

Solution for Final Sample Problems

1.

- (a) The rank of A is 3 since there are three leading ones in $\mathbf{rref}A$.
- (b) The nullity of A is 2. $\text{nullity} + \text{rank} = \# \text{ of columns} \implies \text{nullity} + 3 = 5 \implies \text{nullity} = 2$.
Or, $\text{nullity} = \# \text{ of columns without leading one} = 2$.
- (c) The leading ones in $\mathbf{rref}A$ lie on the first, second, and fifth columns. Therefore, the corresponding columns from \mathbf{A} (not $\mathbf{rref}A$.) form a basis of the column space.

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 1 \\ 4 \end{bmatrix} \right\}.$$

- (d) The other basis for the column space of A can be obtained by the survived rows in $\mathbf{rref}A^T$. Of course, we need to take the transpose again to make them as column vectors.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

- (e) The leading ones in $\mathbf{rref}A^T$ lie on the first, second, and third columns. Therefore, the corresponding rows from \mathbf{A} (not $\mathbf{rref}A^T$.) form a basis of the row space.

$$\{[1 \ 0 \ -2 \ 1 \ 3], [-1 \ 1 \ 5 \ -1 \ -3], [0 \ 2 \ 6 \ 0 \ 1]\}.$$

- (f) The other basis for the row space of A can be obtained by the survived columns in $\mathbf{rref}A$. Of course, we need to take the transpose again to make them as row vectors.

$$\{[1 \ 0 \ -2 \ 1 \ 0], [0 \ 1 \ 3 \ 0 \ 0], [0 \ 0 \ 0 \ 0 \ 1]\}.$$

- (g) Look at $\mathbf{rref}A$. The null space of A can be expressed as

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \begin{cases} x_1 = 2x_3 - x_4 \\ x_2 = -3x_3 \\ x_5 = 0 \end{cases} \right\} \implies \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_3 - x_4 \\ -3x_3 \\ x_3 \\ x_4 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} = \text{span} \left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

- (h) The orthogonal complement of the row space of A is the same as the null space of A . Therefore,

$$\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

- (i) Set up the homogeneous system $A\mathbf{x} = \mathbf{0}$. If we would like to write the third column as a linear combination of the other columns, it is the same as we set $x_3 = 1$. Then

$$\begin{cases} x_1 = 2x_3 - x_4 \\ x_2 = -3x_3 \\ x_5 = 0 \end{cases} \implies \begin{cases} x_1 = 2 - x_4 \\ x_2 = -3 \\ x_5 = 0 \end{cases}.$$

For instance, we can set $x_4 = 0$. Then $x_1 = 2, x_2 = -3$. It means that

$$2 \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} - 3 \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} -2 \\ 5 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} -2 \\ 5 \\ 6 \\ 1 \end{bmatrix} = -2 \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + 3 \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}.$$

2.

- (a) The rank of A is 2 since there are two leading ones in $\mathbf{rref}A$.
 (b) The nullity of A is 2. $\text{nullity} + \text{rank} = \# \text{ of columns} \implies \text{nullity} + 2 = 4 \implies \text{nullity} = 2$.
 Or, $\text{nullity} = \# \text{ of columns without leading one} = 2$.
 (c) The leading ones in $\mathbf{rref}A$ lie on the first and third columns. Therefore, the corresponding columns from \mathbf{A} (not $\mathbf{rref}A$), form a basis of the column space.

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

- (d) The other basis for the column space of A can be obtained by the survived rows in $\mathbf{rref}A^T$. Of course, we need to take the transpose again to make them as column vectors.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

- (e) The leading ones in $\mathbf{rref}A^T$ lie on the first, second, and third columns. Therefore, the corresponding rows from \mathbf{A} (not $\mathbf{rref}A^T$), form a basis of the row space.

$$\{[1 \ 2 \ 1 \ -1], [2 \ 4 \ 1 \ -4]\}.$$

- (f) The other basis for the row space of A can be obtained by the survived columns in $\mathbf{rref}A$. Of course, we need to take the transpose again to make them as row vectors.

$$\{[1 \ 2 \ 0 \ -3], [0 \ 0 \ 1 \ 2]\}.$$

- (g) Look at $\mathbf{rref}A$. The null space of A can be expressed as

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \begin{cases} x_1 = -2x_2 + 3x_4 \\ x_3 = -2x_4 \end{cases} \right\} \implies \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 + 3x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\} = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

(h) The orthogonal complement of the row space of A is the same as the null space of A . Therefore,

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

(i) Since the rank of A is 2, the dimension of the row space of A is 2. If row vectors span \mathbb{R}^4 , then the dimension of the row space must be 4. This is not true.

3.

(a) The $\text{ref}[A|\mathbf{b}]$ is

$$\left[\begin{array}{ccc|c} 1 & 2a & a & 1 \\ 1 & 2a & 1 & -1 \\ 0 & 1 & a & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2a & a & 1 \\ 0 & 0 & 1-a & -1 \\ 0 & 1 & a & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2a & a & 1 \\ 0 & 1 & a & 1 \\ 0 & 0 & 1-a & -1 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

If $1 - a = 0$, then the third equation becomes $0 = 1$, which is inconsistent. This is not the case that we want. Assume $1 - a \neq 0$. Then we can solve $z = -1/(1 - a)$, and backward substitute to solve y and x ; the system has a unique solution. Therefore, the condition on a such that the system has a unique solution is $a \neq 1$.

There is the other way to do it. A system has a unique solution if and only if the rank is equal to the number of variables; in this case, the rank is 3. This leads that the determinant of the coefficient matrix is non-zero.

$$\begin{vmatrix} 1 & 2a & a \\ 1 & 2a & 1 \\ 0 & 1 & a \end{vmatrix} \neq 0 \Rightarrow -(a-1) \neq 0 \Rightarrow a \neq 1.$$

(b) If $x = 3, y = -1, z = 1$ is a solution, then

$$\begin{cases} 1 \cdot 3 + (2a)(-1) + a \cdot 1 = 1 \\ 1 \cdot 3 + (2a)(-1) + 1 \cdot 1 = 0 \\ 0 \cdot 3 + 1 \cdot (-1) + a \cdot 1 = 1 \\ 0 \cdot 3 + 0 \cdot (-1) + 0 \cdot 1 = 0 \end{cases} \Rightarrow \begin{cases} -a = -2 \\ -2a = -4 \\ a = 2 \\ 0 = 0 \end{cases} \Rightarrow \begin{cases} a = 2 \\ a = 2 \\ a = 2 \\ 0 = 0 \end{cases} \Rightarrow a = 2.$$

4.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 5 \\ 2 & 3 & (a^2-3) & a+3 \end{array} \right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & (a^2-5) & a-1 \end{array} \right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & (a^2-4) & a-2 \end{array} \right].$$

Look at the last equation.

$$(a^2 - 4)z = a - 2.$$

If $a^2 - 4 = 0, a = -2, 2$, we have a consistent problem.

(a) If $a = -2$, then the equation becomes $0 = -2 - 2 = -4$; the system is not consistent. Hence, there is no solution.

(b) If $a \neq 2, -2$, we can solve $z = 1/(a + 2)$ and backward substitute to get y, x . Therefore, the system has a unique solution.

There is the other way to do it. A system has a unique solution if and only if the rank is equal to the number of variables; in this case, the rank is 3. This leads that the determinant of the coefficient matrix is non-zero.

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 2 & 3 & (a^2-3) \end{vmatrix} \neq 0 \Rightarrow a^2 - 4 \neq 0 \Rightarrow a \neq 2, -2.$$

(c) If $a = 2$, then the equation becomes $0 = 2 - 2 = 0$; the system is consistent and z is arbitrary. Hence, there are infinitely many solutions.

5. The standard matrix A for L is

$$A = [L(\mathbf{e}_1) \quad L(\mathbf{e}_2) \quad L(\mathbf{e}_3)] = \left[L \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) \quad L \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) \quad L \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \right] = \begin{bmatrix} -3 & 2 & -1 \\ 1 & -2 & 2 \\ 2 & 1 & 4 \end{bmatrix}.$$

6.

(a) The augmented matrix is

$$\begin{aligned} \left[\begin{array}{ccc|c} -2 & 3 & 1 & -1 \\ 1 & 2 & 1 & -3 \\ -2 & -1 & -1 & 3 \end{array} \right] &\implies \left[\begin{array}{ccc|c} 1 & 2 & 1 & -3 \\ -2 & 3 & 1 & -1 \\ -2 & -1 & -1 & 3 \end{array} \right] \implies \left[\begin{array}{ccc|c} 1 & 2 & 1 & -3 \\ 0 & 7 & 3 & -7 \\ 0 & 3 & 1 & -3 \end{array} \right] \\ &\implies \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array} \right] \implies \begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix} = - \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}. \end{aligned}$$

(b)

$$\begin{aligned} L \left(\begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix} \right) &= L \left(- \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right) \\ &= -L \left(\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \right) - L \left(\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right) + 0 \cdot \left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right) = - \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}. \end{aligned}$$

7. The standard matrix A for L is

$$A = [L(\mathbf{e}_1) \quad L(\mathbf{e}_2) \quad L(\mathbf{e}_3)] = \left[L \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) \quad L \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) \quad L \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \right] = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 2 & 1 \\ 3 & -1 & 0 \end{bmatrix}.$$

8.

$$A \cdot \text{adj } A = (\det A) \cdot I_5 \implies A^2 \cdot (\text{adj } A)^2 = A \cdot A \cdot \text{adj } A \cdot \text{adj } A$$

$$= A \cdot \det AI_5 \cdot \text{adj } A = (\det AI_5) \cdot A \cdot \text{adj } A = (\det AI_5) \cdot (\det AI_5) = (\det A)^2 I_5 = 9 \cdot I_5 = \begin{bmatrix} 9 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 9 \end{bmatrix}.$$

$$\det (A^2 \cdot (\text{adj } A)^2) = \det ((\det A)^2 I_5) = ((\det A)^2)^5 \cdot \det I_5 = (3^2)^5 \cdot 1 = 3^{10}.$$

9.

(a) $8 + 0 + (-2) - 3 - 8 - 0 = -5$.

(b) $2 + 12 + (-8) - 12 - (-8) - 2 = 0$.

10.

- (a) The vectors $t^2 + 2t, 3t^2 + t - 1$ are linearly independent. Therefore, S is a basis. There is no solution for the system $a_1(t^2 + 2t) + a_2(3t^2 + t - 1) = 6t^2 - 1$. Hence, the vector $6t^2 - 1$ does not belong to $\text{span } S$.
- (b) At first, we check W is a subspace. We only need to check two conditions; one is W closed under addition, and the other is W closed under scalar multiplication.

(1) Addition.

$$\mathbf{w}_1 = \begin{bmatrix} a_1 & b_1 \\ 0 & 3b_1 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} a_2 & b_2 \\ 0 & 3b_2 \end{bmatrix} \implies \mathbf{w}_1 + \mathbf{w}_2 = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ 0 & 3(b_1 + b_2) \end{bmatrix}.$$

Let $a = a_1 + a_2, b = b_1 + b_2$.

$$\mathbf{w}_1 + \mathbf{w}_2 = \begin{bmatrix} a & b \\ 0 & 3b \end{bmatrix} \in \mathbf{W}.$$

Therefore, W is closed under addition.

(2) Scalar multiplication.

$$c \in \mathbb{R}, \mathbf{w} = \begin{bmatrix} a' & b' \\ 0 & 3b' \end{bmatrix} \in W \implies c\mathbf{w} = \begin{bmatrix} ca' & cb' \\ 0 & 3cb' \end{bmatrix}.$$

Choose $a = ca', b = cb'$. We have

$$c\mathbf{w} = \begin{bmatrix} ca' & cb' \\ 0 & 3cb' \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 3b \end{bmatrix} \in W.$$

Therefore, W is closed under scalar multiplication.

$$\begin{bmatrix} a & b \\ 0 & 3b \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} \implies W = \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} \right\}.$$

These two matrices are linearly independent. Therefore they form a basis for W and $\dim W = 2$.

11. Let A be the 3×3 matrix. The column vectors of A are linearly independent if and only if the determinant of A is non-zero.

$$\begin{vmatrix} 1 & 1 & 0 \\ 3 & 1 & 1 \\ d & 0 & 1 \end{vmatrix} \neq 0 \implies 1 + 0 + d - 0 - 0 - 3 \neq 0 \implies d \neq 2.$$

12.

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = 4, \quad A_{12} = (-1)^{1+2} \begin{vmatrix} -2 & 3 \\ 2 & 1 \end{vmatrix} = 8, \quad A_{13} = (-1)^{1+3} \begin{vmatrix} -2 & 1 \\ 2 & -1 \end{vmatrix} = 0,$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} = -1, \quad A_{22} = (-1)^{2+2} \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} = 0, \quad A_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 0 \\ 2 & -1 \end{vmatrix} = 2,$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix} = -1, \quad A_{32} = (-1)^{3+2} \begin{vmatrix} 2 & 1 \\ -2 & 3 \end{vmatrix} = -8, \quad A_{33} = (-1)^{3+3} \begin{vmatrix} 2 & 0 \\ -2 & 1 \end{vmatrix} = 2.$$

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 4 & -1 & -1 \\ 8 & 0 & -8 \\ 0 & 2 & 2 \end{bmatrix}.$$

13.

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A = -\frac{1}{13} \begin{bmatrix} 2 & 3 & -2 \\ 3 & -2 & -3 \\ -11 & -10 & -2 \end{bmatrix}.$$

14. A is non-singular if and only if $\det A \neq 0$.

$$\begin{vmatrix} 1 & 2 & 1 \\ 0 & -3 & 5 \\ 1 & a & 6 \end{vmatrix} \neq 0 \implies -5a - 5 \neq 0 \implies a \neq -1.$$

15. It is not a vector space. It fails the following condition:

$$(c_1 + c_2) \odot (x, y) = (c_1 \odot (x, y)) \oplus (c_2 \odot (x, y)).$$

Let $c_1 = 2, c_2 = 3, x = 1, y = 1$

$$(2 + 3) \odot (1, 1) = 5 \odot (1, 1) = (5 \cdot 1, 1) = (5, 1)$$

On the other hand,

$$(2 \odot (1, 1)) \oplus (3 \odot (1, 1)) = (2, 1) \oplus (3, 1) = (5, 2) \neq (5, 1).$$

16. Let λ be an eigenvalue of A and \mathbf{v} an eigenvector associated with λ . Then

$$A^2 \mathbf{v} = A(A\mathbf{v}) = A(\lambda \mathbf{v}) = \lambda(A\mathbf{v}) = \lambda(\lambda \mathbf{v}) = \lambda^2 \mathbf{v}.$$

This means that λ^2 is an eigenvalue of A^2 and \mathbf{v} an eigenvector associated with λ^2 . The same argument can imply that λ^3 is an eigenvalue of A^3 and \mathbf{v} an eigenvector associated with λ^3 .

Hence, $1^3, 3^3, 5^3$ are eigenvalues for A^3 and \mathbf{v} is an eigenvector for A^3 .

17. Let

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 3 \\ -3 \\ 0 \\ -2 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ -3 \end{bmatrix}.$$

$$\mathbf{v}_1 = \mathbf{u}_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \mathbf{u}_2 - \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -3/2 \\ 0 \\ 1 \end{bmatrix}.$$

It is no hurt that we use $2 \begin{bmatrix} 3/2 \\ -3/2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 0 \\ 2 \end{bmatrix}$ as \mathbf{v}_2 .

$$\mathbf{v}_3 = \mathbf{u}_3 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \right) \mathbf{v}_1 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \right) \mathbf{v}_2 = \begin{bmatrix} 3 \\ -3 \\ 0 \\ -2 \end{bmatrix} - \begin{pmatrix} 0 \\ 2 \end{pmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \left(\frac{14}{22} \right) \begin{bmatrix} 3 \\ -3 \\ 0 \\ 2 \end{bmatrix} = \frac{4}{11} \begin{bmatrix} 1 \\ -1 \\ 0 \\ -3 \end{bmatrix}.$$

Again, it is no hurt that we use $\begin{bmatrix} 1 \\ -1 \\ 0 \\ -3 \end{bmatrix}$ as \mathbf{v}_3 .

$$\mathbf{v}_4 = \mathbf{u}_4 - \left(\frac{\mathbf{u}_4 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right) \mathbf{v}_1 - \left(\frac{\mathbf{u}_4 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right) \mathbf{v}_2 - \left(\frac{\mathbf{u}_4 \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3}\right) \mathbf{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ -3 \end{bmatrix} - \left(\frac{-1}{2}\right) \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \left(\frac{3}{22}\right) \begin{bmatrix} 3 \\ -3 \\ 0 \\ 2 \end{bmatrix} - \left(\frac{12}{11}\right) \begin{bmatrix} 1 \\ -1 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ -3 \end{bmatrix} \right\}.$$

Note that the fourth vector is the zero vector; we can not put it into a basis.

18.

(a)

$$W^\perp = \left\{ \mathbf{w} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, x_1 + x_2 - x_3 = 0 \right\} \implies x_1 = -x_2 + x_3$$

$$\implies \mathbf{w} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \implies W^\perp = \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(b) Use the Gram-Schmidt process to produce an orthogonal basis and then normalize them to be orthonormal.

$$\left\{ \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix} \right\}.$$

19.

(a) First, use the Gram-Schmidt to find an orthogonal basis for W first. An orthogonal basis for W is

$$\left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

$$\text{Proj}_W \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 + \frac{\mathbf{v} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2 + \frac{\mathbf{v} \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3} \mathbf{v}_3 = \frac{-9}{9} \begin{bmatrix} 1 \\ 2 \\ 0 \\ -2 \end{bmatrix} + \frac{3}{6} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} + \frac{9}{18} \begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 2 \end{bmatrix}.$$

(b)

$$\mathbf{w} = \text{Proj}_{\mathbf{W}} \mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 2 \end{bmatrix}. \quad \mathbf{u} = \text{Proj}_{\mathbf{W}^\perp} \mathbf{v} = \mathbf{v} - \mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix}.$$

(c) The distance d from \mathbf{v} to W is the length of $\text{Proj}_{W^\perp} \mathbf{v} = \mathbf{u}$.

$$d = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}.$$

20.

- (a) False. The uniquely solvable case only happens when the rank is the same as the number of variables (columns).
- (b) False. The uniquely solvable case only happens when the rank is the same as the number of variables (columns).
- (c) False. The system always has solutions if and only if the rank is equal to the number of equations (rows).
- (d) True. Since the rank is less than the number of variables, we have extra freedom. Therefore, if there is a solution, we have infinitely many solutions.
- (e) True. Since the rank is less than the number of equations (rows), there are some \mathbf{b} such that the system is not solvable.

21.

- (a) False. For example $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.
- (b) True.
- (c) True.
- (d) True.
- (e) False. $(A - A^T)^T = A^T - A \neq A - A^T$.
- (f) False. $\det(\text{adj } A) = (\det A)^{n-1} = (\det A)^2$.

22.

- (a) Yes.
- (b) No. No zero vector here.
- (c) Yes.
- (d) Yes.
- (e) No. No zero vector here.

23.

- (a) Yes.
- (b) Yes
- (c) No. $\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 1 & 4 \end{vmatrix} = 0$.
- (d) No. The number of vectors is greater than the dimension.
- (e) Yes.

24.

- (a) 3.

- (b) $21 = 3 \cdot 7$.
 (c) 0, since two columns are the same.
 (d) 3.
 (e) $\frac{1}{5^3 \cdot 3}$.

$$\det((5A)^{-1}) = \frac{1}{\det(5A)} = \frac{1}{5^3 \det(A)} = \frac{1}{5^3 \cdot 3}.$$

25.

- (a) 0. Two rows are the same.
 (b) 0. Two columns are the same.
 (c) 0. There is a zero row.
 (d) $-40 = 2 \cdot 2 \cdot 5 \cdot (-1) \cdot 2$.
 (e) -24 .

$$\begin{vmatrix} 2 & 3 & 2 & -1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} \cdot \begin{vmatrix} 5 & 1 \\ 1 & -1 \end{vmatrix} = 4 \cdot (-5 - 1) = -24.$$

26.

$$\begin{aligned} p(\lambda) = \det(\lambda I_3 - A) &= \begin{vmatrix} \lambda - 2 & -1 & -2 \\ 3 & \lambda - 1 & 1 \\ 0 & 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1) \begin{vmatrix} \lambda - 2 & -1 \\ 3 & \lambda - 1 \end{vmatrix} \\ &= (\lambda - 1)((\lambda - 2)(\lambda - 1) + 3) = (\lambda - 1)(\lambda^2 - 3\lambda + 5). \end{aligned}$$

The characteristic equation is $(\lambda - 1)(\lambda^2 - 3\lambda + 5) = 0$. $\lambda = 1, \frac{3 \pm \sqrt{11}i}{2}$.

27.

- (a) $\lambda = 1, 1, 4$.
 (b) For $\lambda = 1$,

$$\lambda I_3 - A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 = -x_2 - x_3 \\ x_2, x_3 \text{ arb.} \end{cases} \Rightarrow x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

For $\lambda = 4$,

$$\lambda I_3 - A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 = x_3 \\ x_2 = x_3 \\ x_3 \text{ arb.} \end{cases} \Rightarrow x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$1 \rightsquigarrow \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad 4 \rightsquigarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(c)

$$1 \rightsquigarrow \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, 4 \rightsquigarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow 1 \rightsquigarrow \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, 4 \rightsquigarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow 1 \rightsquigarrow \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}, 4 \rightsquigarrow \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}.$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix},$$

where the diagonal entries of D are from the eigenvalues of A , and the column vectors of P are from the associated orthonormal eigenvectors.

28.

$$\hat{A} = A^T \cdot A = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}, \quad \hat{\mathbf{b}} = A^T \cdot \mathbf{b} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

29. Consider a line defined by

$$y = a + bx.$$

We would like to determine the best a and b .

$$\begin{cases} a + b = 2 \\ a + 2b = 1 \\ a + 3b = 3 \\ a + 4b = 3 \end{cases} \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 3 \end{bmatrix}.$$

Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 3 \end{bmatrix}.$$

The new system is

$$(A^T \cdot A)\hat{\mathbf{x}} = A^T \cdot \mathbf{b} \Rightarrow \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 9 \\ 25 \end{bmatrix} \Rightarrow \hat{\mathbf{x}} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}.$$

Hence, the least square fit line is

$$y = 1 + \frac{1}{2}x.$$

30.

(a) $\lambda = 6, 4$

(b)

$$\mathbf{x}(t) = c_1 \begin{bmatrix} -1 \\ 4 \end{bmatrix} e^{6t} + c_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{4t},$$

where c_1, c_2 are arbitrary.

(c) Set

$$\begin{bmatrix} 3 \\ -10 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} \implies \begin{cases} -c_1 - c_2 = 3 \\ 4c_1 + 2c_2 = -10 \end{cases} \implies c_1 = -2, c_2 = -1.$$

Therefore,

$$\mathbf{x}(t) = -2 \begin{bmatrix} -1 \\ 4 \end{bmatrix} e^{6t} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{4t}.$$

31.

(a) The characteristic equation is $\lambda^2 + 3 = 0 \implies \lambda = \pm\sqrt{3}i$.

(b) For $\lambda = i$, assume that $\mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector associated to i . Then

$$(\lambda I_2 - A)\mathbf{v} = \mathbf{0} \implies \begin{bmatrix} \sqrt{3}i - 1 & 2 \\ -2 & \sqrt{3}i + 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The augmented matrix is

$$\left[\begin{array}{cc|c} \sqrt{3}i - 1 & 2 & 0 \\ -2 & \sqrt{3}i + 1 & 0 \end{array} \right] \implies \left[\begin{array}{cc|c} -2 & \sqrt{3}i + 1 & 0 \\ \sqrt{3}i - 1 & 2 & 0 \end{array} \right] \implies \left[\begin{array}{cc|c} 1 & -(\sqrt{3}i + 1)/2 & 0 \\ 0 & 0 & 0 \end{array} \right] \implies x_1 = \frac{\sqrt{3}i + 1}{2} x_2$$

$$\implies \mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}i + 1}{2} x_2 \\ x_2 \end{bmatrix} = \frac{x_2}{2} \begin{bmatrix} \sqrt{3}i + 1 \\ 2 \end{bmatrix}.$$

Therefore, an eigenvector can be

$$\begin{bmatrix} \sqrt{3}i + 1 \\ 2 \end{bmatrix}.$$

For the complex root cases, we do not need to compute the other eigenvalue and eigenvector. The real part and the imaginary part of the solution $\mathbf{v}e^{\lambda t}$ will be two linearly independent solutions.

$$\begin{aligned} \mathbf{v}e^{\lambda t} &= \begin{bmatrix} \sqrt{3}i + 1 \\ 2 \end{bmatrix} e^{\sqrt{3}it} = \begin{bmatrix} \sqrt{3}i + 1 \\ 2 \end{bmatrix} (\cos \sqrt{3}t + i \sin \sqrt{3}t) = \begin{bmatrix} \cos \sqrt{3}t - \sqrt{3} \sin \sqrt{3}t + i(\sqrt{3} \cos \sqrt{3}t + \sin \sqrt{3}t) \\ 2 \cos \sqrt{3}t + i2 \sin \sqrt{3}t \end{bmatrix} \\ &= \begin{bmatrix} \cos \sqrt{3}t - \sqrt{3} \sin \sqrt{3}t \\ 2 \cos \sqrt{3}t \end{bmatrix} + i \begin{bmatrix} \sqrt{3} \cos \sqrt{3}t + \sin \sqrt{3}t \\ 2 \sin \sqrt{3}t \end{bmatrix}. \end{aligned}$$

Two independent solutions are

$$\begin{bmatrix} \cos \sqrt{3}t - \sqrt{3} \sin \sqrt{3}t \\ 2 \cos \sqrt{3}t \end{bmatrix}, \quad \begin{bmatrix} \sqrt{3} \cos \sqrt{3}t + \sin \sqrt{3}t \\ 2 \sin \sqrt{3}t \end{bmatrix}.$$

The general solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} \cos \sqrt{3}t - \sqrt{3} \sin \sqrt{3}t \\ 2 \cos \sqrt{3}t \end{bmatrix} + c_2 \begin{bmatrix} \sqrt{3} \cos \sqrt{3}t + \sin \sqrt{3}t \\ 2 \sin \sqrt{3}t \end{bmatrix},$$

where c_1, c_2 are arbitrary.

(c) The initial value is

$$\begin{aligned} \mathbf{x}(0) &= \begin{bmatrix} 5 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} \cos \sqrt{3} \cdot 0 - \sqrt{3} \sin \sqrt{3} \cdot 0 \\ 2 \cos \sqrt{3} \cdot 0 \end{bmatrix} + c_2 \begin{bmatrix} \sqrt{3} \cos \sqrt{3} \cdot 0 + \sin \sqrt{3} \cdot 0 \\ 2 \sin \sqrt{3} \cdot 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} \sqrt{3} \\ 0 \end{bmatrix} \\ \implies \begin{cases} c_1 + \sqrt{3}c_2 = 5 \\ 2c_1 = 1 \end{cases} &\implies c_1 = \frac{1}{2}, c_2 = \frac{3\sqrt{3}}{2} \implies \mathbf{x}(t) = \frac{1}{2} \begin{bmatrix} \cos \sqrt{3}t - \sqrt{3} \sin \sqrt{3}t \\ 2 \cos \sqrt{3}t \end{bmatrix} + \frac{3\sqrt{3}}{2} \begin{bmatrix} \sqrt{3} \cos \sqrt{3}t + \sin \sqrt{3}t \\ 2 \sin \sqrt{3}t \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 10 \cos \sqrt{3}t + 2\sqrt{3} \sin \sqrt{3}t \\ 2 \cos \sqrt{3}t + 6\sqrt{3} \sin \sqrt{3}t \end{bmatrix} = \begin{bmatrix} 5 \cos \sqrt{3}t + \sqrt{3} \sin \sqrt{3}t \\ \cos \sqrt{3}t + 3\sqrt{3} \sin \sqrt{3}t \end{bmatrix} \\ \implies x_1(t) &= 5 \cos \sqrt{3}t + \sqrt{3} \sin \sqrt{3}t, x_2(t) = \cos \sqrt{3}t + 3\sqrt{3} \sin \sqrt{3}t. \end{aligned}$$

32.

- (a) $\lambda = 1, 2, 3$
 (b)

$$\mathbf{x}(t) = c_1 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} e^t + c_2 \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} e^{3t},$$

where c_1, c_2, c_3 are arbitrary.

(c) Set

$$\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} \implies c_1 = 1, c_2 = 1, c_3 = -1$$

Therefore,

$$\mathbf{x}(t) = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} e^t + \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} e^{2t} - \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} e^{3t}.$$