Math 265, Spring 2002

Solution of Practice Exam for Second Midterm

1.(4 points)

$$\cos\theta = \frac{\mathbf{v_1}\cdot\mathbf{v_2}}{\|\mathbf{v_1}\|\|\mathbf{v_2}\|} = \frac{2}{\sqrt{0^2+(-2)^2+(-2)^2}\sqrt{1^2+(-1)^2+0^2}} = \frac{2}{2\sqrt{2}\cdot\sqrt{2}} = \frac{1}{2} \Longrightarrow \theta = \cos^{-1}(\frac{1}{2}) = \frac{\pi}{3}.$$

2.(4 points) We can use the rule that determinant does not change its value if we apply the third type of row (column) operations on it. Use a_{23} as a pivot to put 0s in the other positions of the third column, that is, apply the row operations $-2R_2+R_1 \to R_1, 3R_2+R_3 \to R_3, R_2+R_4 \to R_4$; that is

$$|A| = \begin{vmatrix} 5 & 4 & 2 & 1 \\ 2 & 3 & 1 & -2 \\ -5 & -7 & -3 & 9 \\ 1 & -2 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 0 & 5 \\ 2 & 3 & 1 & -2 \\ 1 & 2 & 0 & 3 \\ 3 & 1 & 0 & 2 \end{vmatrix} = (-1)^{2+3}\cdot 1 \cdot \begin{vmatrix} 1 & -2 & 5 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & -2 & 5 \\ 0 & 4 & -2 \\ 0 & 7 & -13 \end{vmatrix} = -\begin{vmatrix} 4 & -2 \\ 7 & -13 \end{vmatrix} = 38$$

3.(4 points)

$$W^{\perp} = \left\{ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix},\ \mathbf{x}\cdot\begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \mathbf{x}\cdot\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \mathbf{x}\cdot\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 0 \right\} \Longrightarrow \begin{cases} x_1+2x_2+x_3+2x_4+x_5 = & 0 \\ x_1+x_2+x_4+x_5 = & 0 \\ x_2+x_4 = & 0 \end{cases}$$

$$\Longrightarrow \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & -1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

$$\Longrightarrow \begin{cases} x_1 = & 0 \\ x_2 = & -x_4-x_5 \\ x_3 = & x_5 \\ x_4, x_5 & \text{arbitrary} \end{cases} \Longrightarrow \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ -x_4-x_5 \\ x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_5\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + x_4\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Let $\mathbf{u_1} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{u_2} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. Then $\operatorname{span}\{\mathbf{u_1}, \mathbf{u_2}\}$ is a basis for $W^{\perp}$. We can use the Gram-Schmidt to find an orthogonal basis for $W^{\perp}$.

$$\mathbf{v_1} = \mathbf{u_1} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 1 \end{bmatrix},\ \mathbf{v_2} = \mathbf{u_2} - \frac{\mathbf{u_2}\cdot\mathbf{v_1}}{\mathbf{v_1}\cdot\mathbf{v_1}\mathbf{v_1}} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{\mathbf{1}}{\mathbf{3}}\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{2}{3} \\ -\frac{1}{3} \\ 1 \\ -\frac{1}{3} \end{bmatrix} \Longrightarrow \{\mathbf{v_1}, \mathbf{v_2}\} = \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{2}{3} \\ -\frac{1}{3} \\ 1 \\ -\frac{1}{3} \end{bmatrix} \right\}$$

is an orthogonal basis of $W^{\perp}$.

4.(4 points)

(a) **(2 points)** Let

$$\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e_4} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e_5} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$A = \begin{bmatrix} L(\mathbf{e_1}) & L(\mathbf{e_2}) & L(\mathbf{e_3}) & L(\mathbf{e_4}) & L(\mathbf{e_5}) \end{bmatrix} = \begin{bmatrix} 1 & -2 & 5 & 0 & -3 \\ 5 & 2 & -6 & 1 & 0 \\ -2 & 0 & 2 & -1 & 2 \end{bmatrix}.$$

(b) **(2 points)**

$$L\left(\begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1+2\cdot 1+5\cdot 2-3\cdot 1 \\ 5\cdot 1+2\cdot(-1)-6\cdot 2+(-2) \\ 2\cdot 2+2\cdot 1-2\cdot 1-(-2) \end{bmatrix} = \begin{bmatrix} 10 \\ -11 \\ 6 \end{bmatrix},$$

$$\text{or } L\left(\begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \\ 1 \end{bmatrix}\right) = A \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 5 & 0 & -3 \\ 5 & 2 & -6 & 1 & 0 \\ -2 & 0 & 2 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ -11 \\ 6 \end{bmatrix}.$$

5.(4 points) Since the eigenvalues are 0,1 and 2, the characteristic polynomial is

$$\lambda(\lambda-1)(\lambda-2) = \lambda^3 - 3\lambda + 2\lambda$$

On the other hand, the characteristic polynomial of A is

$$\begin{vmatrix} \lambda-1 & 0 & 1 \\ -a & \lambda-b & -c \\ 0 & -1 & \lambda+1 \end{vmatrix} = (\lambda-1)\begin{vmatrix} \lambda-b & -c \\ -1 & \lambda+1 \end{vmatrix} - (-a)\begin{vmatrix} 0 & 1 \\ -1 & \lambda+1 \end{vmatrix} = (\lambda-1)\big(\lambda^2+(1-b)\lambda-b-c\big)+a$$

$$= \lambda^3 + (1-b-1)\lambda^2 + (b-1-b-c)\lambda + (a+b+c) = \lambda^3 - b\lambda^2 + (-1-c)\lambda + (a+b+c)$$

Hence,

$$-b=-3, -1-c=2, a+b+c=0 \Longrightarrow b=3, c=-3, a=0$$

6.(4 points)

$$\det(F) = \begin{vmatrix} -5a & -5b & -5c & -5d \\ 2y & 2x & 2z & 2w \\ 2-3y & 3-3x & 2-3z & 3-3w \\ 1 & 1 & 1 & 1 \end{vmatrix} = (-5)\cdot 2 \cdot \begin{vmatrix} a & b & c & d \\ y & x & z & w \\ 2-3y & 3-3x & 2-3z & 3-3w \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

$$= (-1)\cdot(-10)\cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ y & x & z & w \\ 2-3y & 3-3x & 2-3z & 3-3w \\ a & b & c & d \end{vmatrix} = 10\cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ y & x & z & w \\ 2 & 3 & 2 & 3 \\ a & b & c & d \end{vmatrix} = 10\cdot\det(E) = 10\cdot 5 = 50.$$

7.(4 points)

$$\det(C) = \det(B^3)\cdot\det(A^{-1})\cdot\det\big((A^T)^4\big)\cdot\det(3B)\cdot\det\big((B^T)^{-1}\big) = 2^3\cdot 5^{-1}\cdot 5^4\cdot 3^3\cdot 2\cdot 2^{-1} = 27000.$$

8.(4 points) The entries in D are the eigenvalues of A. To find the eigenvalues, we need to solve the roots for the characteristic equation of A. The characteristic equation is

$$\det(\lambda I_3 - A) = 0 \Longrightarrow \begin{vmatrix} \lambda+3 & -1 & 1 \\ 7 & \lambda-5 & 1 \\ 6 & -6 & \lambda+2 \end{vmatrix} = 0 \Longrightarrow \lambda^3 - 12\lambda - 16 = 0$$

By testing all factors of 16, we get $-2, 4$ are roots.

$$\Longrightarrow (\lambda+2)\cdot(\lambda^2-2\lambda-8) = 0 \Longrightarrow (\lambda+2)\cdot(\lambda+2)(\lambda-4) = 0 \Longrightarrow \lambda = -2,-2,4 \Longrightarrow D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

9.(6 points)

(a) **(4 points)**

$$\operatorname{adj} A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} +\begin{vmatrix} -4 & 2 \\ -1 & 5 \end{vmatrix} & -\begin{vmatrix} 3 & -4 \\ -1 & 5 \end{vmatrix} & +\begin{vmatrix} 3 & -4 \\ -4 & 2 \end{vmatrix} \\ -\begin{vmatrix} 0 & -2 \\ 1 & 5 \end{vmatrix} & +\begin{vmatrix} 2 & -4 \\ 1 & 5 \end{vmatrix} & -\begin{vmatrix} 2 & -4 \\ 0 & 2 \end{vmatrix} \\ +\begin{vmatrix} 0 & -4 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 0 & -4 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -18 & -11 & -10 \\ 2 & 14 & -4 \\ 4 & 5 & -8 \end{bmatrix}.$$

(b) **(2 points)**

$$\det A = \begin{bmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{bmatrix} = -46, \quad A^{-1} = \frac{1}{\det A}\cdot\operatorname{adj} = \begin{bmatrix} \frac{9}{23} & \frac{11}{23} & \frac{5}{23} \\ -\frac{1}{23} & -\frac{7}{23} & \frac{2}{23} \\ -\frac{2}{23} & -\frac{5}{23} & \frac{4}{23} \end{bmatrix}.$$

10.(6 points)

(a) **(3 points)**

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}, 2x+y+2z+w=0, x+2y+z+2w=0 \right\} \Longrightarrow \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & 0 & -3 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \Longrightarrow \begin{cases} x = & -z \\ y = & -w \end{cases} \Longrightarrow \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -z \\ w \\ z \\ w \end{bmatrix} = z\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + w\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \Longrightarrow W = \operatorname{span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Let $\mathbf{u_1} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u_2} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$. Since $\{\mathbf{u_1}, \mathbf{u_2}\}$ is orthogonal already, it is what we want.

(b) **(3 points)**

$$\text{Proj}_W\,\mathbf{b} = \frac{\mathbf{b}\cdot\mathbf{u_1}}{\mathbf{u_1}\cdot\mathbf{u_1}}\mathbf{u_1} + \frac{\mathbf{b}\cdot\mathbf{u_2}}{\mathbf{u_2}\cdot\mathbf{u_2}}\mathbf{u_2} = \frac{-2}{2}\begin{bmatrix}-1\\0\\1\\0\end{bmatrix} + \frac{-2}{2}\begin{bmatrix}0\\-1\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix},$$

$$d = \|\mathbf{b} - \text{Proj}_W\,\mathbf{b}\| = \left\|\begin{bmatrix}4\\3\\2\\1\end{bmatrix} - \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}\right\| = \left\|\begin{bmatrix}3\\2\\3\\2\end{bmatrix}\right\| = \sqrt{3^2+2^2+3^2+2^2} = \sqrt{26}.$$

11.(6 points)

(a) **(1 points)** False.

(b) **(1 points)** False. All eigenvalues of a real symmetry matrix are real but not distinct.

(c) **(1 points)** True.

(d) **(1 points)** True. if $\det(A) \neq 0$.

(e) **(1 points)** True.

(f) **(1 points)** True. If A is a real orthogonal matrix, then $A^T = A^{-1}$. Hence, $\det(A^T) = \det(A^{-1})$; it implies that $\det(A^T) = \det(A) = \det(A^{-1})$ and $\big(\det(A)\big)^2 = 1$.

12.(4 points) Let μ be an eigenvalue of A and $\mathbf{w}$ the associated eigenvector. Then we have

$$A\cdot\mathbf{w} = \mu\mathbf{w} \Longrightarrow A\big(A(A\cdot\mathbf{w})\big) = A\big(A\cdot(\mu\mathbf{w})\big) = \mu\big(A(A\cdot\mathbf{w})\big) = \mu\big(A(\mu\mathbf{w})\big) = \mu^2(A\cdot\mathbf{w}) = \mu^3\mathbf{w}.$$

It tells us that μ^3 is an eigenvalue for A^3 and the associated eigenvector is $\mathbf{w}$. Any non-zero scalar multiple of an eigenvector is still an eigenvector. Hence, the answer E is correct.

13.(4 points)

$$(\text{adj}\,A)\cdot A = \det(A)\cdot I_4 \Longrightarrow \det\big((\text{adj}\,A)\cdot A\big) = \det\big(\det A\cdot I_5\big) = \big(\det(A)\big)^4\cdot\det(I_4) = 3^4\cdot 1 = 81.$$

$$\Longrightarrow \det(\text{adj}\,A)\cdot\det(A) = 81 \Longrightarrow \det(\text{adj}\,A)\cdot 3 = 81 \Longrightarrow \big(\det(\text{adj}\,A)\big)^{-1} = \frac{1}{27}.$$

The answer is A.

14.(6 points)

(a) **(2 points)** True. It is a real symmetric matrix. By theorem, it is diagonalizable.

(b) **(2 points)** False. The eigenvalue is 2 and only one linear independent eigenvector $\begin{bmatrix}1\\0\end{bmatrix}$. We need two linear independent eigenvectors to make it diagonalizable.

(c) **(2 points)** False. There are two eigenvalues; one is 2 and the other is 1. For 2, the associated eigenvector is $\begin{bmatrix}1\\0\\0\end{bmatrix}$; for 1, the associated eigenvector is $\begin{bmatrix}0\\1\\0\end{bmatrix}$. The dimension is 3 but we only have two linearly independent eigenvectors. Hence, it is not diagonalizable.

15.(4 points)

$$\mathbf{u} = \mathbf{u_1} + \mathbf{u_2} = \text{Proj}_{\mathbf{W}}\,\mathbf{u} + \text{Proj}_{\mathbf{W}^\perp}\,\mathbf{u}.$$

$$\text{Proj}_W\,\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v_1}}{\mathbf{v_1}\cdot\mathbf{v_1}}\mathbf{v_1} + \frac{\mathbf{u}\cdot\mathbf{v_2}}{\mathbf{v_2}\cdot\mathbf{v_2}}\mathbf{v_2} = \frac{9}{9}\begin{bmatrix}1\\2\\2\end{bmatrix} + \frac{9}{9}\begin{bmatrix}2\\-2\\1\end{bmatrix} = \begin{bmatrix}3\\0\\3\end{bmatrix} \Longrightarrow \mathbf{u_1} = \begin{bmatrix}3\\0\\3\end{bmatrix},\ \mathbf{u_2} = \mathbf{u} - \mathbf{u_1} = \begin{bmatrix}2\\1\\-2\end{bmatrix}.$$

Hence, the answer is B.

16.(10 points)

(a) **(3 points)** Use the Gram-Schmidt process to find an orthogonal basis for W.

$$\mathbf{v_1} = \mathbf{u_1} = \begin{bmatrix}1\\-3\\5\\1\end{bmatrix},\ \mathbf{v_2} = \mathbf{u_2} - \frac{\mathbf{u_2}\cdot\mathbf{v_1}}{\mathbf{v_1}\cdot\mathbf{v_1}}\mathbf{v_1} = \begin{bmatrix}5\\-5\\9\\7\end{bmatrix} - \frac{72}{36}\begin{bmatrix}1\\-3\\5\\1\end{bmatrix} = \begin{bmatrix}3\\1\\-1\\5\end{bmatrix}.$$

$$\mathbf{v_3} = \mathbf{u_3} - \frac{\mathbf{u_3}\cdot\mathbf{v_1}}{\mathbf{v_1}\cdot\mathbf{v_1}}\mathbf{v_1} - \frac{\mathbf{u_3}\cdot\mathbf{v_2}}{\mathbf{v_2}\cdot\mathbf{v_2}}\mathbf{v_2} = \begin{bmatrix}7\\-1\\3\\11\end{bmatrix} - \frac{36}{36}\begin{bmatrix}1\\-3\\5\\1\end{bmatrix} - \frac{72}{36}\begin{bmatrix}3\\1\\-1\\5\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}.$$

$$\mathbf{v_4} = \mathbf{u_4} - \frac{\mathbf{u_4}\cdot\mathbf{v_1}}{\mathbf{v_1}\cdot\mathbf{v_1}}\mathbf{v_1} - \frac{\mathbf{u_4}\cdot\mathbf{v_2}}{\mathbf{v_2}\cdot\mathbf{v_2}}\mathbf{v_2} = \begin{bmatrix}4\\-2\\4\\6\end{bmatrix} - \frac{36}{36}\begin{bmatrix}1\\-3\\5\\1\end{bmatrix} - \frac{36}{36}\begin{bmatrix}3\\1\\-1\\5\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}.$$

Hence,

$$\{\mathbf{v_1}, \mathbf{v_2}\} = \left\{\begin{bmatrix}1\\-3\\5\\1\end{bmatrix}, \begin{bmatrix}3\\1\\-1\\5\end{bmatrix}\right\}$$

is an orthogonal basis of W. We can make it as an orthonormal basis

$$\mathbf{w_1} = \frac{\mathbf{v_1}}{\|\mathbf{v_1}\|} = \frac{1}{\sqrt{36}}\begin{bmatrix}1\\-3\\5\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{6}\\-\frac{1}{2}\\\frac{5}{6}\\\frac{1}{6}\end{bmatrix},\ \mathbf{w_2} = \frac{\mathbf{v_2}}{\|\mathbf{v_2}\|} = \frac{1}{\sqrt{36}}\begin{bmatrix}3\\1\\-1\\5\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\\frac{1}{6}\\-\frac{1}{6}\\\frac{5}{6}\end{bmatrix}.$$

$\{\mathbf{w_1}, \mathbf{w_2}\}$ is an orthonormal basis of W.

(b) **(3 points)**

$$W^\perp = \left\{\mathbf{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}, \mathbf{x}\cdot\mathbf{v_1} = \mathbf{x}\cdot\mathbf{v_2} = \mathbf{x}\cdot\mathbf{u_3} = 0\right\} \Longrightarrow \begin{cases} x_1 - 3x_2 + 5x_3 + x_4 = & 0 \\ 3x_1 + x_2 - x_3 + 5x_4 = & 0 \end{cases} \Longrightarrow \begin{bmatrix}1 & -3 & 5 & 1\\ 3 & 1 & -1 & 5\end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix}1 & -3 & 5 & 1\\ 0 & 10 & -16 & 2\end{bmatrix} \Longrightarrow \begin{bmatrix}1 & 0 & \frac{1}{5} & \frac{8}{5}\\ 0 & 1 & -\frac{8}{5} & \frac{1}{5}\end{bmatrix} \Longrightarrow \begin{cases} x_1 = & -\frac{1}{5}x_3 - \frac{8}{5}x_4 \\ x_2 = & \frac{8}{5}x_3 - \frac{1}{5}x_4 \\ x_3, x_4 & \text{arbitrary} \end{cases}$$

$$\Longrightarrow \mathbf{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}-\frac{1}{5}x_3 - \frac{8}{5}x_4\\ \frac{8}{5}x_3 - \frac{1}{5}x_4\\ x_3\\ x_4\end{bmatrix} = \frac{x_3}{5}\begin{bmatrix}-1\\8\\5\\0\end{bmatrix} + \frac{x_4}{5}\begin{bmatrix}-8\\-1\\0\\5\end{bmatrix} \Longrightarrow W^{\perp} = \text{span}\left\{\begin{bmatrix}-1\\8\\5\\0\end{bmatrix}, \begin{bmatrix}-8\\-1\\0\\5\end{bmatrix}\right\}.$$

Those two vectors are already orthogonal. We just need to normalize them.

$$\mathbf{w_1'} = \frac{1}{\sqrt{90}}\begin{bmatrix}-1\\8\\5\\0\end{bmatrix}, \ \mathbf{w_2'} = \frac{1}{\sqrt{90}}\begin{bmatrix}-8\\-1\\0\\5\end{bmatrix}.$$

$\{\mathbf{w_1'}, \mathbf{w_2'}\}$ is an orthonormal basis of $W^{\perp}$.

(c) **(2 points)**

$$\text{Proj}_W\, \mathbf{b} = \frac{\mathbf{b}\cdot\mathbf{v_1}}{\mathbf{v_1}\cdot\mathbf{v_1}}\mathbf{v_1} + \frac{\mathbf{b}\cdot\mathbf{v_2}}{\mathbf{v_2}\cdot\mathbf{v_2}}\mathbf{v_2} = \frac{72}{36}\begin{bmatrix}1\\-3\\5\\1\end{bmatrix} + \frac{108}{36}\begin{bmatrix}3\\1\\-1\\5\end{bmatrix} = \begin{bmatrix}11\\-3\\7\\17\end{bmatrix} \Longrightarrow \mathbf{u_1} = \begin{bmatrix}11\\-3\\7\\17\end{bmatrix}, \ \mathbf{u_2} = \mathbf{b} - \mathbf{u_1} = \begin{bmatrix}-9\\7\\5\\5\end{bmatrix}.$$

(d) **(2 points)** Let $\hat{\mathbf{x}} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ be the least square solution. Then it satisfies

$$A^T \cdot A\hat{\mathbf{x}} = A^T \cdot \mathbf{b} \Longrightarrow \begin{bmatrix}1 & -3 & 5 & 1\\ 5 & -5 & 9 & 7\\ 7 & -1 & 3 & 11\end{bmatrix}\begin{bmatrix}1 & 5 & 7\\ -3 & -5 & -1\\ 5 & 9 & 3\\ 1 & 7 & 11\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}1 & -3 & 5 & 1\\ 5 & -5 & 9 & 7\\ 7 & -1 & 3 & 11\end{bmatrix}\begin{bmatrix}2\\4\\12\\22\end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix}36 & 72 & 36\\ 72 & 180 & 144\\ 36 & 144 & 180\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}72\\252\\288\end{bmatrix} \Longrightarrow \begin{bmatrix}36 & 72 & 36 & \vdots & 72\\ 72 & 180 & 144 & \vdots & 252\\ 36 & 144 & 180 & \vdots & 288\end{bmatrix} \Longrightarrow \begin{bmatrix}1 & 2 & 1 & \vdots & 2\\ 2 & 5 & 4 & \vdots & 7\\ 1 & 4 & 5 & \vdots & 8\end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix}1 & 2 & 1 & \vdots & 2\\ 0 & 1 & 2 & \vdots & 3\\ 0 & 2 & 4 & \vdots & 6\end{bmatrix} \Longrightarrow \begin{bmatrix}1 & 0 & -3 & \vdots & -4\\ 0 & 2 & 1 & \vdots & 3\\ 0 & 0 & 0 & \vdots & 0\end{bmatrix} \Longrightarrow \begin{cases} x_1 = & 3x_3 - 4 \\ x_2 = & -2x_3 + 3 \\ x_3 & \text{arbitrary} \end{cases}$$

$$\Longrightarrow \hat{\mathbf{x}} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}3x_3 - 4\\ -2x_3 + 3\\ x_3\end{bmatrix} = x_3\begin{bmatrix}3\\-2\\1\end{bmatrix} + \begin{bmatrix}-4\\3\\0\end{bmatrix},$$

where x_3 is arbitrary.

17.(6 points)

(a) **(4 points)**

(i) **(1 points)** We can check the determinant

$$\begin{vmatrix} 1 & 0 & -2 \\ -3 & -1 & 6 \\ 2 & 1 & -5 \end{vmatrix} = 5+6+0-4-6-0 = 1 \neq 0.$$

Since the determinant is non-zero, the column (row) vectors are linearly independent and form a basis for $\mathbb{R}^3$.

(ii) **(3 points)** There are two way to get the solution.

- The first method is to write $\mathbf{v}$ as a linear combination of $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$.

$$\mathbf{v} = \mathbf{a_1 v_1} + \mathbf{a_2 v_2} + \mathbf{a_3 v_3} \Longrightarrow \begin{cases} a_1 - 2a_3 = & -3 \\ -3a_1 - a_2 + 6a_3 = & 5 \\ 2a_1 + a_2 + 5a_3 = & 7 \end{cases} \Longrightarrow \left| \begin{array}{ccccc} 1 & 0 & -2 & \vdots & -3 \\ -3 & -1 & 6 & \vdots & 7 \\ 2 & 1 & -5 & \vdots & -5 \end{array} \right|$$

$$\Longrightarrow \begin{bmatrix} 1 & 0 & -2 & \vdots & -3 \\ 0 & -1 & 0 & \vdots & -2 \\ 0 & 1 & -1 & \vdots & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & -2 & \vdots & -3 \\ 0 & 1 & 0 & \vdots & 2 \\ 0 & 0 & -1 & \vdots & -1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 & \vdots & -1 \\ 0 & 1 & 0 & \vdots & 2 \\ 0 & 0 & 1 & \vdots & 1 \end{bmatrix} \Longrightarrow \begin{cases} a_1 = & -1 \\ a_2 = & 2 \\ a_3 = & 1 \end{cases}.$$

$$L(\mathbf{v}) = L(-\mathbf{v_1} + 2\mathbf{v_2} + \mathbf{v_3}) = \begin{bmatrix} -5 \\ -5 \\ 2 \\ -8 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \\ 2 \\ 8 \\ -2 \end{bmatrix} + \begin{bmatrix} -12 \\ -11 \\ 2 \\ -19 \\ 3 \end{bmatrix} = \begin{bmatrix} -13 \\ -12 \\ 6 \\ -19 \\ 3 \end{bmatrix}$$

- The second way is to find the standard matrix A representing L.

$$\begin{bmatrix} 1 & 0 & -2 \\ -3 & -1 & 6 \\ 2 & 1 & -5 \\ \cdots & \cdots & \cdots \\ 5 & 2 & -12 \\ 5 & 2 & -11 \\ -2 & 1 & 2 \\ 8 & 4 & -19 \\ -2 & -1 & 3 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ -3 & -1 & 0 \\ 2 & 1 & -1 \\ \cdots & \cdots & \cdots \\ 5 & 2 & -2 \\ 5 & 2 & -1 \\ -2 & 1 & -2 \\ 8 & 4 & -3 \\ -2 & -1 & -1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ -3 & -1 & 0 \\ 0 & 0 & 1 \\ \cdots & \cdots & \cdots \\ 1 & 0 & 2 \\ 3 & 1 & 1 \\ -6 & -1 & 2 \\ 2 & 1 & 3 \\ -4 & -2 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \cdots & \cdots & \cdots \\ 1 & 0 & 2 \\ 0 & -1 & 1 \\ -3 & 1 & 2 \\ -1 & -1 & 3 \\ 2 & 2 & 1 \end{bmatrix}$$

Hence

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ -3 & 1 & 2 \\ -1 & -1 & 3 \\ 2 & 2 & 1 \end{bmatrix} \Longrightarrow L(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ -3 & 1 & 2 \\ -1 & -1 & 3 \\ 2 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 7 \\ -5 \end{bmatrix} = \begin{bmatrix} -13 \\ -12 \\ 6 \\ -19 \\ 3 \end{bmatrix}$$

(b) **(4 points)**

(i) **(1 points)** We can check the determinant

$$\begin{vmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1 + 6 + 0 - 2 - 6 - 0 = -1 \neq 0.$$

Since the determinant is non-zero, the row (column) vectors are linearly independent and form a basis for $\mathbb{R}_3$.

(ii) **(3 points)** There are two way to get the solution.

- The first method is to write $\mathbf{v}'$ as a linear combination of $\mathbf{v_1'}, \mathbf{v_2'}, \mathbf{v_3'}$.

$$\mathbf{v}' = \mathbf{a_1'}\mathbf{v_1'} + \mathbf{a_2'}\mathbf{v_2'} + \mathbf{a_3'}\mathbf{v_3'} \Longrightarrow \begin{cases} a_1' + 3a_2' + a_3' = & 0 \\ 2a_1' + a_2' + a_3' = & -3 \\ 2a'_+ a_3' = & -4 \end{cases} \Longrightarrow \begin{bmatrix} 1 & 3 & 1 & \vdots & 0 \\ 2 & 1 & 1 & \vdots & -3 \\ 2 & 0 & 1 & \vdots & -4 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 1 & 3 & 1 & \vdots & 0 \\ 0 & -5 & -1 & \vdots & -3 \\ 0 & -6 & -1 & \vdots & -4 \end{bmatrix} \Longrightarrow \begin{bmatrix} 13 & 1 & \vdots & 0 \\ 0 & 1 & 0 & \vdots & 1 \\ 0 & -6 & -1 & \vdots & -4 \end{bmatrix} \Longrightarrow \begin{bmatrix} 10 & 0 & \vdots & -1 \\ 0 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 1 & \vdots & -2 \end{bmatrix} \Longrightarrow \begin{cases} a_1' = & -1 \\ a_2' = & 2 \\ a_3' = & -2 \end{cases}.$$

$$L'(\mathbf{v}') = L'(-\mathbf{v_1'} + \mathbf{v_2'} - 2\mathbf{v_3'})$$

$$= \begin{bmatrix} 0 & 1 & -7 & -3 \end{bmatrix} + \begin{bmatrix} -6 & -1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 2 & -8 & -4 \end{bmatrix} = \begin{bmatrix} -4 & 2 & -13 & -4 \end{bmatrix}$$

- The second way is to find the standard matrix A' representing L'.

$$\begin{bmatrix} 1 & 2 & 2 & \vdots & 0 & -1 & 7 & 3 \\ 3 & 1 & 0 & \vdots & -6 & -1 & 2 & 3 \\ 1 & 1 & 1 & \vdots & -1 & -1 & 4 & 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 & 2 & \vdots & 0 & -1 & 7 & 3 \\ 0 & -5 & -6 & \vdots & -6 & 2 & -19 & -6 \\ 0 & -1 & -1 & \vdots & -1 & 0 & -3 & -1 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 1 & 2 & 2 & \vdots & 0 & -1 & 7 & 3 \\ 0 & 1 & 0 & \vdots & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & \vdots & 1 & 0 & 3 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 & \vdots & -2 & -1 & 1 & 1 \\ 0 & 1 & 0 & \vdots & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & \vdots & 1 & -2 & 4 & 1 \end{bmatrix}$$

Hence

$$A' = \begin{bmatrix} -2 & -1 & 1 & 1 \\ 0 & 2 & -1 & 0 \\ 1 & -2 & 4 & 1 \end{bmatrix}$$

$$\Longrightarrow L'(\mathbf{v}') = \mathbf{v}' \cdot A' = \begin{bmatrix} 0 & -3 & -4 \end{bmatrix} \cdot \begin{bmatrix} -2 & -1 & 1 & 1 \\ 0 & 2 & -1 & 0 \\ 1 & -2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 2 & -13 & -4 \end{bmatrix}$$

18.(6 points)

$$x = \frac{\begin{vmatrix} 1 & 3 & -1 \\ 8 & 5 & 2 \\ -1 & -2 & -3 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & -1 \\ 3 & 5 & 2 \\ 1 & -2 & -3 \end{vmatrix}} = \frac{66}{22} = 3,\ y = \frac{\begin{vmatrix} 2 & 1 & -1 \\ 3 & 8 & 2 \\ 1 & -1 & -3 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & -1 \\ 3 & 5 & 2 \\ 1 & -2 & -3 \end{vmatrix}} = \frac{-22}{22} = -1,\ z = \frac{\begin{vmatrix} 2 & 3 & 1 \\ 3 & 5 & 8 \\ 1 & -2 & -1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & -1 \\ 3 & 5 & 2 \\ 1 & -2 & -3 \end{vmatrix}} = \frac{44}{22} = -2.$$

19.(8 points)

(a) **(3 points)** The characteristic equation is

$$\det(\lambda I_3 - A) = 0 \Longrightarrow \lambda^3 - 9\lambda^2 + 24\lambda - 20 = 0.$$

Testing all possible rational roots, $\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20$, we find $\lambda = 2, 5$

$$\Longrightarrow \lambda^3 - 9\lambda^2 + 24\lambda - 20 = (\lambda - 5)(\lambda^2 - 4\lambda + 4) = (\lambda - 5)(\lambda - 2)^2 = 0 \Longrightarrow \lambda = 2, 2, 5$$

(b) **(3 points)** For $\lambda = 2$, let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigenvector. Then

$$(\lambda I_3 - A)\mathbf{x} = \mathbf{0} \Longrightarrow \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} -1 & -1 & -1 & \vdots & 0 \\ -1 & -1 & -1 & \vdots & 0 \\ -1 & -1 & -1 & \vdots & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

$$\Longrightarrow \begin{cases} x_1 = & -x_2 - x_3 \\ x_2, x_3 & \text{arbitrary} \end{cases} \Longrightarrow \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\Longrightarrow \mathbf{x} \in \operatorname{span}\left\{ \mathbf{u_1} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

For $\lambda = 5$, let $\mathbf{x}' = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigenvector. Then

$$(\lambda I_3 - A)\mathbf{x}' = \mathbf{0} \Longrightarrow \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 2 & -1 & -1 & \vdots & 0 \\ -1 & 2 & -1 & \vdots & 0 \\ -1 & -1 & 2 & \vdots & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & -2 & 1 & \vdots & 0 \\ 0 & 3 & -3 & \vdots & 0 \\ 0 & -3 & 3 & \vdots & 0 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 1 & 0 & -1 & \vdots & 0 \\ 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix} \Longrightarrow \begin{cases} x_1 = & x_3 \\ x_2 = & x_3 \\ x_3 & \text{arbitrary} \end{cases} \Longrightarrow \mathbf{x_2} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Longrightarrow \mathbf{x}' \parallel \mathbf{u_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(c) **(2 points)** According the theorem, we know a choice is

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

To find P, we need to find three orthonormal eigenvectors. From theorems, we know $\mathbf{u_3} \perp \mathrm{span}\{\mathbf{u_1}, \mathbf{u_2}\}$; therefore, we only need to find an orthogonal basis for $\mathrm{span}\{\mathbf{u_1}, \mathbf{u_2}\}$.

$$\mathbf{v_1} = \mathbf{u_1} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v_2} = \mathbf{u_2} - \frac{\mathbf{u_2} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}} \mathbf{v_1} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}.$$

Let $\mathbf{v_3} = \mathbf{u_3}$. Then $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is an orthogonal basis.

$$\mathbf{w_1} = \frac{\mathbf{v_1}}{\|\mathbf{v_1}\|} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \ \mathbf{w_2} = \frac{\mathbf{v_2}}{\|\mathbf{v_2}\|} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}, \ \mathbf{w_3} = \frac{\mathbf{v_3}}{\|\mathbf{v_3}\|} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}.$$

$$P = \begin{bmatrix} \mathbf{w_1} & \mathbf{w_2} & \mathbf{w_3} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}.$$

The diagonal entries of D are from the eigenvalues of A, and the column vectors of P are from the orthonormal eigenvectors.