylow p-subgroup of G.

6. Let H be a normal subgroup of a finite group G, and let $N \subset H$ be a normal Sylow subgroup of H. Prove that N is a normal subgroup of G.

7. Let G be a finite group, $p > 0$ a prime number, and H a normal p-subgroup of G. Prove the following assertions.

   (a) H is contained in each Sylow p-subgroup of G.

   (b) If K is any normal p-subgroup of G, then HK is a normal p-subgroup of G.

8. Let $q > 0$ be a prime integer of the form $6n - 1$. Show that any group of order $27q$ is the direct product of the cyclic group Z_q with a group of order 27.

9. Prove that the order of the automorphism group of $(Z/3)^4$ is $80 \times 78 \times 72 \times 54$

10. Describe an example of a group of order 405 which is not the direct product of Z_5 with a group of order 81. Justify your answer.

11. Prove, for fixed n, that the following conditions are equivalent:

    (a) Every abelian group of order n is cyclic.

    (b) n is squarefree (i.e., not divisible by any square integer > 1).

12. Prove that there is no simple group of order 4125.

13. Let G be a group of order 105.

    (a) Show that G has a normal subgroup of order 5 or 7.

    (b) Show that G has a cyclic normal subgroup of order 35.

    (c) Show that the Sylow 5- and 7-subgroups of G are both normal.

    (d) Classify groups of order 105.

14. Let G be a group of order 66.

    (a) Show that G has a unique subgroup of order 11.

    (b) Show that G has a cyclic subgroup-call it H-of order 33.

(c) Show that G has a unique subgroup of order 3.

(d) How many elements of order ≤ 2 does the automorphism group of H have? (Justify your answer.)

(e) Any group of order 66 is isomorphic to one and only one of Z_{66}, $S_3 \times Z_{11}$, $Z_3 \times D_{22}$, or D_{66},

15. (a) Show that a simple group which has a subgroup of index $n > 2$ is isomorphic to a subgroup of the alternating group A_n.

    (b) What is the smallest index $[A_n : G]$ occurring for a subgroup $G \subsetneq A_n$? (Hint A_n is simple if $n \geq 5$)

    (c) Show that there is no simple group of order 112. (Hint: Use b)

    (d) Show that there is no simple group of order 120. Hint: Consider the normalizer of a Sylow 5-subgroup and use (b).

    (e) Is every group of order 120 solvable?

16. Let G be a finite group, let p be a prime divisor of the order $|G|$, and let P be a Sylow p-subgroup of G. Let $N_G(P)$ be the normalizer of P, and $C_G(P) \subset N_G(P)$ the centralizer of P.

    (a) Show that the index $[N_G(P) : C_G(P)]$ is the order of a subgroup of the automorphism group of P.

    (b) Show that p divides $[N_G(P) : C_G(P)]$ if, and only if, P is non-abelian. (Hint: P is abelian iff $C_G(P) \supset P$.)

    (c) Show that if P is cyclic and the $\gcd(|G|, p-1) = 1$ then $C_G(P) = N_G(P)$. (Hint: Consider conjugation morphism $N_G(P) \rightarrow \text{Aut}(P)$)

17. Show that P is abelian whenever $\text{Aut}(P)$ is cyclic. (Hint: A subgroup of cyclic group is cyclic)

18. Fill in the blanks with positive integers in all possible ways which make the resulting statement true. (Justify your answer.) There are exactly $---$ distinct abelian groups of order 2800 having exactly $---$ elements of order 28.