**Math 265, Spring 2002**

# Solution for Practice Exam of First Midterm

**1.(6 points)**

(a) The rank is 3. The number of columns is 3. Therefore, the nullity of $A$ is $3-3=0$.

(b) The leading ones in rref$(A^T)$ are contained in the first row, the second one and the third one. We can pick them up as a basis; i.e., $\{\begin{bmatrix}1 & 2 & 0\end{bmatrix}, \begin{bmatrix}3 & 6 & -1\end{bmatrix}, \begin{bmatrix}5 & 9 & 6\end{bmatrix}\}$.

(c) The dimension of row space is 3. Therefore, it spans the whole $\mathbb{R}_3$. Any vector is in its row space.

**2.(6 points)** The augmented matrix is

$$\begin{bmatrix}1 & 2 & -a^2 & \vdots & 1\\ 1 & 1 & -5 & \vdots & a\\ 0 & 1 & 1 & \vdots & -1\end{bmatrix} \Longrightarrow \begin{bmatrix}1 & 2 & -a^2 & \vdots & 1\\ 0 & -1 & a^2-5 & \vdots & a-1\\ 0 & 1 & 1 & \vdots & -1\end{bmatrix} \Longrightarrow \begin{bmatrix}1 & 1 & -a^2 & \vdots & 1\\ 0 & 1 & -a^2+5 & \vdots & -a+1\\ 0 & 0 & a^2-4 & \vdots & a-2\end{bmatrix}.$$

Look at the last equation.

(a) If $a \neq 2, -2$, we can solve $x_3$, then $x_2, x_1$. The solution is unique. Therefore, when $a \neq 2, -2$, the system has the unique solution.

(b) If $a=-2$, then the third equation becomes

$$0 = -4.$$

Therefore, the system has no solution.

(c) If $a = 2$, then the system becomes

$$\begin{cases} x_1 + x_2 = & 2.\\ x_2 + 4x_3 = & 1.\\ 0 = & 0.\end{cases}$$

There are infinitely many solutions.

**3.(3 points)** The augmented matrix is

$$\begin{bmatrix}1 & -1 & 1 & \vdots & -1\\ 2 & -3 & 1 & \vdots & -4\\ 3 & -5 & 1 & \vdots & -7\\ -2 & 6 & 2 & \vdots & t\end{bmatrix} \Longrightarrow \begin{bmatrix}1 & -1 & 1 & \vdots & -1\\ 0 & -1 & -1 & \vdots & -2\\ 0 & -2 & -2 & \vdots & -4\\ 0 & 4 & 4 & \vdots & t-2\end{bmatrix} \Longrightarrow \begin{bmatrix}1 & -1 & 1 & \vdots & -1\\ 0 & -1 & -1 & \vdots & -2\\ 0 & 0 & 0 & \vdots & 0\\ 0 & 0 & 0 & \vdots & t-10\end{bmatrix} \Longrightarrow t = 10.$$

**4.(4 points)** The column matrix is

$$\begin{bmatrix} 1 & -1 & -2 & 1 \\ 2 & -1 & -4 & 3 \\ 3 & -7 & -6 & -1 \\ 2 & 0 & -4 & 4 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & -1 & -2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -4 & 0 & -4 \\ 0 & 2 & 0 & 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & -1 & -2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The columns containing leading ones are the first and the second. Therefore, one of bases for $W$ is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -7 \\ 0 \end{bmatrix} \right\}.$$

**5.(6 points)**

(a) **(2 points)**

$$\begin{bmatrix} -1 & -2 & -3 & \vdots & 1 \\ 2 & 5 & 6 & \vdots & 0 \\ 1 & 3 & 3 & \vdots & 0 \end{bmatrix}$$

(b) **(4 points)**

$$\begin{bmatrix} -1 & -2 & -3 & \vdots & 1 \\ 2 & 5 & 6 & \vdots & 0 \\ 1 & 3 & 3 & \vdots & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 & 3 & \vdots & -1 \\ 2 & 5 & 6 & \vdots & 0 \\ 1 & 3 & 3 & \vdots & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 & 3 & \vdots & -1 \\ 0 & 1 & 0 & \vdots & 2 \\ 0 & 1 & 0 & \vdots & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 & 3 & \vdots & -1 \\ 0 & 1 & 0 & \vdots & 2 \\ 0 & 0 & 0 & \vdots & -1 \end{bmatrix}.$$

The last equation is

$$0 = -1.$$

Hence, there is no solution.

**6.(4 points)**

$$\begin{bmatrix} -1 & -2 & -3 & \vdots & 1 & 0 & 0 \\ 2 & 5 & 6 & \vdots & 0 & 1 & 0 \\ 1 & 2 & 4 & \vdots & 0 & 0 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 & 3 & \vdots & -1 & 0 & 0 \\ 2 & 5 & 6 & \vdots & 0 & 1 & 0 \\ 1 & 2 & 4 & \vdots & 0 & 0 & 1 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 1 & 2 & 3 & \vdots & -1 & 0 & 0 \\ 0 & 1 & 0 & \vdots & 2 & 1 & 0 \\ 0 & 0 & 1 & \vdots & 1 & 0 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 & \vdots & -8 & -2 & -3 \\ 0 & 1 & 0 & \vdots & 2 & 1 & 0 \\ 0 & 0 & 1 & \vdots & 1 & 0 & 1 \end{bmatrix} \Longrightarrow A^{-1} = \begin{bmatrix} -8 & -2 & -3 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

**7.(4 points)**

(a) **(2 points)**

$$L\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}1-2\\2\\1-1\\1+1\end{bmatrix}=\begin{bmatrix}-1\\2\\0\\2\end{bmatrix}.$$

(b) **(2 points)**

Solve the system in variables $a_1$ and $a_2$

$$\begin{bmatrix}a_1-2a_2\\2a_2\\a_1-a_2\\a_1+a_2\end{bmatrix}=\begin{bmatrix}1\\0\\1\\0\end{bmatrix}.$$

The augmented matrix for this system is

$$\begin{bmatrix}1&-2&\vdots&1\\0&2&\vdots&0\\1&-1&\vdots&1\\1&1&\vdots&0\end{bmatrix}\Longrightarrow\begin{bmatrix}1&-2&\vdots&1\\0&2&\vdots&0\\0&1&\vdots&0\\0&3&\vdots&-1\end{bmatrix}\Longrightarrow\begin{bmatrix}1&-2&\vdots&1\\0&1&\vdots&0\\0&1&\vdots&0\\0&3&\vdots&-1\end{bmatrix}\Longrightarrow\begin{bmatrix}1&-2&\vdots&1\\0&1&\vdots&0\\0&0&\vdots&0\\0&0&\vdots&-1\end{bmatrix}.$$

The last equation is $0=-1$. Therefore, there is no solution.

**8.(4 points)**

$$A^T=\begin{bmatrix}1&0&0\\-1&2&3\\-2&-2&-3\\2&4&6\end{bmatrix}\Longrightarrow\begin{bmatrix}1&0&0\\0&2&3\\0&-2&-3\\0&4&6\end{bmatrix}\Longrightarrow\begin{bmatrix}1&0&0\\0&1&3/2\\0&0&0\\0&0&0\end{bmatrix}\Longrightarrow \text{row rank }=2.$$

**9.(4 points)**

$$\begin{bmatrix}1&-2&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&-3\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\-1&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}-1&-2&-3\\2&5&6\\1&2&4\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

Therefore,

$$A^{-1}=\begin{bmatrix}1&-2&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&-3\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\-1&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

**10.(3 points)**

$$AB=\begin{bmatrix}2&3&5\\5&-2&9\\7&2&0\end{bmatrix}\cdot\begin{bmatrix}0&1&1\\0&0&1\\1&0&0\end{bmatrix}=\begin{bmatrix}5&2&5\\9&5&3\\0&7&9\end{bmatrix},\ BA=\begin{bmatrix}0&1&1\\0&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}2&3&5\\5&-2&9\\7&2&0\end{bmatrix}=\begin{bmatrix}12&0&9\\7&2&0\\2&3&5\end{bmatrix}.$$

Hence, $AB\neq BA$.

**11.(6 points)**

(a) **(1 points)** True.
(b) **(1 points)** True.
(c) **(1 points)** True.
(d) **(1 points)** True.
(e) **(1 points)** True.
(f) **(1 points)** True.

**12.(4 points)**

(a) **(1 points)** False. The vectors in the set do not belongs to $\mathbb{R}^3$.
(b) **(1 points)** True.
(c) **(1 points)** False. The number of vectors in the set is more than the dimension.
(d) **(1 points)** True.

**13.(6 points)**

(a) **(1 points)** True. It is closed under $+, \cdot$.
(b) **(1 points)** False. It is not closed under $+, \cdot$.
(c) **(1 points)** False. It is not closed under $+, \cdot$.
(d) **(1 points)** False. It is not closed under $+, \cdot$.
(e) **(1 points)** True. It is closed under $+, \cdot$.
(f) **(1 points)** True. It is closed under $+, \cdot$.

**14.(6 points)** First of all, the null space $W$ is

$$\left\{ r \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 6 \\ 0 \\ 1 \end{bmatrix}, r, s \in \mathbb{R} \right\}.$$

The dimension of $W$ is 2.

(a) **(1 points)** False. The number of elements of a basis is 2.
(b) **(1 points)** True.
(c) **(1 points)** False. The vector $\begin{bmatrix} -4 \\ 6 \\ 0 \\ -1 \end{bmatrix}$ does not belongs to the null space.
(d) **(1 points)** True. The first vector is a scalar multiple of the first vector in $(b)$.
(e) **(1 points)** True. Both vectors belongs to the null space. They are linearly independent, too. Hence, it is a maximal linearly independent;i.e., a basis.
(f) **(1 points)** False. The vectors do not belong to $\mathbb{R}^4$.

**15.(2 points)** True. They has the same dimension (3).
**16.(8 points)**

$$\begin{bmatrix} 1 & 0 & 1 & 1 & \vdots & 1 \\ 2 & 1 & 2 & 1 & \vdots & 1 \\ -1 & -1 & 1 & -2 & \vdots & 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 1 & 1 & \vdots & 1 \\ 0 & 1 & 0 & -1 & \vdots & -1 \\ 0 & -1 & 2 & -1 & \vdots & 3 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 1 & 1 & \vdots & 1 \\ 0 & 1 & 0 & -1 & \vdots & -1 \\ 0 & 0 & 2 & -2 & \vdots & 2 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 1 & 0 & 1 & 1 & \vdots & 1 \\ 0 & 1 & 0 & -1 & \vdots & -1 \\ 0 & 0 & 1 & -1 & \vdots & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 & -2 & \vdots & 0 \\ 0 & 1 & 0 & -1 & \vdots & -1 \\ 0 & 0 & 1 & -1 & \vdots & 1 \end{bmatrix} \Longrightarrow \begin{cases} x_1 = & 2x_4. \\ x_2 = & x_4 - 1. \\ x_3 = & x_4 + 1. \end{cases}$$

where $x_4$ is arbitrary.

**17.(8 points)**

(a) **(3 points)**

$$A^T = \begin{bmatrix} 1 & -1 & -2 \\ -2 & 2 & 4 \\ 1 & -2 & -3 \\ 0 & -1 & -1 \\ 2 & -3 & -5 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The leading ones are contained in the first and the second. Therefore,

$$\{\begin{bmatrix} 1 & -2 & 1 & 0 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 2 & -2 & -1 & -3 \end{bmatrix}\}$$

is a basis.

(b) **(3 points)** The solution for the homogeneous system is

$$\begin{cases} x_1 = & 2x_2 + x_4 + x_5. \\ x_3 = & -x_4 - x_5. \end{cases}$$

The null space of $A$ is

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 + x_5 \\ x_2 \\ -x_4 - x_5 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} x_4 \\ 0 \\ -x_4 \\ x_4 \\ 0 \end{bmatrix} + \begin{bmatrix} x_5 \\ 0 \\ -x_5 \\ 0 \\ x_5 \end{bmatrix} = x_2 \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_5 \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Therefore,

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis of the null space of $A$.

(c) **(2 points)**

$$\begin{bmatrix} 1 & -2 & 1 & 0 & 2 \\ -1 & 2 & -2 & -1 & -3 \\ -2 & 4 & -3 & -1 & -5 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & -2 & 1 & 0 & 2 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & -2 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A basis for the column space is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix} \right\}.$$

**18.(8 points)**

(a) **(2 points)** Consider the homogeneous system

$$a_1(-t^3 + t^2 - t - 1) + a_2(t^3 - t^2 + 2t - 1) + a_3(-t^3 + t^2 + t - 5) = 0.$$

The augmented matrix is

$$\begin{bmatrix} -1 & 1 & -1 & \vdots & 0 \\ 1 & -1 & 1 & \vdots & 0 \\ -1 & 2 & 1 & \vdots & 0 \\ -1 & -1 & -5 & \vdots & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} -1 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \\ 0 & 1 & 2 & \vdots & 0 \\ 0 & -2 & -4 & \vdots & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & -1 & 1 & \vdots & 0 \\ 0 & 1 & 2 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

The system has non-trivial solution. Hence, they are not linearly independent.

(b) **(4 points)** The leading ones are contained in the first column and the second one. Hence,

$$\{-t^3 + t^2 - t - 1, t^3 - t^2 + 2t - 1\}$$

is a basis.

(c) **(2 points)** Since the number of elements of a basis is 2, the dimension of $\operatorname{span} S$ is 2. It is impossible to span the whole $P_3$ since the dimension of $P_3$ is 4.

**19.(8 points)**

(a) **(2 points)** Let $\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{v_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Consider the system $a_1\mathbf{v_1} + a_2\mathbf{v_2} + a_3\mathbf{v_3} = \mathbf{v}$. The augmented matrix is

$$\left[\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3} | \mathbf{v}\right];$$

i.e., $\mathbf{v}$ attached to the column matrix $\left[\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\right]$. Then

$$\begin{bmatrix} 1 & 0 & 1 & \vdots & 1 \\ 0 & 1 & 1 & \vdots & 1 \\ 1 & 2 & 1 & \vdots & 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 1 & \vdots & 1 \\ 0 & 1 & 1 & \vdots & 1 \\ 0 & 2 & 0 & \vdots & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 1 & \vdots & 1 \\ 0 & 1 & 0 & \vdots & 1 \\ 0 & 0 & -2 & \vdots & -1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 & \vdots & 1/2 \\ 0 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 1 & \vdots & 1/2 \end{bmatrix}.$$

Hence, the column matrix is row equivalent to $I_3$. It is a basis for $\mathbb{R}^3$.

(b) **(4 points)** As we just did above,

$$\left[\mathbf{v}\right]_S = \begin{bmatrix} 1/2 \\ 1 \\ 1/2 \end{bmatrix}.$$

(c) **(2 points)** The row rank of the matrix is the same as the column rank, which is 3.