**Math 265, Spring 2002**

# Solution for First Midterm

**1.(6 points)**

(a) The rank is 2. The number of columns is 5. Therefore, the nullity of $A$ is $5-2=3$.

(b) The leading ones are contained in the first column and the third column. We can pick them up as a basis; i.e., $\{\begin{bmatrix}1\\3\\-1\end{bmatrix}, \begin{bmatrix}1\\2\\5\end{bmatrix}\}$.

(c) Consider the augmented matrix

$$\begin{bmatrix}1 & 1 & \vdots & 0\\ 3 & 2 & \vdots & 1\\ -1 & 5 & \vdots & -6\end{bmatrix} \Longrightarrow \begin{bmatrix}1 & 1 & \vdots & 0\\ 0 & -1 & \vdots & 1\\ 0 & 6 & \vdots & -6\end{bmatrix} \Longrightarrow \begin{bmatrix}1 & 1 & \vdots & 0\\ 0 & 1 & \vdots & -1\\ 0 & 6 & \vdots & -6\end{bmatrix} \Longrightarrow \begin{bmatrix}1 & 1 & \vdots & 0\\ 0 & 1 & \vdots & -1\\ 0 & 0 & \vdots & 0\end{bmatrix}.$$

It is solvable. Therefore, the vector is in the column space of $A$.

**2.(6 points)** The augmented matrix is

$$\begin{bmatrix}1 & 1 & 0 & \vdots & 2\\ 1 & 0 & a^2-5 & \vdots & a\\ 0 & 1 & 4 & \vdots & 1\end{bmatrix} \Longrightarrow \begin{bmatrix}1 & 1 & 0 & \vdots & 2\\ 0 & -1 & a^2-5 & \vdots & a-2\\ 0 & 1 & 4 & \vdots & 1\end{bmatrix} \Longrightarrow \begin{bmatrix}1 & 1 & 0 & \vdots & 2\\ 0 & 1 & -a^2+5 & \vdots & -a+2\\ 0 & 0 & a^2-1 & \vdots & a-1\end{bmatrix}.$$

Look at the last equation.

(a) If $a \neq 1, -1$, we can solve $x_3$, then $x_2, x_1$. The solution is unique. Therefore, when $a \neq 1, -1$, the system has the unique solution.

(b) If $a=-1$, then the third equation becomes

$$0=-2.$$

Therefore, the system has no solution.

(c) If $a=1$, then the system becomes

$$\begin{cases} x_1+x_2 = & 2.\\ x_2+4x_3 = & 1.\\ 0 = & 0.\end{cases}$$

There are infinitely many solutions.

**3.(3 points)** The augmented matrix is

$$\begin{bmatrix} -1 & 2 & 3 & \vdots & 0 \\ 3 & -5 & -8 & \vdots & 1 \\ 5 & 6 & 1 & \vdots & s \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & -2 & -3 & \vdots & 0 \\ 0 & 1 & 1 & \vdots & 1 \\ 0 & 16 & 16 & \vdots & s \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & -2 & -3 & \vdots & 0 \\ 0 & 1 & 1 & \vdots & 1 \\ 0 & 0 & 0 & \vdots & s-16 \end{bmatrix} \Longrightarrow s = 16.$$

**4.(4 points)** The column matrix is

$$\begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ -4 & 12 & -8 & -7 & -3 \\ 3 & -9 & 6 & 5 & 2 \\ 2 & -6 & 4 & 1 & -1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -3 & -3 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The columns containing leading ones are the first and the fourth. Therefore, one of bases for $W$ is

$$\left\{ \begin{bmatrix} 1 \\ -4 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -7 \\ 5 \\ 1 \end{bmatrix} \right\}.$$

**5.(6 points)**

(a) **(3 points)**

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = & 1, \\ x_1 + 2x_2 + 3x_3 + 4x_4 = & 0, \\ 2x_1 + 3x_2 + 4x_3 + 5x_4 = & 0. \end{cases} \Longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

(b) **(3 points)**

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The nullity of $A$ is equal to $4 - 2 = 2$.

**6.(4 points)**

$$\begin{bmatrix} 4 & 7 & \vdots & 1 & 0 \\ 5 & 9 & \vdots & 0 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 7/4 & \vdots & 1/4 & 0 \\ 5 & 9 & \vdots & 0 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 7/4 & \vdots & 1/4 & 0 \\ 0 & 1/4 & \vdots & -5/4 & 1 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 1 & 7/4 & \vdots & 1/4 & 0 \\ 0 & 1 & \vdots & -5 & 4 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 7/4 & \vdots & 9 & -7 \\ 0 & 1 & \vdots & -5 & 4 \end{bmatrix} \Longrightarrow A^{-1} = \begin{bmatrix} 9 & -7 \\ -5 & 4 \end{bmatrix}$$

**7.(4 points)**

(a) **(2 points)**

$$L\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 2+3 \\ 2 \cdot 3 \\ 2-3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix}.$$

(b) **(2 points)**

Given a vector $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ in $\mathbb{R}^3$. Solve $a_1$ and $a_2$

$$\begin{bmatrix} a_1 + a_2 \\ 2a_2 \\ a_1 - a_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

The augmented matrix for this system is

$$\begin{bmatrix} 1 & 1 & \vdots & b_1 \\ 0 & 2 & \vdots & b_2 \\ 1 & -1 & \vdots & b_3 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & \vdots & b_1 \\ 0 & 2 & \vdots & b_2 \\ 0 & -2 & \vdots & b_3 - b_1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & \vdots & b_1 \\ 0 & 2 & \vdots & b_2 \\ 0 & 0 & \vdots & b_3 + b_2 - b_1 \end{bmatrix}.$$

The condition for solvability is

$$b_3 + b_2 - b_1 = 0.$$

It means that some $b_1, b_2, b_3$ will make the system unsolvable (for instance, $b_1 = 1, b_2 = 1, b_3 = 1$). Therefore, it is not onto.

**8.(4 points)**

$$\begin{bmatrix} 1 & -1 & 3 & 4 \\ -2 & 2 & -5 & -7 \\ -6 & 6 & -1 & -7 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 17 & 17 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow \text{column rank } = 2 = \text{ row rank }.$$

**9.(4 points)**

$$\begin{bmatrix} 1 & -5/7 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -10 & 1 \end{bmatrix} \begin{bmatrix} 1/7 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 7 & 5 \\ 10 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore,

$$A^{-1} = \begin{bmatrix} 1 & -5/7 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -7 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -10 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/7 & 0 \\ 0 & 1 \end{bmatrix}.$$

**10.(1 points)**

$$AB = \begin{bmatrix} 1 & 3 \\ 5 & -2 \end{bmatrix} \cdot \begin{bmatrix} -4 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ -22 & 3 \end{bmatrix} \Longrightarrow (AB)^T = \begin{bmatrix} -1 & -22 \\ 4 & 3 \end{bmatrix}.$$

**11.(6 points)**

(a) **(2 points)** False.
(b) **(2 points)** True.
(c) **(2 points)** False.

**12.(4 points)**

(a) **(1 points)** False. The number of vectors in the set is less than the dimension.
(b) **(1 points)** False. The fourth column is a linear combination of the previous three.
(c) **(1 points)** False. The vectors in the set do not belongs to $\mathbb{R}^4$.

(d) **(1 points)** False. The number of vectors in the set is more than the dimension.

**13.(6 points)**

(a) **(2 points)** True. It is closed under $+, \cdot$.

(b) **(2 points)** False. It is not closed under $+, \cdot$.

(c) **(2 points)** False. It is not closed under $+, \cdot$.

**14.(6 points)** First of all, the null space $W$ is

$$\left\{ r \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -4 \\ -1 \\ 1 \\ 0 \end{bmatrix}, r, s \in \mathbb{R} \right\}.$$

The dimension of $W$ is 2.

(a) **(2 points)** True. The vector $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ belongs to $W$ and the first two vectors have already spanned $W$.

(b) **(2 points)** False. The vector $\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ does not belong to $W$.

(c) **(2 points)** True. Both vectors belongs to $W$ and they are linearly independent. The set is a maximal linearly independent; i.e., a basis. Hence, it spans $W$.

**15.(2 points)** True. They has the same dimension ($6 \times 6 = 36 = 4 \times 9$).

**16.(8 points)**

(a) **(1 points)** The augmented matrix is

$$\begin{bmatrix} 1 & -1 & 1 & 0 & -2 & \vdots & -1 \\ 5 & -5 & 6 & 1 & -9 & \vdots & -7 \\ 2 & -2 & 3 & 1 & -3 & \vdots & -4 \\ 1 & -1 & 2 & 1 & -1 & \vdots & -3 \end{bmatrix}.$$

(b) **(7 points)**

$$\begin{bmatrix} 1 & -1 & 1 & 0 & -2 & \vdots & -1 \\ 5 & -5 & 6 & 1 & -9 & \vdots & -7 \\ 2 & -2 & 3 & 1 & -3 & \vdots & -4 \\ 1 & -1 & 2 & 1 & -1 & \vdots & -3 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & -1 & 1 & 0 & -2 & \vdots & -1 \\ 0 & 0 & 1 & 1 & 1 & \vdots & -2 \\ 0 & 0 & 1 & 1 & 1 & \vdots & -2 \\ 0 & 0 & 1 & 1 & 1 & \vdots & -2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & -1 & 1 & 0 & -2 & \vdots & -1 \\ 0 & 0 & 1 & 1 & 1 & \vdots & -2 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 1 & -1 & 1 & 0 & -2 & \vdots & -1 \\ 0 & 0 & 1 & 1 & 1 & \vdots & -2 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & -1 & 0 & -1 & -3 & \vdots & 1 \\ 0 & 0 & 1 & 1 & 1 & \vdots & -2 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix} \Longrightarrow \begin{cases} x_1 = & x_2 + x_4 + 3x_5 + 1. \\ x_3 = & -x_4 - x_5 - 2, \end{cases}$$

where $x_2, x_4, x_5$ are arbitrary.

**17.(8 points)**

(a) **(3 points)**

$$\begin{bmatrix} 2 & -6 & 4 & 6 & -2 \\ -1 & 3 & -3 & -4 & 2 \\ -2 & 6 & -3 & -5 & 1 \\ 1 & -3 & -1 & 0 & 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & -3 & 2 & 3 & -1 \\ -1 & 3 & -3 & -4 & 2 \\ -2 & 6 & -3 & -5 & 1 \\ 1 & -3 & -1 & 0 & 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & -3 & 2 & 3 & -1 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & -3 & -3 & 3 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 1 & -3 & 2 & 3 & -1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & -3 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The rank is 2.

(b) **(3 points)** The solution for the homogeneous system is

$$\begin{cases} x_1 = & 3x_2 - x_4 - x_5. \\ x_3 = & -x_4 + x_5. \end{cases}$$

The null space of $A$ is

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3x_2 - x_4 - x_5 \\ x_2 \\ -x_4 + x_5 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -x_4 \\ 0 \\ -x_4 \\ x_4 \\ 0 \end{bmatrix} + \begin{bmatrix} -x_5 \\ 0 \\ x_5 \\ 0 \\ x_5 \end{bmatrix} = x_2 \cdot \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \cdot \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_5 \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Therefore,

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis of the null space of $A$.

(c) **(2 points)** The rank of $A^T$ is the same as that of $A$. Hence, it is 2, too.

**18.(8 points)**

(a) **(2 points)** Consider the homogeneous system

$$a_1(t^2 - t - 1) + a_2(-t^2 + 2t - 1) + a_3(t^2 + t - 5) = 0.$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 2 & 1 & 0 \\ -1 & -1 & -5 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -2 & -4 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The system has non-trivial solution. Hence, they are not linearly independent.

(b) **(4 points)** The leading ones are contained in the first column and the second one. Hence,

$$\{t^2 - t - 1, -t^2 + 2t - 1\}$$

is a basis.

(c) **(2 points)** Since the number of elements of a basis is 2, the dimension of $\operatorname{span} S$ is 2. It is impossible to span the whole $P_2$ since the dimension of $P_2$ is 3.

**19.(8 points)**

(a) **(2 points)** Let $\mathbf{v_1} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 2 & 4 & 5 \end{bmatrix}$, $\mathbf{v_3} = \begin{bmatrix} 4 & 7 & 12 \end{bmatrix}$. Consider the system $a_1\mathbf{v_1} + a_2\mathbf{v_2} + a_3\mathbf{v_3} = \mathbf{v}$. The augmented matrix is

$$\begin{bmatrix} \mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3} | \mathbf{v} \end{bmatrix};$$

i.e., $\mathbf{v}$ attached to the column matrix $\begin{bmatrix} \mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3} \end{bmatrix}$. Then

$$\begin{bmatrix} 1 & 2 & 4 & \vdots & 1 \\ 2 & 4 & 7 & \vdots & 1 \\ 3 & 5 & 12 & \vdots & 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 & 4 & \vdots & 1 \\ 0 & 0 & -1 & \vdots & -1 \\ 0 & -1 & 0 & \vdots & -1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 & 4 & \vdots & 1 \\ 0 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 1 & \vdots & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 & \vdots & -5 \\ 0 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 1 & \vdots & 1 \end{bmatrix}.$$

Hence, the column matrix is row equivalent to $I_3$. It is a basis for $\mathbb{R}_3$.

(b) **(4 points)** As we just did above,

$$\begin{bmatrix} \mathbf{v} \end{bmatrix}_S = \begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}.$$

(c) **(2 points)** The row rank of the matrix is the same as the column rank, which is 3.