Math 265 Quiz#6 : 3.1-3.4

For **Division 7, Section 3**:
Let

$$S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 4 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \\ 3 \end{bmatrix} \right\}, \quad W = \operatorname{span} S, \quad \mathbf{v} = \begin{bmatrix} -3 \\ 3 \\ -1 \\ -1 \end{bmatrix} \in W.$$

1. **5 points.** Find an orthogonal basis for W.
2. **2 points.** Find an orthonormal basis T for W.
3. **1 points.** Find $\dim W$.
4. **1 points.** Find $[\mathbf{v}]_T$, where T is an ordered basis in (2).

SOLUTION.

1. **5 points.** Use the Gram-Schmidt process.

$$\mathbf{v_1} = \mathbf{u_1} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{v_2} = \mathbf{u_2} - \frac{\mathbf{u_2} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}} \mathbf{v_1} = \begin{bmatrix} -4 \\ 4 \\ -2 \\ -2 \end{bmatrix} - \frac{\mathbf{-12}}{\mathbf{4}} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}. \quad \mathbf{v_3} = \mathbf{v_4} = \mathbf{0}.$$

Therefore, $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is an orthogonal basis for W.

2. **2 points.**

$$T = \left\{ \mathbf{w_1} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{w_2} = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is an orthonormal basis for W.

3. **1 points.** The dimension is 2 since there are two members in a basis for W.
4. **2 points.**

$$[\mathbf{v}]_T = \begin{bmatrix} \mathbf{v} \cdot \mathbf{w_1} \\ \mathbf{v} \cdot \mathbf{w_2} \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}.$$

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For **Division 8, Section 2**:
Let

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \\ 3 \end{bmatrix} \right\}, \quad W = \operatorname{span} S, \quad \mathbf{v} = \begin{bmatrix} -2 \\ 0 \\ 2 \\ 0 \end{bmatrix} \in W.$$

1. **5 points.** Find an orthogonal basis for W.
2. **2 points.** Find an orthonormal basis T for W.
3. **1 points.** Find $\dim W$.
4. **1 points.** Find $[\mathbf{v}]_T$, where T is an ordered basis in (2).

SOLUTION.

1. **5 points.** Use the Gram-Schmidt process.

$$\mathbf{v_1} = \mathbf{u_1} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \; \mathbf{v_2} = \mathbf{u_2} - \frac{\mathbf{u_2} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}} \mathbf{v_1} = \begin{bmatrix} -3 \\ -1 \\ 3 \\ -1 \end{bmatrix} - \frac{\mathbf{-8}}{\mathbf{4}} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}. \quad \mathbf{v_3} = \mathbf{v_4} = \mathbf{0}.$$

Therefore, $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is an orthogonal basis for W.

2. **2 points.**

$$T = \left\{ \mathbf{w_1} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \; \mathbf{w_2} = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is an orthonormal basis for W.

3. **1 points.** The dimension is 2 since there are two members in a basis for W.
4. **2 points.**

$$[\mathbf{v}]_T = \begin{bmatrix} \mathbf{v} \cdot \mathbf{w_1} \\ \mathbf{v} \cdot \mathbf{w_2} \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}.$$

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