Math 265 Quiz#7: 3.5, 3.6

For **Division 7, Section 3** and **Division 8, Section 2**:
Let

$$\mathbf{u_1} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{u_3} \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \mathbf{A} = \begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} & \mathbf{u_3} \end{bmatrix}, \ \mathbf{S} = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}, \ \mathbf{W} = \operatorname{span} \mathbf{S}, \ \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

1. **5 points.** Find an orthogonal basis for $W^\perp$.
2. **1 points.** Find the sum of the dimension W and the dimension $W^\perp$.
3. **3 points.** Find the least square solution for the system $A\mathbf{x} = \mathbf{b}$.
4. **1 points.** Find the distance between $\mathbf{b}$ and W.

SOLUTION.

1. **5 points.** $W^\perp =$ null space of A^T. To find the null space of A^T, we need to solve the homogeneous system $A^T\mathbf{x} = \mathbf{0}$.

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 1 & 1 & -1 \\ 1 & 0 & 1 & -1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -3 & 3 & -3 \\ 0 & -2 & 2 & -2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The solution is

$$\begin{cases} x_1 = & -x_3 + x_4 \\ x_2 = & x_3 - x_4 \end{cases} \Longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 + x_4 \\ x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\Longrightarrow W^\perp = \operatorname{span} \left\{ \mathbf{u_1} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{u_2} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

We can use the Gram-Schmidt process to find an orthogonal basis for $W^\perp$ and normalize it to an orthonormal basis.

$$W^\perp = \operatorname{span} \left\{ \mathbf{w_1} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{w_2} = \frac{1}{\sqrt{3}} \frac{1}{\sqrt{15}} \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix} \right\}.$$

2. **1 points.** $\dim W + \dim W^{\perp} = n = 4$.
3. **3 points.** Just solve

$$(A^T \cdot A)\mathbf{x} = A^T\mathbf{b} \Longrightarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 1 & 1 & -1 \\ 1 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 1 & 1 & -1 \\ 1 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$\Longrightarrow \begin{bmatrix} 7 & 2 & -1 \\ 2 & 7 & 4 \\ -1 & 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} \Longrightarrow \mathbf{x} = \begin{cases} x_1 = & (x_3+1)/3 \\ x_2 = & (-2x_3+1)/3 \\ x_3 & \text{arb.} \end{cases}$$

4. **1 points.**

$$\mathbf{b} = \mathrm{Proj}_{\mathbf{W}}\,\mathbf{b} + \mathrm{Proj}_{\mathbf{W}^{\perp}}\,\mathbf{b} \Longrightarrow \mathbf{b} - \mathrm{Proj}_{\mathbf{W}}\,\mathbf{b} = \mathrm{Proj}_{\mathbf{W}^{\perp}}\,\mathbf{b}\frac{\mathbf{b}\cdot\mathbf{w_1}}{\mathbf{w_1}\cdot\mathbf{w_1}}\mathbf{w_1} + \frac{\mathbf{b}\cdot\mathbf{w_2}}{\mathbf{w_2}\cdot\mathbf{w_2}}\mathbf{w_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

$$\mathrm{Proj}_W\,\mathbf{b} = A\mathbf{x} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} (x_3+1)/3 \\ (-2x_3+1)/3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow \mathbf{b} - \mathrm{Proj}_W\,\mathbf{b} = \mathrm{Proj}_{W^{\perp}}\,\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Therefore, the distance between $\mathbf{b}$ and W is

$$\|\mathbf{b} - \mathrm{Proj}_W\,\mathbf{b}\| = \|\mathrm{Proj}_{W^{\perp}}\,\mathbf{b}\| = \left\| \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\| = \sqrt{1^2+1^2} = \sqrt{2}.$$

∎