COMPLEXITY OF THE HIRONAKA RESOLUTION ALGORITHM

DIMA GRIGORIEV, PIERRE MILMAN, JAROSŁAW WŁODARCZYK

Abstract. Building upon works of Hironaka Bierstone-Milman, and Villamayor and the third author we give estimate for the complexity of the simplified Hironaka algoritm.

Contents

0. Introduction

In the present paper we discuss the complexity of the Hironaka theorem on resolution of singularities. Recall that approach to the problem of embedded resolution was originated by Hironaka (see [24]) and later developed and simplified by Bierstone-Milman (see [10]) and Villamayor (see [32]), and others

1. Formulation of the main theorems

All algebraic varieties in this paper are defined over a ground field of characteristic zero. The assumption of characteristic zero is only needed for the local existence of a hypersurface of maximal contact (Lemma 2.6.4).

We give a proof of the following Hironaka Theorems (see [24]):

(1) **Canonical Principalization**

Theorem 1.0.1. *Let $\mathcal{I}$ be a sheaf of ideals on a smooth algebraic variety X. There exists a principalization of $\mathcal{I}$, that is, a sequence*

$$X = X_0 \stackrel{\sigma_1}{\longleftarrow} X_1 \stackrel{\sigma_2}{\longleftarrow} X_2 \longleftarrow \ldots \longleftarrow X_i \longleftarrow \ldots \longleftarrow X_r = \widetilde{X}$$

of blow-ups $\sigma_i : X_{i-1} \leftarrow X_i$ of smooth centers $C_{i-1} \subset X_{i-1}$ such that

(a) *The exceptional divisor E_i of the induced morphism $\sigma^i = \sigma_1 \circ \ldots \circ \sigma_i : X_i \to X$ has only simple normal crossings and C_i has simple normal crossings with E_i.*

(b) *The total transform $\sigma^{r*}(\mathcal{I})$ is the ideal of a simple normal crossing divisor $\widetilde{E}$ which is a natural combination of the irreducible components of the divisor E_r.*

The morphism $(\widetilde{X}, \widetilde{\mathcal{I}}) \to (X, \mathcal{I})$ defined by the above principalization commutes with smooth morphisms and embeddings of ambient varieties. It is equivariant with respect to any group action not necessarily preserving the ground field K.

(2) **Weak-Strong Hironaka Embedded Desingularization**

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Theorem 1.0.2. *Let Y be a subvariety of a smooth variety X over a field of characteristic zero. There exists a sequence*

$$X_0 = X \xleftarrow{\sigma_1} X_1 \xleftarrow{\sigma_2} X_2 \longleftarrow \ldots \longleftarrow X_i \longleftarrow \ldots \longleftarrow X_r = \widetilde{X}$$

of blow-ups $\sigma_i : X_{i-1} \longleftarrow X_i$ of smooth centers $C_{i-1} \subset X_{i-1}$ such that

(a) *The exceptional divisor E_i of the induced morphism $\sigma^i = \sigma_1 \circ \ldots \circ \sigma_i : X_i \to X$ has only simple normal crossings and C_i has simple normal crossings with E_i.*
(b) *Let $Y_i \subset X_i$ be the strict transform of Y. All centers C_i are disjoint from the set $\mathrm{Reg}(Y) \subset Y_i$ of points where Y (not Y_i) is smooth (and are not necessarily contained in Y_i).*
(c) *The strict transform $\widetilde{Y} := Y_r$ of Y is smooth and has only simple normal crossings with the exceptional divisor E_r.*
(d) *The morphism $(X, Y) \leftarrow (\widetilde{X}, \widetilde{Y})$ defined by the embedded desingularization commutes with smooth morphisms and embeddings of ambient varieties. It is equivariant with respect to any group action not necessarily preserving K.*

(3) **Canonical Resolution of Singularities**

Theorem 1.0.3. *Let Y be an algebraic variety over a field of characteristic zero.*

There exists a canonical desingularization of Y that is a smooth variety $\widetilde{Y}$ together with a proper birational morphism $\mathrm{res}_Y : \widetilde{Y} \to Y$ which is functorial with respect to smooth morphisms. For any smooth morphism $\phi : Y' \to Y$ there is a natural lifting $\widetilde{\phi} : \widetilde{Y'} \to \widetilde{Y}$ which is a smooth morphism.

In particular $\mathrm{res}_Y : \widetilde{Y} \to Y$ is an isomorphism over the nonsingular part of Y. Moreover res_Y is equivariant with respect to any group action not necessarily preserving the ground field.

2. Preliminaries

To simplify our considerations we shall assume that the ground field is algebraically closed. At the end of the paper we deduce the theorem for an arbitrary ground field of characteristic zero.

2.1. **Resolution of marked ideals.**

For any sheaf of ideals $\mathcal{I}$ on a smooth variety X and any point $x \in X$ we denote by

$$\mathrm{ord}_x(\mathcal{I}) := \max\{i \mid \mathcal{I} \subset m_x^i\}$$

the *order* of $\mathcal{I}$ at x. (Here m_x denotes the maximal ideal of x.)

Definition 2.1.1. (Hironaka (see [24], [26]), Bierstone-Milman (see [10]),Villamayor (see [32])) A *marked ideal* (originally a *basic object* of Villamayor) is a collection $(X, \mathcal{I}, E, \mu)$, where X is a smooth variety, $\mathcal{I}$ is a sheaf of ideals on X, μ is a nonnegative integer and E is a totally ordered collection of divisors whose irreducible components are pairwise disjoint and all have multiplicity one. Moreover the irreducible components of divisors in E have simultaneously simple normal crossings.

Definition 2.1.2. (Hironaka ([24], [26]), Bierstone-Milman (see [10]),Villamayor (see [32])) By the *support* (originally *singular locus*) of $(X, \mathcal{I}, E, \mu)$ we mean

$$\mathrm{supp}(X, \mathcal{I}, E, \mu) := \{x \in X \mid \mathrm{ord}_x(\mathcal{I}) \geq \mu\}.$$

Remarks. (1) The ideals with assigned orders or functions with assigned multiplicities and their supports are key objects in proofs of Hironaka, Villamayor and Bierstone-Milman (see [24]. Hironaka introduced the notion of *idealistic exponent*. Then various modifications of this definition were considered in the papers of Bierstone-Milman (*presentation of invariant*) and Villamayor (*basic objects*). In our proof we stick to Villamayor's presentation of his basic objects (and their resolutions). Our marked ideals are essentially the same notion as basic objects. However because of some technical differences and in order to introduce more suggestive terminology we shall call them marked ideals.
(2) Sometimes for simplicity we shall represent marked ideals $(X, \mathcal{I}, E, \mu)$ as couples $(\mathcal{I}, \mu)$ or even ideals $\mathcal{I}$.
(3) For any sheaf of ideals $\mathcal{I}$ on X we have $\mathrm{supp}(\mathcal{I}, 1) = \mathrm{supp}(\mathcal{I})$.
(4) For any marked ideals $(\mathcal{I}, \mu)$ on X, $\mathrm{supp}(\mathcal{I}, \mu)$ is a closed subset of X (Lemma 2.5.2).

Definition 2.1.3. (Hironaka (see [24], [26]), Bierstone-Milman (see [10]),Villamayor (see [32])) By a *resolution* of $(X, \mathcal{I}, E, \mu)$ we mean a sequence of blow-ups $\sigma_i : X_i \to X_{i-1}$ of disjoint unions of smooth centers $C_{i-1} \subset X_{i-1}$,

$$X_0 = X \xleftarrow{\sigma_1} X_1 \xleftarrow{\sigma_2} X_2 \xleftarrow{\sigma_3} \ldots X_i \longleftarrow \ldots \xleftarrow{\sigma_r} X_r,$$

which defines a sequence of marked ideals $(X_i, \mathcal{I}_i, E_i, \mu)$ where

(1) $C_i \subset \mathrm{supp}(X_i, \mathcal{I}_i, E_i, \mu)$.
(2) C_i has simple normal crossings with E_i.
(3) $\mathcal{I}_i = \mathcal{I}(D_i)^{-\mu}\sigma_i^*(\mathcal{I}_{i-1})$, where $\mathcal{I}(D_i)$ is the ideal of the exceptional divisor D_i of σ_i.
(4) $E_i = \sigma_i^{\mathrm{c}}(E_{i-1}) \cup \{D_i\}$, where $\sigma_i^{\mathrm{c}}(E_{i-1})$ is the set of strict transforms of divisors in E_{i-1}.
(5) The order on $\sigma_i^{\mathrm{c}}(E_{i-1})$ is defined by the order on E_{i-1} while D_i is the maximal element of E_i.
(6) $\mathrm{supp}(X_r, \mathcal{I}_r, E_r, \mu) = \emptyset$.

Definition 2.1.4. The sequence of morphisms which are either isomorphisms or blow-ups satisfying conditions (1)-(5) is called a *multiple blow-up*. The number of morphisms in a multiple blow-up will be called its *length*.

Definition 2.1.5. An *extension* of a multiple blow-up (or a resolution) $(X_i)_{0\leq i\leq m}$ is a sequence $(X'_j)_{0\leq j\leq m'}$ of blow-ups and isomorphisms $X'_0 = X'_{j_0} = \ldots = X'_{j_1-1} \leftarrow X'_{j_1} = \ldots = X'_{j_2-1} \leftarrow \ldots X'_{j_m} = \ldots = X'_{m'}$, where $X'_{j_i} = X_i$.

Remarks. (1) The definition of extension arises naturally when we pass to open subsets of the considered ambient variety X.
(2) The notion of a *multiple blow-up* is analogous to the notions of *test* or *admissible* blow-ups considered by Hironaka, Bierstone-Milman and Villamayor.

2.2. **Transforms of marked ideal and controlled transforms of functions.** In the setting of the above definition we shall call

$$(\mathcal{I}_i, \mu) := \sigma_i^{\mathrm{c}}(\mathcal{I}_{i-1}, \mu)$$

a *transform of the marked ideal* or *controlled transform* of $(\mathcal{I}, \mu)$. It makes sense for a single blow-up in a multiple blow-up as well as for a multiple blow-up. Let $\sigma^i := \sigma_1 \circ \ldots \circ \sigma_i : X_i \to X$ be a composition of consecutive morphisms of a multiple blow-up. Then in the above setting

$$(\mathcal{I}_i, \mu) = \sigma^{i\mathrm{c}}(\mathcal{I}, \mu).$$

We shall also denote the controlled transform $\sigma^{i\mathrm{c}}(\mathcal{I}, \mu)$ by $(\mathcal{I}, \mu)_i$ or $[\mathcal{I}, \mu]_i$.

The controlled transform can also be defined for local sections $f \in \mathcal{I}(U)$. Let $\sigma : X \leftarrow X'$ be a blow-up of a smooth center $C \subset \mathrm{supp}(\mathcal{I}, \mu)$ defining transformation of marked ideals $\sigma^{\mathrm{c}}(\mathcal{I}, \mu) = (\mathcal{I}', \mu)$. Let $f \in \mathcal{I}(U)$ be a section of a sheaf of ideals. Let $U' \subseteq \sigma^{-1}(U)$ be an open subset for which the sheaf of ideals of the exceptional divisor is generated by a function y. The function

$$g = y^{-\mu}(f \circ \sigma) \in \mathcal{I}(U')$$

is a *controlled transform* of f on U' (defined up to an invertible function). As before we extend it to any multiple blow-up.

The following lemma shows that the notion of controlled transform is well defined.

Lemma 2.2.1. *Let $C \subset \mathrm{supp}(\mathcal{I}, \mu)$ be a smooth center of the blow-up $\sigma : X \leftarrow X'$ and let D denote the exceptional divisor. Let $\mathcal{I}_C$ denote the sheaf of ideals defined by C. Then*

(1) $\mathcal{I} \subset \mathcal{I}_C^{\mu}$.
(2) $\sigma^*(\mathcal{I}) \subset (\mathcal{I}_D)^{\mu}$.

Proof. (1) We can assume that the ambient variety X is affine. Let $u_1, \ldots, u_k$ be parameters generating $\mathcal{I}_C$ Suppose $f \in \mathcal{I} \setminus \mathcal{I}_C^{\mu}$. Then we can write $f = \sum_\alpha c_\alpha u^\alpha$, where either $|\alpha| \geq \mu$ or $|\alpha| < \mu$ and $c_\alpha \notin \mathcal{I}_C$. By the assumption there is α with $|\alpha| < \mu$ such that $c_\alpha \notin \mathcal{I}_C$. Take α with the smallest $|\alpha|$. There is a point $x \in C$ for which $c_\alpha(x) \neq 0$ and in the Taylor expansion of f at x there is a term $c_\alpha(x)u^\alpha$. Thus $\mathrm{ord}_x(\mathcal{I}) < \mu$. This contradicts to the assumption $C \subset \mathrm{supp}(\mathcal{I}, \mu)$.

(2) $\sigma^*(\mathcal{I}) \subset \sigma^*(\mathcal{I}_C)^{\mu} = (\mathcal{I}_D)^{\mu}$. □

2.3. **Hironaka resolution principle.** Our proof is based upon the following principle which can be traced back to Hironaka and was used by Villamayor in his simplification of Hironaka's algorithm:

(1) **(Canonical) Resolution of marked ideals** $(X, \mathcal{I}, E, \mu)$

$$\Downarrow$$

(2) **(Canonical) Principalization of the sheaves** $\mathcal{I}$ **on** X

$$\Downarrow$$

(3) **(Canonical) Weak Embedded Desingularization of subvarieties** $Y \subset X$

$$\Downarrow$$

(4) **(Canonical) Desingularization**

(1)⇒(2) It follows immediately from the definition that a resolution of $(X, \mathcal{I}, \emptyset, 1)$ determines a principalization of $\mathcal{I}$. Denote by $\sigma : X \leftarrow \widetilde{X}$ the morphism defined by a resolution of $(X, \mathcal{I}, \emptyset, 1)$. The controlled transform $(\widetilde{\mathcal{I}}, 1) := \sigma^{\mathrm{c}}(\mathcal{I}, 1)$ has the empty support. Consequently, $V(\widetilde{\mathcal{I}}) = \emptyset$, and thus $\widetilde{\mathcal{I}}$ is equal to the structural sheaf $\mathcal{O}_{\widetilde{X}}$. This implies that the full transform $\sigma^*(\mathcal{I})$ is principal and generated by the sheaf of ideal of a divisor whose components are the exceptional divisors. The actual process of desingularization is controlled by some invariant and is often achieved before $(X, \mathcal{I}, E, 1)$ has been resolved (Proposition **??**) .

(2)⇒(3) Let $Y \subset X$ be an irreducible subvariety. Assume there is a principalization of sheaves of ideals $\mathcal{I}_Y$ subject to conditions (a) and (b) from Theorem 1.0.1. Then in the course of the principalization of $\mathcal{I}_Y$ the strict transform Y_i of Y on some X_i is the center of a blow-up. At this stage Y_i is nonsingular and has simple normal crossing with the exceptional divisors. In the algorithm this moment is detected by some invariant.

(3)⇒(4) Every algebraic variety admits locally an embedding into an affine space. Thus we can show that the existence of canonical embedded desingularization independent of the embedding defines a canonical desingularization. The patching of local desingularizations is controlled by an invariant independent of embeddings into smooth ambient varieties, provided the dimensions of the ambient varieties are the same.

2.4. **Equivalence relation for marked ideals.** Let us introduce the following equivalence relation for marked ideals:

Definition 2.4.1. Let $(X, \mathcal{I}, E_{\mathcal{I}}, \mu_{\mathcal{I}})$ and $(X, \mathcal{J}, E_{\mathcal{J}}, \mu_{\mathcal{J}})$ be two marked ideals on the smooth variety X. Then

$$(X, \mathcal{I}, E_{\mathcal{I}}, \mu_{\mathcal{I}}) \simeq (X, \mathcal{J}, E_{\mathcal{J}}, \mu_{\mathcal{J}})$$

if

(1) $E_{\mathcal{I}} = E_{\mathcal{J}}$ and the orders on $E_{\mathcal{I}}$ and on $E_{\mathcal{J}}$ coincide.
(2) $\mathrm{supp}(X, \mathcal{I}, E_{\mathcal{I}}, \mu_{\mathcal{I}}) = \mathrm{supp}(X, \mathcal{J}, E_{\mathcal{J}}, \mu_{\mathcal{J}})$.
(3) All the multiple blow-ups $X_0 = X \xleftarrow{\sigma_1} X_1 \xleftarrow{\sigma_2} \ldots \longleftarrow X_i \xleftarrow{\sigma_r} \ldots \longleftarrow X_r$ of $(X, \mathcal{I}, E_{\mathcal{I}}, \mu_{\mathcal{I}})$ are exactly the multiple blow-ups of $(X, \mathcal{J}, E_{\mathcal{J}}, \mu_{\mathcal{J}})$ and moreover we have
$$\mathrm{supp}(X_i, \mathcal{I}_i, E_i, \mu_{\mathcal{I}}) = \mathrm{supp}(X_i, \mathcal{J}_i, E_i, \mu_{\mathcal{J}}).$$

Example 2.4.2. For any $k \in \mathbf{N}$, $(\mathcal{I}, \mu) \simeq (\mathcal{I}^k, k\mu)$.

Remark. The marked ideals considered in this paper satisfy a stronger equivalence condition: For any smooth morphisms $\phi : X' \to X$, $\phi^*(\mathcal{I}, \mu) \simeq \phi^*(\mathcal{J}, \mu)$. This condition will follow and is not added in the definition.

2.5. **Ideals of derivatives.** Ideals of derivatives were first introduced and studied in the resolution context by Giraud. Villamayor developed and applied this language to his *basic objects*.

Definition 2.5.1. (Giraud, Villamayor) Let $\mathcal{I}$ be a coherent sheaf of ideals on a smooth variety X. By the *first derivative* (originally *extension*) $\mathcal{D}(\mathcal{I})$ of $\mathcal{I}$ we mean the coherent sheaf of ideals generated by all functions $f \in \mathcal{I}$ with their first derivatives. Then the *i-th derivative* $\mathcal{D}^i(\mathcal{I})$ is defined to be $\mathcal{D}(\mathcal{D}^{i-1}(\mathcal{I}))$. If $(\mathcal{I}, \mu)$ is a marked ideal and $i \le \mu$ then we define

$$\mathcal{D}^i(\mathcal{I}, \mu) := (\mathcal{D}^i(\mathcal{I}), \mu - i).$$

Recall that on a smooth variety X there is a locally free sheaf of differentials $\Omega_{X/K}$ over K generated locally by $du_1, \ldots, du_n$ for a set of local parameters $u_1, \ldots, u_n$. The dual sheaf of derivations $\mathrm{Der}_K(\mathcal{O}_X)$ is locally generated by the derivations $\frac{\partial}{\partial u_i}$. Immediately from the definition we observe that $\mathcal{D}(\mathcal{I})$ is a coherent sheaf defined locally by generators f_j of $\mathcal{I}$ and all their partial derivatives $\frac{\partial f_j}{\partial u_i}$. We see by induction that $\mathcal{D}^i(\mathcal{I})$ is a coherent sheaf defined locally by the generators f_j of $\mathcal{I}$ and their derivatives $\frac{\partial^{|\alpha|} f_j}{\partial u^\alpha}$ for all multiindices $\alpha = (\alpha_1, \ldots, \alpha_n)$, where $|\alpha| := \alpha_1 + \ldots + \alpha_n \leq i$.

Remark. In characteristic p the partial derivatives $\frac{1}{\alpha!}\frac{\partial^{|\alpha|}}{\partial u^\alpha}$ (where $\alpha! = \alpha_1! \cdot \ldots \cdot \alpha_n!$) are well defined and are called the Hasse-Dieudonné derivatives. They should be used in the definition of the derivatives of marked ideals. One of the major sources of problems is that unlike in characteristic zero

$$\mathcal{D}^i(\mathcal{D}^j(\mathcal{I})) \subsetneq \mathcal{D}^{i+j}(\mathcal{I}).$$

Lemma 2.5.2. *(Giraud, Villamayor) For any $i \leq \mu - 1$,*

$$\mathrm{supp}(\mathcal{I}, \mu) = \mathrm{supp}(\mathcal{D}^i(\mathcal{I}), \mu - i)).$$

In particular $\mathrm{supp}(\mathcal{I}, \mu) = \mathrm{supp}(\mathcal{D}^{\mu-1}(\mathcal{I}), 1) = V(\mathcal{D}^{\mu-1}(\mathcal{I})$ *is a closed set.* □

We write $(\mathcal{I}, \mu) \subset (\mathcal{J}, \mu)$ if $\mathcal{I} \subset \mathcal{J}$.

Lemma 2.5.3. *(Giraud, Villamayor) Let $(\mathcal{I}, \mu)$ be a marked ideal and $C \subset \mathrm{supp}(\mathcal{I}, \mu)$ be a smooth center and $r \leq \mu$. Let $\sigma : X \leftarrow X'$ be a blow-up at C. Then*

$$\sigma^{\mathrm{c}}(\mathcal{D}^r(\mathcal{I}, \mu)) \subseteq \mathcal{D}^r(\sigma^{\mathrm{c}}(\mathcal{I}, \mu)).$$

Proof. See simple computations in

2.6. **Hypersurfaces of maximal contact.** The concept of the *hypersurfaces of maximal contact* is one of the key points of this proof. It was originated by Hironaka, Abhyankhar and Giraud and developed in the papers of Bierstone-Milman and Villamayor.

In our terminology we are looking for a smooth hypersurface containing the supports of marked ideals and whose strict transforms under multiple blow-ups contain the supports of the induced marked ideals. Existence of such hypersurfaces allows a reduction of the resolution problem to codimension 1.

First we introduce marked ideals which locally admit hypersurfaces of maximal contact.

Definition 2.6.1. (Villamayor (see [32])) We say that a marked ideal $(\mathcal{I}, \mu)$ is of *maximal order* (originally *simple basic object*) if $\max\{\mathrm{ord}_x(\mathcal{I}) \mid x \in X\} \leq \mu$ or equivalently $\mathcal{D}^\mu(\mathcal{I}) = \mathcal{O}_X$.

Lemma 2.6.2. *(Villamayor (see [32])) Let $(\mathcal{I}, \mu)$ be a marked ideal of maximal order and $C \subset \mathrm{supp}(\mathcal{I}, \mu)$ be a smooth center. Let $\sigma : X \leftarrow X'$ be a blow-up at $C \subset \mathrm{supp}(\mathcal{I}, \mu)$. Then $\sigma^{\mathrm{c}}(\mathcal{I}, \mu)$ is of maximal order.*

Proof. If $(\mathcal{I}, \mu)$ is a marked ideal of maximal order then $\mathcal{D}^\mu(\mathcal{I}) = \mathcal{O}_X$. Then by Lemma 2.5.3, $\mathcal{D}^\mu(\sigma^{\mathrm{c}}(\mathcal{I}, \mu)) \supset \sigma^{\mathrm{c}}(\mathcal{D}^\mu(\mathcal{I}), 0) = \mathcal{O}_X$. □

Lemma 2.6.3. *(Villamayor (see [32])) If $(\mathcal{I}, \mu)$ is a marked ideal of maximal order and $0 \leq i \leq \mu$ then $\mathcal{D}^i(\mathcal{I}, \mu)$ is of maximal order.*

Proof. $\mathcal{D}^{\mu-i}(\mathcal{D}^i(\mathcal{I}, \mu)) = \mathcal{D}^\mu(\mathcal{I}, \mu) = \mathcal{O}_X$. □

Lemma 2.6.4. *(Giraud) Let $(\mathcal{I}, \mu)$ be the marked ideal of maximal order. Let $\sigma : X \leftarrow X'$ be a blow-up at a smooth center $C \subsetneq \mathrm{supp}(\mathcal{I}, \mu)$. Let $u \in \mathcal{D}^{\mu-1}(\mathcal{I}, \mu)(U)$ be a function of multiplicity one on U, that is, for any $x \in V(u)$, $\mathrm{ord}_x(u) = 1$. In particular $\mathrm{supp}(\mathcal{I}, \mu) \cap U \subset V(u)$. Let $U' \subset \sigma^{-1}(U) \subset X'$ be an open set where the exceptional divisor is described by y. Let $u' := \sigma^{\mathrm{c}}(u) = y^{-1}\sigma^*(u)$ be the controlled transform of u. Then*

1. *$u' \in \mathcal{D}^{\mu-1}(\sigma^{\mathrm{c}}(\mathcal{I}_{|U'}, \mu))$.*
2. *u' is a function of multiplicity one on U'.*
3. *$V(u')$ is the restriction of the strict transform of $V(u)$ to U'.*

Proof. (1) $u' = \sigma^{\mathrm{c}}(u) = u/y \in \sigma^{\mathrm{c}}(\mathcal{D}^{\mu-1}(\mathcal{I})) \subset \mathcal{D}^{\mu-1}(\sigma^{\mathrm{c}}(\mathcal{I}))$.

(2) Since u was one of the local parameters describing the center of blow-ups, $u' = u/y$ is a parameter, that is, a function of order one.

(3) follows from (2). □

Definition 2.6.5. We shall call a function

$$u \in T(\mathcal{I})(U) := \mathcal{D}^{\mu-1}(\mathcal{I}(U))$$

of multiplicity one a *tangent direction* of $(\mathcal{I}, \mu)$ on U.

As a corollary from the above we obtain the following lemma:

Lemma 2.6.6. *(Giraud) Let $u \in T(\mathcal{I})(U)$ be a tangent direction of $(\mathcal{I}, \mu)$ on U. Then for any multiple blow-up (U_i) of $(\mathcal{I}_{|U}, \mu)$ all the supports of the induced marked ideals* $\mathrm{supp}(\mathcal{I}_i, \mu)$ *are contained in the strict transforms $V(u)_i$ of $V(u)$.* □

Remarks.
1. Tangent directions are functions defining locally hypersurfaces of maximal contact.
2. The main problem leading to complexity of the proofs is that of noncanonical choice of the tangent directions. We overcome this difficulty by introducing *homogenized ideals*.

Lemma 2.6.7. *(Villamayor) Let $(\mathcal{I}, \mu)$ be the marked ideal of maximal order whose support is of codimension* 1*. Then all codimension one components of* $\mathrm{supp}(\mathcal{I}, \mu)$ *are smooth and isolated. After the blow-up $\sigma : X \leftarrow X'$ at such a component $C \subset \mathrm{supp}(\mathcal{I}, \mu)$ the induced support* $\mathrm{supp}(\mathcal{I}', \mu)$ *does not intersect the exceptional divisor of σ.*

Proof. By the previous lemma there is a tangent direction $u \in \mathcal{D}^{\mu-1}(\mathcal{I})$ whose zero set is smooth and contains $\mathrm{supp}(\mathcal{I}, \mu)$. Then $\mathcal{D}^{\mu-1}(\mathcal{I}) = (u)$ and $\mathcal{I}$ is locally described as $\mathcal{I} = (u^\mu)$. The blow-up at the component C locally defined by u transforms $(\mathcal{I}, \mu)$ to $(\mathcal{I}', \mu)$, where $\sigma^*(\mathcal{I}) = y^\mu \mathcal{O}_X$, and $\mathcal{I}' = \sigma^{\mathrm{c}}(\mathcal{I}) = y^{-\mu}\sigma^*(\mathcal{I}) = \mathcal{O}_X$, where $y = u$ describes the exceptional divisor. □

Remark. Note that the blow-up of codimension one components is an isomorphism. However it defines a nontrivial transformation of marked ideals. In the actual desingularization process this kind of blow-up may occur for some marked ideals induced on subvarieties of ambient varieties. Though they define isomorphisms of those subvarieties they determine blow-ups of ambient varieties which are not isomorphisms.

2.7. **Arithmetical operations on marked ideals.** In this sections all marked ideals are defined for the smooth variety X and the same set of exceptional divisors E. Define the following operations of addition and multiplication of marked ideals:

1. $(\mathcal{I}, \mu_{\mathcal{I}}) + (\mathcal{J}, \mu_{\mathcal{J}}) := (\mathcal{I}^{\mu_{\mathcal{I}}} + \mathcal{J}^{\mu_{\mathcal{I}}}, \mu_{\mathcal{I}}\mu_{\mathcal{J}})$, or more generally (the operation of addition is not associative)
$$(\mathcal{I}_1, \mu_1) + \ldots + (\mathcal{I}_m, \mu_m) := (\mathcal{I}_1^{\mu_2 \cdot \ldots \cdot \mu_m} + \mathcal{I}_2^{\mu_1\mu_3 \cdot \ldots \cdot \mu_m} + \ldots + \mathcal{I}_m^{\mu_1 \ldots \mu_{k-1}}, \mu_1\mu_2 \ldots \mu_m).$$
2. $(\mathcal{I}, \mu_{\mathcal{I}}) \cdot (\mathcal{J}, \mu_{\mathcal{J}}) := (\mathcal{I} \cdot J, \mu_{\mathcal{I}} + \mu_{\mathcal{J}})$.

Lemma 2.7.1.
1. $\mathrm{supp}((\mathcal{I}_1, \mu_1) + \ldots + (\mathcal{I}_m, \mu_m)) = \mathrm{supp}(\mathcal{I}_1, \mu_1) \cap \ldots \cap \mathrm{supp}(\mathcal{I}_m, \mu_m)$. *Moreover multiple blow-ups (X_k) of $(\mathcal{I}_1, \mu_1) + \ldots + (\mathcal{I}_m, \mu_m)$ are exactly those which are simultaneous multiple blow-ups for all $(\mathcal{I}_j, \mu_j)$ and for any k we have the equality for the controlled transforms $(\mathcal{I}_j, \mu_{\mathcal{I}})_k$*
$$(\mathcal{I}_1, \mu_1)_k + \ldots + (\mathcal{I}_m, \mu_m)_k = [(\mathcal{I}_1, \mu_1) + \ldots + (\mathcal{I}_m, \mu_m)]_k$$
2. $$\mathrm{supp}(\mathcal{I}, \mu_{\mathcal{I}}) \cap \mathrm{supp}(\mathcal{J}, \mu_{\mathcal{J}}) \supseteq \mathrm{supp}((\mathcal{I}, \mu_{\mathcal{I}}) \cdot (\mathcal{J}, \mu_{\mathcal{J}})).$$
Moreover any simultaneous multiple blow-up X_i of both ideals $(\mathcal{I}, \mu_{\mathcal{I}})$ and $(\mathcal{J}, \mu_{\mathcal{J}})$ is a multiple blow-up for $(\mathcal{I}, \mu_{\mathcal{I}}) \cdot (\mathcal{J}, \mu_{\mathcal{J}})$, and for the controlled transforms $(\mathcal{I}_k, \mu_{\mathcal{I}})$ and $(\mathcal{J}_k, \mu_{\mathcal{J}})$ we have the equality
$$(\mathcal{I}_k, \mu_{\mathcal{I}}) \cdot (\mathcal{J}_k, \mu_{\mathcal{J}}) = [(\mathcal{I}, \mu_{\mathcal{I}}) \cdot (\mathcal{J}, \mu_{\mathcal{J}})]_k.$$

Proof.

2.8. **Homogenized ideals and tangent directions.** Let $(\mathcal{I}, \mu)$ be a marked ideal of maximal order. Set $T(\mathcal{I}) := \mathcal{D}^{\mu-1}\mathcal{I}$. By the *homogenized ideal* we mean

$$\mathcal{H}(\mathcal{I}, \mu) := (\mathcal{H}(\mathcal{I}), \mu) = (\mathcal{I} + \mathcal{D}\mathcal{I} \cdot T(\mathcal{I}) + \ldots + \mathcal{D}^i\mathcal{I} \cdot T(\mathcal{I})^i + \ldots + \mathcal{D}^{\mu-1}\mathcal{I} \cdot T(\mathcal{I})^{\mu-1}, \mu)$$

Remark. A homogenized ideal features two important properties:

1. It is equivalent to the given ideal.
2. It "looks the same" from all possible tangent directions.

By the first property we can use the homogenized ideal to construct resolution via the Giraud Lemma 2.6.6. By the second property such a construction does not depend on the choice of tangent directions.

Lemma 2.8.1. *Let $(\mathcal{I},\mu)$ be a marked ideal of maximal order. Then*

(1) $(\mathcal{I},\mu)\simeq(\mathcal{H}(\mathcal{I}),\mu)$.
(2) *For any multiple blow-up (X_k) of $(\mathcal{I},\mu)$,*
$(\mathcal{H}(\mathcal{I}),\mu)_k=(\mathcal{I},\mu)_k+[\mathcal{D}(\mathcal{I},\mu)]_k\cdot[(T(\mathcal{I}),1)]_k+\ldots[\mathcal{D}^{\mu-1}(\mathcal{I},\mu)]_k\cdot+[(T(\mathcal{I}),1)]_k^{\mu-1}$.

Although the following Lemmas 2.8.2 and 2.8.3 are used in this paper only in the case $E=\emptyset$ we formulate them in slightly more general versions.

Lemma 2.8.2. *Let $(X,\mathcal{I},E,\mu)$ be a marked ideal of maximal order. Assume there exist tangent directions $u,v\in T(\mathcal{I},\mu)_x=\mathcal{D}^{\mu-1}(\mathcal{I},\mu)_x$ at $x\in\operatorname{supp}(\mathcal{I},\mu)$ which are transversal to E. Then there exists an automorphism $\widehat{\phi}_{uv}$ of $\widehat{X}_x:=\operatorname{Spec}(\widehat{\mathcal{O}}_{x,X})$ such that*

(1) $\widehat{\phi}^*_{uv}(\mathcal{H}\widehat{\mathcal{I}})_x=(\mathcal{H}\widehat{\mathcal{I}})_x$.
(2) $\widehat{\phi}^*_{uv}(E)=E$.
(3) $\widehat{\phi}^*_{uv}(u)=v$.
(4) $\operatorname{supp}(\widehat{\mathcal{I}},\mu):=V(T(\widehat{\mathcal{I}},\mu))$ *is contained in the fixed point set of ϕ.*

Proof. (0) **Construction of the automorphism $\widehat{\phi}_{uv}$.**

Find parameters $u_2,\ldots,u_n$ transversal to u and v such that $u=u_1,u_2,\ldots,u_n$ and $v,u_2,\ldots,u_n$ form two sets of parameters at x and divisors in E are described by some parameters u_i where $i\geq 2$. Set

$$\widehat{\phi}_{uv}(u_1)=v,\quad \widehat{\phi}_{uv}(u_i)=u_i\quad\text{for}\quad i>1.$$

(1) Let $h:=v-u\in T(\mathcal{I})$. For any $f\in\widehat{\mathcal{I}}$,

$$\widehat{\phi}^*_{uv}(f)=f(u_1+h,u_2,\ldots,u_n)=f(u_1,\ldots,u_n)+\frac{\partial f}{\partial u_1}\cdot h+\frac{1}{2!}\frac{\partial^2 f}{\partial u_1^2}\cdot h^2+\ldots+\frac{1}{i!}\frac{\partial^i f}{\partial u_1^i}\cdot h^i+\ldots$$

The latter element belongs to

$$\widehat{\mathcal{I}}+\mathcal{D}\widehat{\mathcal{I}}\cdot\widehat{T(\mathcal{I})}+\ldots+\mathcal{D}^i\widehat{\mathcal{I}}\cdot\widehat{T(\mathcal{I})}^i+\ldots+\mathcal{D}^{\mu-1}\widehat{\mathcal{I}}\cdot\widehat{T(\mathcal{I})}^{\mu-1}=\mathcal{H}\widehat{\mathcal{I}}.$$

Hence $\widehat{\phi}^*_{uv}(\widehat{\mathcal{I}})\subset\mathcal{H}\widehat{\mathcal{I}}$. (2)(3) Follow from the construction.

(4) The fixed point set of $\widehat{\phi}^*_{uv}$ is defined by $u_i=\widehat{\phi}^*_{uv}(u_i)$, $i=1,\ldots,n$, that is, $h=0$. But $h\in\mathcal{D}^{\mu-1}(\mathcal{I})$ is 0 on $\operatorname{supp}(\mathcal{I},\mu)$. ☐

Lemma 2.8.3. *(**Glueing Lemma**) Let $(X,\mathcal{I},E,\mu)$ be a marked ideal of maximal order for which there exist tangent directions $u,v\in T(\mathcal{I},\mu)$ at $x\in\operatorname{supp}(\mathcal{I},\mu)$ which are transversal to E. Then there exist étale neighborhoods $\phi_u,\phi_v:\overline{X}\to X$ of $x=\phi_u(\overline{x})=\phi_v(\overline{x})\in X$, where $\overline{x}\in\overline{X}$, such that*

(1) $\phi^*_u(X,\mathcal{H}(\mathcal{I}),E,\mu)=\phi^*_v(X,\mathcal{H}(\mathcal{I}),E,\mu)$.
(2) $\phi^*_u(u)=\phi^*_v(v)$.
Set $(\overline{X},\overline{\mathcal{I}},\overline{E},\mu):=\phi^*_u(X,\mathcal{H}(\mathcal{I}),E,\mu)=\phi^*_v(X,\mathcal{H}(\mathcal{I},E,\mu))$.
(3) *For any* $\overline{y}\in\operatorname{supp}(\overline{X},\overline{\mathcal{I}},\overline{E},\mu)$, $\phi_u(\overline{y})=\phi_v(\overline{y})$.
(4) *For any multiple blow-up (X_i) of $(X,\mathcal{I},\emptyset,\mu)$ the induced multiple blow-ups $\phi^*_u(X_i)$ and $\phi^*_v(X_i)$ of $(\overline{X},\overline{\mathcal{I}},\overline{E},\mu)$ are the same (defined by the same centers). Set $(\overline{X}_i):=\phi^*_u(X_i)=\phi^*_v(X_i)$. For any $\overline{y}_i\in\operatorname{supp}(\overline{X}_i,\overline{\mathcal{I}}_i,\overline{E}_i,\mu)$ and the induced morphisms $\phi_{ui},\phi_{vi}:\overline{X}_i\to X_i$, $\phi_{ui}(\overline{y}_i)=\phi_{vi}(\overline{y}_i)$.*

Proof. (0) **Construction of étale neighborhoods $\phi_u,\phi_v:U\to X$.**

Let $U\subset X$ be an open subset for which there exist $u_2,\ldots,u_n$ which are transversal to u and v on U such that $u=u_1,u_2,\ldots,u_n$ and $v,u_2,\ldots,u_n$ form two sets of parameters on U and divisors in E are described by some u_i, where $i\geq 2$. Let $\mathbf{A}^n$ be the affine space with coordinates $x_1,\ldots,x_n$. Construct first étale morphisms $\phi_1,\phi_2:U\to\mathbf{A}^n$ with

$$\phi^*_1(x_i)=u_i\quad\text{for all } i\qquad\text{and}\quad \phi^*_2(x_1)=v,\quad \phi^*_2(x_i)=u_i\quad\text{for}\quad i>1.$$

Then

$$\overline{X}:=U\times_{\mathbf{A}^n}U$$

is a fiber product for the morphisms ϕ_1 and ϕ_2. The morphisms ϕ_u, ϕ_v are defined to be the natural projections $\phi_u, \phi_v : \overline{X} \to U$ such that $\phi_1\phi_u = \phi_2\phi_v$. Set

$$w_1 := \phi_u^*(u) = (\phi_1\phi_u)^*(x_1) = (\phi_2\phi_v)^*(x_1) = \phi_v^*(v),$$

$$w_i = \phi_u^*(u_i) = \phi_v^*(u_i) \quad \text{for } i \geq 2. \tag{5}$$

(1) Let $h := v - u$. By the above the morphisms ϕ_u and ϕ_v coincide on $\phi_u^{-1}(V(h)) = \phi_v^{-1}(V(h))$.

2.9. **Coefficient ideals and Giraud Lemma.** The idea of coefficient ideals was originated by Hironaka and then developed in papers of Villamayor and Bierstone-Milman. The following definition modifies and generalizes the definition of Villamayor.

Definition 2.9.1. Let $(\mathcal{I}, \mu)$ be a marked ideal of maximal order. By the *coefficient ideal* we mean

$$\mathcal{C}(\mathcal{I}, \mu) = \sum_{i=1}^{\mu} (\mathcal{D}^i\mathcal{I}, \mu - i).$$

Remark. The coefficient ideals $\mathcal{C}(\mathcal{I})$ feature two important properties.

(1) $\mathcal{C}(\mathcal{I})$ is equivalent to $\mathcal{I}$.
(2) The intersection of the support of $(\mathcal{I}, \mu)$ with any smooth subvariety S is the support of the restriction of $\mathcal{C}(\mathcal{I})$ to S:
$$\mathrm{supp}(\mathcal{I}) \cap S = \mathrm{supp}(\mathcal{C}(\mathcal{I})_{|S}).$$
Moreover this condition is persistent under relevant multiple blow-ups.

These properties allow one to control and modify the part of support of $(\mathcal{I}, \mu)$ contained in S by applying multiple blow-ups of $\mathcal{C}(\mathcal{I})_{|S}$.

Lemma 2.9.2. $\mathcal{C}(\mathcal{I}, \mu) \simeq (\mathcal{I}, \mu)$.

Proof. By Lemma 2.7.1 multiple blow-ups of $\mathcal{C}(\mathcal{I}, \mu)$ are simultaneous multiple blow-ups of $\mathcal{D}^i(\mathcal{I}, \mu)$ for $0 \leq i \leq \mu-1$. By Lemma **??** multiple blow-ups of $(\mathcal{I}, \mu)$ define the multiple blow-up of all $\mathcal{D}^i(\mathcal{I}, \mu)$. Thus multiple blow-ups of $(\mathcal{I}, \mu)$ and $\mathcal{C}(\mathcal{I}, \mu)$ are the same and $\mathrm{supp}(\mathcal{C}(\mathcal{I}, \mu))_k = \bigcap \mathrm{supp}(\mathcal{D}^i\mathcal{I}, \mu - i)_k = \mathrm{supp}(\mathcal{I}_k, \mu)$. □

Lemma 2.9.3. *Let $(X, \mathcal{I}, E, \mu)$ be a marked ideal of maximal order whose support $\mathrm{supp}(\mathcal{I}, \mu)$ does not contain a smooth subvariety S of X. Assume that S has only simple normal crossings with E. Then*

$$\mathrm{supp}(\mathcal{I}, \mu) \cap S = \mathrm{supp}(\mathcal{C}(\mathcal{I}, \mu)_{|S}).$$

Moreover let (X_i) be a multiple blow-up with centers C_i contained in the strict transforms $S_i \subset X_i$ of S. Then

(1) *The restrictions $\sigma_{i|S_i} : S_i \to S_{i-1}$ of the morphisms $\sigma_i : X_i \to X_{i-1}$ define a multiple blow-up (S_i) of $\mathcal{C}(\mathcal{I}, \mu)_{|S}$.*
(2) $\mathrm{supp}(\mathcal{I}_i, \mu) \cap S_i = \mathrm{supp}[\mathcal{C}(\mathcal{I}, \mu)_{|S}]_i$.
(3) *Every multiple blow-up (S_i) of $\mathcal{C}(\mathcal{I}, \mu)_{|S}$ defines a multiple blow-up (X_i) of $(\mathcal{I}, \mu)$ with centers C_i contained in the strict transforms $S_i \subset X_i$ of $S \subset X$.*

Proof. By Lemmas 2.9.2 and **??**, $\mathrm{supp}(\mathcal{I}, \mu) \cap S = \mathrm{supp}(\mathcal{C}(\mathcal{I}, \mu)) \cap S \subseteq \mathrm{supp}(\mathcal{C}(\mathcal{I}, \mu)_{|S})$.

Let $x_1, \ldots, x_k, y_1, \ldots, y_{n-k}$ be local parameters at x such that $\{x_1 = 0, \ldots, x_k = 0\}$ describes S. Then any function $f \in \mathcal{I}$ can be written as

$$f = \sum c_{\alpha f}(y) x^\alpha,$$

where $c_{\alpha f}(y)$ are formal power series in y_i.

Now $x \in \mathrm{supp}(\mathcal{I}, \mu) \cap S$ iff $\mathrm{ord}_x(c_\alpha) \geq \mu - |\alpha|$ for all $f \in \mathcal{I}$ and $|\alpha| \leq \mu$. Note that

$$c_{\alpha f|S} = \left(\frac{1}{\alpha!}\frac{\partial^{|\alpha|}(f)}{\partial x^\alpha}\right)_{|S} \in \mathcal{D}^{|\alpha|}(\mathcal{I})_{|S}$$

and consequently $\mathrm{supp}(\mathcal{I}, \mu) \cap S = \bigcap_{f \in \mathcal{I}, |\alpha| \leq \mu} \mathrm{supp}(c_{\alpha f|S}, \mu - |\alpha|) \supset \mathrm{supp}(\mathcal{C}(\mathcal{I}, \mu)_{|S})$.

Assume that all multiple blow-ups of $(\mathcal{I}, \mu)$ of length k with centers $C_i \subset S_i$ are defined by multiple blow-ups of $\mathcal{C}(\mathcal{I}, \mu)_{|S}$ and moreover for $i \leq k$,

$$\mathrm{supp}(\mathcal{I}_i, \mu) \cap S_i = \mathrm{supp}[\mathcal{C}(\mathcal{I}, \mu)_{|S}]_i.$$

For any $f \in \mathcal{I}$ define $f = f_0 \in \mathcal{I}$ and $f_{i+1} = \sigma_i^{\mathrm{c}}(f_i) = y_i^{-\mu}\sigma^*(f_i) \in \mathcal{I}_{i+1}$. Assume moreover that for any $f \in \mathcal{I}$,

$$f_k = \sum c_{\alpha f k}(y)x^\alpha,$$

where $c_{\alpha f k|S_k} \in (\sigma^k_{|S_k})^{\mathrm{c}}(\mathcal{D}^{\mu-|\alpha|}(\mathcal{I})_{|S})$. Consider the effect of the blow-up of C_k at a point x' in the strict transform of $S_{k+1} \subset X_{k+1}$. By Lemmas 2.9.2 and **??**,

$$\mathrm{supp}(\mathcal{I}_{k+1}, \mu) \cap S_{k+1} = \mathrm{supp}[\mathcal{C}(\mathcal{I}, \mu)]_{k+1} \cap S_{k+1} \subseteq \mathrm{supp}[\mathcal{C}(\mathcal{I}, \mu)]_{k+1|S_{k+1}} = \mathrm{supp}[\mathcal{C}(\mathcal{I}, \mu)_{|S}]_{k+1}$$

Let $x_1, \ldots, x_k$ describe the subvariety S_k of X_k. We can find coordinates $x_1, \ldots, x_k, y_1, \ldots, y_{n-k}$, by taking if necessary linear combinations of $y_1, \ldots, y_{n-k}$, such that the center of the blow-up is described by $x_1, \ldots, x_k, y_1, \ldots, y_m$ and the coordinates at x' are given by

$$x_1' = x_1/y_m, \ldots, x_k' = x_k/y_m, y_1' = y_1/y_m, \ldots, y_m' = y_m, y_{m+1}' = y_{m+1}, \ldots, y_n' = y_n.$$

Note that replacing $y_1, \ldots, y_{n-k}$ with their linear combinations does not modify the form $f_k = \sum c_{\alpha f k}(y)x^\alpha$. Then the function $f_{k+1} = \sigma^{\mathrm{c}}(f_k)$ can be written as

$$f_{k+1} = \sum c_{\alpha f k+1}(y){x'}^\alpha,$$

where $c_{\alpha f k+1} = y_m^{-\mu+|\alpha|}\sigma_{k+1}^*(c_{\alpha f k})$. Thus

$$c_{\alpha f k+1|S_{k+1}} = (\sigma_{k+1|S_{k+1}})^{\mathrm{c}}(c_{\alpha f k|S_k}) \in (\sigma^{k+1}_{|S_{k+1}})^{\mathrm{c}}(\mathcal{D}^{\mu-|\alpha|}(\mathcal{I})_{|S}) = (\sigma^{k+1})^{\mathrm{c}}(\mathcal{D}^{\mu-|\alpha|}(\mathcal{I}))_{|S_{k+1}}$$

and consequently

$$\mathrm{supp}(\mathcal{I}_{k+1}, \mu) \cap S_{k+1} = \bigcap_{f \in \mathcal{I}, |\alpha| \le \mu} \mathrm{supp}(c_{\alpha f k+1|S_{k+1}}, \mu - |\alpha|) \supseteq \mathrm{supp}[\mathcal{C}(\mathcal{I}, \mu)_{|S}]_{k+1} = \mathrm{supp}(\mathcal{C}(\mathcal{I}, \mu)_{k+1})_{|S_{k+1}}.$$

□

A direct consequence of the above lemma is the following result:

Lemma 2.9.4. *Let $(X, \mathcal{I}, E, \mu)$ be a marked ideal of maximal order whose support* $\mathrm{supp}(\mathcal{I}, \mu)$ *does not contain a smooth subvariety S of X. Assume that S has only simple normal crossings with E. Let (X_i) be its multiple blow-up such that all centers C_i are either contained in the strict transforms $S_i \subset X_i$ of S or are disjoint from them. Then the restrictions $\sigma_{i|S_i} : S_i \to S_{i-1}$ of the morphisms $\sigma_i : X_i \to X_{i-1}$ define a multiple blow-up (S_i) of $\mathcal{C}(\mathcal{I}, \mu)_{|S}$ and*

$$\mathrm{supp}(\mathcal{I}_i, \mu) \cap S_i = \mathrm{supp}[\mathcal{C}(\mathcal{I}, \mu)_{|S}]_i.$$

As a simple consequence of the Lemma 2.9.3 we formulate the following refinement of the Giraud Lemma.

Lemma 2.9.5. *Let $(X, \mathcal{I}, \emptyset, \mu)$ be a marked ideal of maximal order whose support* $\mathrm{supp}(\mathcal{I}, \mu)$ *has codimension at least 2 at some point x. Let $U \ni x$ be an open subset for which there is a tangent direction $u \in T(\mathcal{I})$ and such that* $\mathrm{supp}(\mathcal{I}, \mu) \cap U$ *is of codimension 2. Let $V(u)$ be the regular subscheme of U defined by u. Then for any multiple blow-up X_i of X,*

1. $\mathrm{supp}(\mathcal{I}_i, \mu)$ *is contained in the strict transform $V(u)_i$ of $V(u)$ as a proper subset.*
2. *The sequence $(V(u)_i)$ is a multiple blow-up of $\mathcal{C}(\mathcal{I}, \mu)_{|V(u)}$.*
3. $\mathrm{supp}(\mathcal{I}_i, \mu) \cap V(u)_i = \mathrm{supp}[\mathcal{C}(\mathcal{I}, \mu)_{|V(u)}]_i$.
4. *Every multiple blow-up $(V(u)_i)$ of $\mathcal{C}(\mathcal{I}, \mu)_{|V(u)}$ defines a multiple blow-up (X_i) of $(\mathcal{I}, \mu)$.*

□

Lemma 2.9.6. *Let $\phi : X' \to X$ be a smooth morphism of smooth varieties and let $(X, \mathcal{I}, \emptyset, \mu)$ be a marked ideal. Then*

$$\phi^*(\mathcal{C}(\mathcal{I})) = \mathcal{C}(\phi^*(\mathcal{I})).$$

Proof. A direct consequence of Lemma **??**. □

3. Resolution algorithm

The presentation of the following Hironaka resolution algorithm builds upon Villamayor's and Bierstone-Milman's proofs.

Inductive assumption For any marked ideal $(X, \mathcal{I}, E, \mu)$ such that $\mathcal{I} \neq 0$ there is an associated resolution $(X_i)_{0 \leq i \leq m_X}$, called canonical, satisfying the following conditions:

Proof. Induction on the dimension of X. If X is 0-dimensional, $\mathcal{I} \neq 0$ and $\mu > 0$ then $\operatorname{supp}(X, \mathcal{I}, \mu) = \emptyset$ and all resolutions are trivial.

Step 1 Resolving a marked ideal $(X, \mathcal{J}, E, \mu)$ of maximal order.

Before we start our resolution algorithm for the marked ideal $(\mathcal{J}, \mu)$ of maximal order we shall replace it with the equivalent homogenized ideal $\mathcal{C}(\mathcal{H}(\mathcal{J}, \mu))$. Resolving the ideal $\mathcal{C}(\mathcal{H}(\mathcal{J}, \mu))$ defines a resolution of $(\mathcal{J}, \mu)$ at this step. To simplify notation we shall denote $\mathcal{C}(\mathcal{H}(\mathcal{J}, \mu))$ by $(\overline{\mathcal{J}}, \overline{\mu})$.

Step 1a Reduction to the nonboundary case. Moving $\operatorname{supp}(\overline{\mathcal{J}}, \overline{\mu})$ and H^s_α apart . For any multiple blow-up (X_i) of $(X, \overline{\mathcal{J}}, E, \overline{\mu})$ we shall identify (for simplicity) strict transforms of E on X_i with E.

For any $x \in X_i$, let $s(x)$ denote the number of divisors in E through x and set

$$s_i = \max\{s(x) \mid x \in \operatorname{supp}(\overline{\mathcal{J}}_i)\}.$$

Let $s = s_0$. By assumption the intersections of any $s > s_0$ components of the exceptional divisors are disjoint from $\operatorname{supp}(\overline{\mathcal{J}}, \overline{\mu})$. Each intersection of divisors in E is locally defined by intersection of some irreducible components of these divisors. Find all intersections $H^s_\alpha, \alpha \in A$, of s irreducible components of divisors E such that $\operatorname{supp}(\overline{\mathcal{J}}, \overline{\mu}) \cap H^s_\alpha \neq \emptyset$. By the maximality of s, the supports $\operatorname{supp}(\overline{\mathcal{J}}_{|H^s_\alpha}) \subset H^s_\alpha$ are disjoint from $H^s_{\alpha'}$, where $\alpha' \neq \alpha$.

Construct the canonical resolutions of $\overline{\mathcal{J}}_{p|H^s_\alpha}$. By Lemma 2.9.3 each such resolution defines a multiple blow-up of $(\overline{\mathcal{J}}_p, \overline{\mu})$ (and of $(\overline{\mathcal{J}}, \overline{\mu})$) . Since the supports $\operatorname{supp}(\overline{\mathcal{J}}_{|H^s_\alpha}) \subset H^s_\alpha$ are disjoint from $H^s_{\alpha'}$, where $\alpha' \neq \alpha$, these resolutions glue to a unique multiple blow-up $(X_i)_{i \leq j_1}$ of $(\overline{\mathcal{J}}, \overline{\mu})$ such that $s_{j_1} < s$.

Note that by the maximality condition for any H^s_α the irreducible components of the centers are contained in H^s_α or are disjoint from them. Therefore by Lemma 2.9.4,

$$\operatorname{supp}(\overline{\mathcal{J}}_i, \overline{\mu})_{|H^s_\alpha} = \operatorname{supp}(\overline{\mathcal{J}}_i, \overline{\mu}) \cap H^s_\alpha.$$

By applying this multiple blow-up we create a marked ideal $(\overline{\mathcal{J}}_{j_1}, \overline{\mu})$ with support disjoint from all H^s_α. Summarizing the above we construct a multiple blow-up $(X_i)_{0 \leq i \leq j_1}$ subject to the conditions:

Conclusion of the algorithm in Step 1a. After performing the blow-ups in Steps 1aa and 1ab for the marked ideal $(\overline{\mathcal{J}}, \overline{\mu})$ we arrive at a marked ideal $(\overline{\mathcal{J}}_{j_1}, \overline{\mu})$ with $s_{j_1} < s_0$. Now we put $s = s_{j_1}$ and repeat the procedure of Steps 1a for $(\overline{\mathcal{J}}_{j_1}, \overline{\mu})$. Note that any $H^s_{\alpha j_1}$ on X_{j_1} is the strict transform of some intersection $H^{s_{j_1}}_\alpha$ of $s = s_{j_1}$ divisors in E on X. Moreover by the maximality condition for all s_i, where $i \leq j_1$ and $\alpha \neq \alpha'$, the set $\operatorname{supp}(\overline{\mathcal{J}}_i, \overline{\mu}) \cap H^{s_i}_{\alpha' i}$ is either disjoint from $H^{s_{j_1}}_{\alpha i}$ or contained in it. Thus for $0 \leq i \leq j_1$, all centers C_i have components either contained in $H^{s_{j_1}}_{\alpha i} = H^s_{\alpha i}$ or disjoint from them and by Lemma 2.9.4,

$$\operatorname{supp}(\overline{\mathcal{J}}_i, \overline{\mu})_{|H^s_{\alpha i}} = \operatorname{supp}(\overline{\mathcal{J}}_i, \overline{\mu}) \cap H^s_{\alpha i}.$$

Moreover if we repeat the procedure in Steps 1a the above property will still be satisfied until either $(\overline{\mathcal{J}}_i, \overline{\mu})_{|H^s_\alpha}$ are resolved as in Step 1ab or H^s_α disappear as in Step 1aa.

We continue the above process till $s_{j_k} = s_r = 0$. Then $(X_j)_{0 \leq j \leq r}$ is a multiple blow-up of $(X, \overline{\mathcal{J}}, E, \overline{\mu})$ such that $\operatorname{supp}(\overline{\mathcal{J}}_r, \overline{\mu})$ does not intersect any divisor in E. Therefore $(X_j)_{0 \leq j \leq r}$ and further longer multiple blow-ups $(X_j)_{0 \leq j \leq r_0}$ for any $r \leq r_0$ can be considered as multiple blow-ups of $(X, \overline{\mathcal{J}}, \emptyset, \overline{\mu})$ since starting from X_r the strict transforms of E play no further role in the resolution process since they do not intersect $\operatorname{supp}(\overline{\mathcal{J}}_j, \overline{\mu})$ for $j \geq r$.

Step 1b. Nonboundary case

Let $(X_j)_{0 \leq j \leq r}$ be the multiple blow-up of $(X, \overline{\mathcal{J}}, \emptyset, \overline{\mu})$ defined in Step 1a.

If $\operatorname{supp}(\overline{\mathcal{J}}_r, \overline{\mu})$ is of codimension 1 then by Lemma 2.6.7 all its codimension 1 components are smooth and disjoint from the other components of $\operatorname{supp}(\overline{\mathcal{J}}_r, \overline{\mu})$. These components are strict transforms of the codimension 1 components of $\operatorname{supp}(\overline{\mathcal{J}}, \overline{\mu})$. Moreover the irreducible components of the centers of blow-ups were either contained in the strict transforms or disjoint from them. Therefore E_r will be transversal to all the codimension 1 components. Let $\mathrm{codim}(1)(\mathrm{supp}(\overline{\mathcal{J}}_i,\overline{\mu}))$ be the union of all components of $\mathrm{supp}(\overline{\mathcal{J}}_i,\overline{\mu}))$ of codimension 1. This definition, as we see below, is devised so as to ensure that all codimension 1 components will be blown up first.

By Lemma 2.6.7 blowing up the components reduces the situation to the case when $\mathrm{supp}(\overline{\mathcal{J}},\overline{\mu})$ is of codimension ≥ 2.

For any $x \in \mathrm{supp}(\overline{\mathcal{J}},\overline{\mu}) \setminus \mathrm{codim}(1)(\mathrm{supp}(\overline{\mathcal{J}},\overline{\mu}) \subset X$ find a tangent direction $u \in \mathcal{D}^{\overline{\mu}-1}(\overline{\mathcal{J}})$ on some neighborhood U_u of x. Then $V(u) \subset U_u$ is a hypersurface of maximal contact. By the quasicompactness of X we can assume that the covering defined by U_u is finite. Let $U_{ui} \subset X_i$ be the inverse image of U_u and let $V(u)_i \subset U_u$ denote the strict transform of $V(u)$. By Lemma 2.9.5, $(V(u)_i)_{0\leq i\leq p}$ is a multiple blow-up of $(V(u),\overline{\mathcal{J}}_{|V(u)},\emptyset,\overline{\mu})$. In particular the induced marked ideal for $i=p$ is equal to

$$\overline{\mathcal{J}}_{p|V(u)_p} = (V(u)_p,\overline{\mathcal{J}}_{p|V(u)_p},(E_p\setminus E)_{|V(u)_p},\overline{\mu}).$$

Construct the canonical resolution of $(V(u)_i)_{p\leq i\leq m_u}$ of the marked ideal $\overline{\mathcal{J}}_{p|V(u)_p}$. Then the sequence $(V(u)_i)_{0\leq i\leq m_u}$ is a resolution of $(V(u),\overline{\mathcal{J}}_{|V(u)},\emptyset,\overline{\mu})$ which defines, by Lemma 2.9.5, a resolution $(U_{ui})_{0\leq i\leq m_u}$ of $(U_u,\overline{\mathcal{J}}_{|U_u},\emptyset,\overline{\mu})$. Moreover both resolutions are related by the property

$$\mathrm{supp}(\overline{\mathcal{J}}_{i|U_{ui}}) = \mathrm{supp}(\overline{\mathcal{J}}_{i|V(u)_i}).$$

We shall construct the resolution of $(X,\overline{\mathcal{J}},\emptyset,\overline{\mu})$ by patching together extensions of the local resolutions $(U_{ui})_{0\leq i\leq m_u}$.

Let $x\in \mathrm{supp}(\overline{\mathcal{J}}_p,\overline{\mu})\cap U_{up}\cap U_{vp}$. By Glueing Lemma 2.8.3 for any two different tangent directions u and v we find étale neighborhoods $\phi_u,\phi_v : U^{uv}\to U := U_u\cap U_v$ and their liftings $\phi_{pu},\phi_{pv} : U^{uv}_p \to U_p := U_{up}\cap U_{vp}$ such that

(1) $X^{uv}_p := (\phi_{pu})^{-1}(V(u)_p) = (\phi_{pv})^{-1}(V(v)_p)$.
(2) $(U^{uv}_p,\overline{\mathcal{J}}^{uv}_p,E^{uv}_p,\overline{\mu}) := (\phi_{pu})^*(U_p,\overline{\mathcal{J}}_p,E_p,\overline{\mu}) = (\phi_{pv})^*(U_p,\overline{\mathcal{J}}_p,E_p,\overline{\mu})$.
(3) There exists $y\in \mathrm{supp}(U^{uv}_p,\overline{\mathcal{J}}^{uv}_p,E^{uv}_p,\overline{\mu})$ such that $\phi_{pu}(\overline{x}) = \phi_{pv}(\overline{x})$.

Note that for any tangent direction u, either $C_p\cap U_{up}$ defines the first blow-of the canonical resolution of $(V(u)_p,\overline{\mathcal{J}}_{p|V(u)_p},E_{p|V(u)_p},\overline{\mu})$ or $C_p\cap U_{up}=\emptyset$ and the blow-up of C_p does not change $V(u)_p\subset U_{up}$.

Blowing up C_p defines X_{p+1} and we are in a position to construct the invariants on X_{p+1} and define the center of the blow-up $C_{p+1}\subset X_{p+1}$ as before.

By repeating the same reasoning for $j=p+1,\ldots,m$ we construct the resolution $(X_i)_{p\leq i\leq m}$ of $(X_p,\overline{\mathcal{J}}_p,E_p\setminus E,\overline{\mu})$ satisfying the following properties.

(1) For any u, the restriction of $(X_i)_{p\leq i\leq m}$ to $(V(u)_i)_{p\leq i\leq m}$ is an extension of the canonical resolution of $(V(u)_p,\overline{\mathcal{J}}_{p|V(u)_p},E_{p|V(u)_p},\overline{\mu})$.
(2) $\mathrm{supp}(\overline{\mathcal{J}}_m,\overline{\mu})=\emptyset$.

The resolution $(X_i)_{p\leq i\leq m}$ of $(X_p,\overline{\mathcal{J}}_p,E_p\setminus E,\overline{\mu})$ defines the resolution $(X_i)_{0\leq i\leq m}$ of $(X,\overline{\mathcal{J}},\emptyset,\overline{\mu})$ and of $(X,\overline{\mathcal{J}},E,\overline{\mu})$.

In Step 1b all points $x\in\mathrm{supp}(\overline{\mathcal{J}}_i,\overline{\mu})$ with $s(x)=0$ were assigned the invariants $\overline{\mathrm{inv}}$, ν and ρ. They are upper semicontinuous by the inductive assumption.

Step 2. Resolving of marked ideals $(X,\mathcal{I},E,\mu)$.

For any marked ideal $(X,\mathcal{I},E,\mu)$ write

$$I = \mathcal{M}(\mathcal{I})\mathcal{N}(\mathcal{I}),$$

where $\mathcal{M}(\mathcal{I})$ is the *monomial part* of $\mathcal{I}$, that is, the product of the principal ideals defining the irreducible components of the divisors in E, and $\mathcal{N}(\mathcal{I})$ is a *nonmonomial part* which is not divisible by any ideal of a divisor in E. Let

$$\mathrm{ord}_{\mathcal{N}(\mathcal{I})} := \max\{\mathrm{ord}_x(\mathcal{N}(\mathcal{I}))\mid x\in\mathrm{supp}(\mathcal{I},\mu)\}.$$

Definition 3.0.7. (Hironaka, Bierstone-Milman,Villamayor, Encinas-Hauser) By the *companion ideal* of $(\mathcal{I},\mu)$ where $I=\mathcal{N}(\mathcal{I})\mathcal{M}(\mathcal{I})$ we mean the marked ideal of maximal order

$$O(\mathcal{I},\mu) = \begin{cases} (\mathcal{N}(\mathcal{I}),\mathrm{ord}_{\mathcal{N}(\mathcal{I})}) \quad + \quad (\mathcal{M}(\mathcal{I}),\mu-\mathrm{ord}_{\mathcal{N}(\mathcal{I})}) & \text{if } \mathrm{ord}_{\mathcal{N}(\mathcal{I})}<\mu, \\ (\mathcal{N}(\mathcal{I}),\mathrm{ord}_{\mathcal{N}(\mathcal{I})}) & \text{if } \mathrm{ord}_{\mathcal{N}(\mathcal{I})}\geq\mu. \end{cases}$$

In particular $O(\mathcal{I},\mu)=(\mathcal{I},\mu)$ for ideals $(\mathcal{I},\mu)$ of maximal order.

Step 2a. **Reduction to the monomial case by using companion ideals**

By Step 1 we can resolve the marked ideal of maximal order $(\mathcal{J},\mu_{\mathcal{J}}):=O(\mathcal{I},\mu)$ using the invariant $\overline{\mathrm{inv}}_{O(\mathcal{I},\mu)}$. By Lemma 2.7.1, for any multiple blow-up of $O(\mathcal{I},\mu)$,

$$\begin{aligned}\mathrm{supp}(O(\mathcal{I},\mu))_i = \mathrm{supp}[\mathcal{N}(\mathcal{I}),\mathrm{ord}_{\mathcal{N}(\mathcal{I})}]_i \ \cap\ \mathrm{supp}[M(\mathcal{I}),\mu-\mathrm{ord}_{\mathcal{N}(H\mathcal{I})}]_i =\\ \mathrm{supp}[\mathcal{N}(\mathcal{I}),\mathrm{ord}_{\mathcal{N}(\mathcal{I})}]_i \ \cap\ \mathrm{supp}(\mathcal{I}_i,\mu).\end{aligned}$$

Consequently, such a resolution leads to the ideal $(\mathcal{I}_{r_1},\mu)$ such that $\mathrm{ord}_{\mathcal{N}(\mathcal{I}_{r_1})}<\mathrm{ord}_{\mathcal{N}(\mathcal{I})}$. This resolution is controlled by the invariants inv, ν and ρ defined for all $x\in\mathrm{supp}(\mathcal{N}(\mathcal{I}),\mathrm{ord}_{\mathcal{N}(\mathcal{I})})\cap\mathrm{supp}(\mathcal{I}_i,\mu)_i$,

$$\mathrm{inv}(x)=\left(\frac{\mathrm{ord}_{\mathcal{N}(\mathcal{I})}}{\mu},\overline{\mathrm{inv}}_{O(\mathcal{I},\mu)}(x)\right),\quad \nu(x)=\nu_{O(\mathcal{I},\mu)}(x),\quad \rho(x)=\rho_{O(\mathcal{I},\mu)}(x).$$

Then we repeat the procedure for $(\mathcal{I}_{r_1},\mu)$. We find marked ideals $(\mathcal{I}_{r_0},\mu)=(\mathcal{I},\mu),(\mathcal{I}_{r_1},\mu),\ldots,(\mathcal{I}_{r_m},\mu)$ such that $\mathrm{ord}_{\mathcal{N}(\mathcal{I}_0)}>\mathrm{ord}_{\mathcal{N}(\mathcal{I}_{r_1})}>\ldots>\mathrm{ord}_{\mathcal{N}(\mathcal{I}_{r_m})}$. The procedure terminates after a finite number of steps when we arrive at the ideal $(\mathcal{I}_{r_m},\mu)$ with $\mathrm{ord}_{\mathcal{N}(\mathcal{I}_{r_m})}=0$ or with $\mathrm{supp}(\mathcal{I}_{r_m},\mu)=\emptyset$. In the second case we get the resolution. In the first case $\mathcal{I}_{r_m}=\mathcal{M}(\mathcal{I}_{r_m})$ is monomial.

In Step 2a all points $x\in\mathrm{supp}(\mathcal{I},\mu)$ for which $\mathrm{ord}_c(\mathcal{I})\neq 0$ were assigned the invariants inv,μ,ρ. They are upper semicontinuous by the semicontinuouity of ord_x and of the invariants $\overline{\mathrm{inv}},\mu,\rho$ for the marked ideals of maximal order.

Step 2b. **Monomial case $\mathcal{I}=\mathcal{M}(\mathcal{I})$.**

Define the invariants

$$\mathrm{inv}(x)=(0,\ldots,0,\ldots),\quad \nu(x)=\frac{\mathrm{ord}_x(\mathcal{I})}{\mu}.$$

Let $x_1,\ldots,x_k$ define equations of the components $D^x_1,\ldots,D^x_k\in E$ through $x\in\mathrm{supp}(X,\mathcal{I},E,\mu)$ and $\mathcal{I}$ be generated by the monomial $x^{a_1,\ldots,a_k}$ at x. In particular $\nu(x)=\frac{a_1+\ldots+a_k}{\mu}$.

Let $\rho(x)=\{D_{i_1},\ldots,D_{i_l}\}\in\mathrm{Sub}(E)$ be the maximal subset satisfying the properties

(1) $a_{i_1}+\ldots+a_{i_l}\geq\mu$.
(2) For any $j=1,\ldots,l$, $a_{i_1}+\ldots+\check{a}_{i_j}+\ldots+a_{i_l}<\mu$.

Let $R(x)$ denote the subsets in $\mathrm{Sub}(E)$ satisfying the properties (1) and (2). The maximal components of the $\mathrm{supp}(\mathcal{I},\mu)$ through x are described by the intersections $\bigcap_{D\in A}D$ where $A\in R(x)$. The maximal locus of ρ determines at most o one maximal component of $\mathrm{supp}(\mathcal{I},\mu)$ through each x.

After the blow-up at the maximal locus $C=\{x_{i_1}=\ldots=x_{i_l}=0\}$ of ρ, the ideal $\mathcal{I}=(x^{a_1,\ldots,a_k})$ is equal to $\mathcal{I}'=({x'}^{a_1,\ldots,a_{i_j-1},a,a_{i_j+1},\ldots,a_k})$ in the neighborhood corresponding to x_{i_j}, where $a=a_{i_1}+\ldots+a_{i_l}-\mu<a_{i_j}$. In particular the invariant ν drops for all points of some maximal components of $\mathrm{supp}(\mathcal{I},\mu)$. Thus the maximal value of ν on the maximal components of $\mathrm{supp}(\mathcal{I},\mu)$ which were blown up is bigger than the maximal value of ν on the new maximal components of $\mathrm{supp}(\mathcal{I},\mu)$. Since the set $\frac{1}{\mu}\mathbf{Z}_{\geq 0}$ of values of ν is discrete the algorithm terminates after a finite number of steps. □

3.1. **Summary of the resolution algorithm.** The resolution algorithm can be represented by the following scheme.

Step 2. Resolve $(\mathcal{I},\mu)$.

Step 2a. Reduce $(\mathcal{I},\mu)$ to the monomial marked ideal $\mathcal{I}=\mathcal{M}(\mathcal{I})$.

$$\Downarrow$$

If $\mathcal{I}\neq\mathcal{M}(\mathcal{I})$, decrease the maximal order of the nonmonomial part $\mathcal{N}(\mathcal{I})$ by resolving the companion ideal $O(\mathcal{I},\mu)$. For $x\in\mathrm{supp}(O(\mathcal{I},\mu))$, set

$$\mathrm{inv}(x)=(\mathrm{ord}_x(\mathcal{N}(\mathcal{J}))/\mu,\overline{\mathrm{inv}}_{O(\mathcal{I},\mu)}).$$

Step 1. Resolve the companion ideal $(\mathcal{J},\mu_{\mathcal{J}}):=O(\mathcal{I},\mu)$:

Replace $\mathcal{J}$ with $\overline{\mathcal{J}}:=\mathcal{C}(\mathcal{H}(\mathcal{J}))\simeq\mathcal{J}$. (*)

Step 1a. Move apart all strict transforms of E and $\mathrm{supp}(\overline{\mathcal{J}},\mu)$.

$$\Downarrow$$

Move apart all intersections H^s_α of s divisors in E

(where s is the maximal number of divisors in E through points in $\mathrm{supp}(\mathcal{I},\mu)$).

$$\Updownarrow$$

For any α, resolve all $\overline{\mathcal{J}}_{|H^s_\alpha}$. For $x \in \mathrm{supp}(\overline{\mathcal{J}},\mu) \cap H^s_\alpha$ set

Step 1b If the strict transforms of E do not intersect $\mathrm{supp}(\overline{\mathcal{J}},\mu)$, resolve $(\overline{\mathcal{J}},\mu)$.

$$\Updownarrow$$

Simultaneously resolve all $\overline{\mathcal{J}}_{|V(u)}$ (by induction), where $V(u)$ is a hypersurface of

maximal contact. For $x \in \mathrm{supp}(\overline{\mathcal{J}},\mu) \setminus \mathrm{codim}(1)(\mathrm{supp}(\overline{\mathcal{J}}))$ set

Step 2b. Resolve the monomial marked ideal $\mathcal{I} = \mathcal{M}(\mathcal{I})$.

(Construct the invariants inv, ρ and ν directly for $\mathcal{M}(\mathcal{I})$.)

Remarks. (1) (*) The ideal $\mathcal{J}$ is replaced with $\mathcal{H}(\mathcal{J})$ to ensure that the algprithm constructed in Step 1b is independent of the choice of the tangent direction u.

We replace $\mathcal{H}(\mathcal{J})$ with $\mathcal{C}(\mathcal{H}(\mathcal{J}))$ to ensure the equalities $\mathrm{supp}(\mathcal{J}_{|S}) = \mathrm{supp}(\mathcal{J}) \cap S$, where $S = H^s_\alpha$ in Step 1a and $S = V(u)$ in Step 1b.

(2) If $\mu = 1$ the companion ideal is equal to $O(\mathcal{I},1) = (\mathcal{N}(\mathcal{I}), \mu_{\mathcal{N}(\mathcal{I})})$ so the general strategy of the resolution of $\mathcal{I},\mu$ is to decrease the order of the nonmonomial part and then to resolve the monomial part.

(3) In particular if we desingularize Y we put $\mu = 1$ and $\mathcal{I} = \mathcal{I}_Y$ to be equal to the sheaf of the subvariety Y and we resolve the marked ideal $(X,\mathcal{I},\emptyset,\mu)$. The nonmonomial part $\mathcal{N}(\mathcal{I}_i)$ is nothing but the <u>weak transform</u> $(\sigma^i)^{\mathrm{w}}(\mathcal{I})$ of $\mathcal{I}$.

4. Complexity bound in terms of Grzegorczyk's classes

Our purpose is to estimate the complexity of the of the desingularization algorithm described in the previous sections. The complexity estimate is given in terms of the Grzegorczyk's classes $\mathcal{E}^i$, $i \geq 0$ [**?**] of primitive-recursive functions. For the sake of self-containdness we provide the definition of $\mathcal{E}^i$ by induction on i (informally speaking, $\mathcal{E}^i$ consists of integer functions $\mathbb{Z}^s \to \mathbb{Z}^t$ whose construction requires i primitive recursions).

For the base of definition $\mathcal{E}^0$ contains constant functions $x_j \mapsto c$ and projections $(x_1,\dots,x_n) \mapsto x_j$ for any variables $x_1,\dots,x_n$. Every class $\mathcal{E}^i$, $i \geq 0$ is closed under composition and some natural basic operations (we don't need and therefore don't make precise the list of these operations), in particular, identifying of variables: if $F(\dots,x_i,\dots,x_j,\dots) \in \mathcal{E}^i$ then $F(\dots,x,\dots,x,\dots) \in \mathcal{E}^i$.

Class $\mathcal{E}^1$ contains functions $x_j \mapsto x_j + 1$.

For the inductive step of the definition assume that functions $G(x_1,\dots,x_n), H(x_1,\dots,x_n,y,z) \in \mathcal{E}^i$. Then function $F(x_1,\dots,x_n,y)$ defined by the following primitive recursion

$$F(x_1,\dots,x_n,0) = G(x_1,\dots,x_n)$$

$$F(x_1,\dots,x_n,y+1) = H(x_1,\dots,x_n,y,F(x_1,\dots,x_n,y))$$

belongs to $\mathcal{E}^{i+1}$. To complete the definition of $\mathcal{E}^i$, $i \geq 0$ take the closure with respect to the basic operations from the aforementioned list. Clearly, $\mathcal{E}^{i+1} \supset \mathcal{E}^i$.

Observe that $\mathcal{E}^2$ contains all the polynomials with integer coefficients and $\mathcal{E}^3$ contains all the towers of exponential functions [**?**]. The union $\cup_{i<\infty}\mathcal{E}^i$ coincides with the set of all primitive-recursive functions.

The principal complexity result of the paper states that

Theorem 4.0.1. *The complexity of the resolution algorithm can be bounded by $L \cdot F(d,n,q,\mu)$ for a certain function $F \in \mathcal{E}^{m+2}$.*

Remark. Below in the proof we give a more explicit form of F providing an additional information on its dependance on d, n, q, μ. But the main consequence of the Theorem is that $m = \dim X$ brings the most significant contribution into the complexity bound.

Proof of the Theorem we conduct by induction on m. It is instructive to represent the resolution algorithm in a form of a tree T as in the

Each node a of T corresponds to an intermediate tuple $T_a =$

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Department of Mathematics, Purdue University, West Lafayette, IN-47907, USA

E-mail address: `wlodar@math.purdue.edu, jwlodar@mimuw.edu.pl`