EXCEPTIONAL COLLECTION OF OBJECTS ON SOME FAKE PROJECTIVE PLANES

CHING-JUI LAI, SAI-KEE YEUNG

Abstract. The purpose of the article is to explain a new method to study existence of a sequence of exceptional collection of length three for fake projective planes M with large automorphism group. This provides more examples to a question in [GKMS].

1. Introduction

1.1 A fake projective plane is a compact complex surface with the same Betti numbers as $P^2_{\mathbb{C}}$. This is a notion introduced by Mumford who also constructed the first example. All fake projective planes have recently been classified into twenty-eight non-empty classes by the work of Prasad-Yeung in [PY], which finally leads to 100 fake projective planes along those 28 classes in the work of Cartwright-Steger [CT]. It is known that a fake projective plane is a smooth complex two ball quotient, and has the smallest Euler number among smooth surfaces of general type.

Most of the fake projective planes have the property that the canonical line bundle K_M can be written as $K_M = 3L$, where L is a generator of the Neron-Severi group, see Lemma 1 for the complete list. One motivation of the present article comes from a question of Dolgachev and Prasad, who asked whether $\mathrm{H}^0(M, 2L)$ contains enough sections for geometric purposes, such as embedding of M. It is also questioned in [GKMS] whether $\mathrm{H}^0(M, 2L)$ is non-trivial.

The other motivation comes from the recent research activities surrounding the search of exceptional collections from the point of view of derived category, such as [AO], [BvBS], [F], [GS], [GKNS] and [GO].

1.2 Denote by $D^b(M)$ the bounded derived category of coherent sheaves on M. A sequence of objects $E_1, E_2, ..., E_r$ of $D^b(M)$ is called an exceptional collection if $\mathrm{Hom}(E_j, E_i[k])$ is non-zero for $j \geq i$ and $k \in \mathbb{Z}$ only when $i = j$ and $k = 0$, in which case it is one dimensional. In [GKMS], the authors consider the problem of the existence of a special type exceptional collection on an n-dimensional fake projective space.

Conjecture 1([GKMS]). *Assume that M is an n-dimensional fake projective space with the canonical class divisible by $(n+1)$. Then for some choice of $\mathcal{O}_M(1)$ such*

Key words: Fake projective planes. Exceptional collection.
AMS 2010 Mathematics subject classification: Primary 14J29, 13D09.
The first author was partially supported by a grant from the National Science Foundation.

that $\omega_M = \mathcal{O}_M(n+1)$, the sequence

$$\mathcal{O}_M, \mathcal{O}_M(-1), \ldots, \mathcal{O}_M(-n)$$

is an exceptional collection on M.

In the cases of fake projective planes ($n = 2$), it is easy to see that a necessary and sufficient condition for the above conjecture is to show that $\mathrm{H}^0(M, 2L) = 0$. This is proved in [GKMS] if $\mathrm{Aut}(M)$ has order 21. This is also proved for 2-adically uniformised fake projective planes in [F]. The main result in this note aims to provide more examples to the conjecture of [GKMS]. However, unlike in the above conjecture of [GKMS], our sequence of exceptional collection is generated by a *numerical* cubic root of K_M, see Main Theorem.

Our formulation to the above conjecture is numerical in nature and suggests the following slightly more general problem which seems to be more accessible and still serves the purpose of providing exceptional objects.

Conjecture 2. *Assume that M is an n-dimensional fake projective space with the canonical class* ***numerically*** *divisible by $(n+1)$. Then for some choice of L such that $K_M \equiv (n+1)L$ and a suitable choices of line bundles E_i's with $E_i \equiv -iL$, $1 \leq i \leq n$, the sequence*

$$\mathcal{O}_M, E_1, E_2, \ldots, E_n$$

is an exceptional collection on M.

1.3 The problem is rather subtle, since the conventional Riemann-Roch formula is not useful in this case without an appropriate vanishing theorem. The approach that we take exploits the small intersection numbers involved as well as existence of a finite group action.

We illustrate our approach by proving the above conjecture for fake projective planes with $\mathrm{Aut}(X) = C_3 \times C_3$ and $C_7 : C_3$, the latter case was proved in [GKMS]. The approach is geometric and is different from [GKMS] and [F]. We choose L to be an $\mathrm{Aut}(X)$-invariant numerical cubic root of K_M. The problem is reduced to a study of the geometry of invariant sections of $\mathrm{H}^0(M, 2L)$ if it exists. The flexibility in the choice of E_i in Conjecture 2 allows us to exploit more carefully the geometry of the fake projective planes classified in [PY] and [CS]. In particular, we make use of appropriate torsion line bundles on M.

Main Theorem. *For M a fake projective plane as listed in the Table below, there is a unique line bundle L with $K_M = 3L$. Moreover,*

(1) if $\mathrm{Aut}(M) = C_7 : C_3$, then the sequence $\mathcal{O}_M, -L, -2L$ forms an exceptional collection of M;
(2) if $\mathrm{Aut}(M) = C_3 \times C_3$, then there is a line bundle $L' \equiv L$ such that the sequence $\mathcal{O}_M, -L', -2L'$ forms an exceptional collection of M.

class	M	Aut(M)	$\mathrm{H}_1(M,\mathbb{Z})$
$(a=7,p=2,\emptyset)$	$(a=7,p=2,\emptyset,D_3,2_7)$	$C_7:C_3$	C_2^4
$(a=7,p=2,\{7\})$	$(a=7,p=2,\{7\},D_3,2_7)$	$C_7:C_3$	C_2^3
$(\mathcal{C}_2,p=2,\emptyset)$	$(\mathcal{C}_2,p=2,\emptyset,d_3,D_3)$	$C_3\times C_3$	$C_2\times C_7$
$(\mathcal{C}_2,p=2,\{3\})$	$(\mathcal{C}_2,p=2,\{3\},d_3,D_3)$	$C_3\times C_3$	C_7
$(\mathcal{C}_{18},p=3,\emptyset)$	$(\mathcal{C}_{18},p=3,\emptyset,d_3,D_3)$	$C_3\times C_3$	$C_2^2\times C_{13}$
$(\mathcal{C}_{20},\{v_2\},\emptyset)$	$(\mathcal{C}_{20},\{v_2\},\emptyset,D_3,2_7)$	$C_7:C_3$	C_2^6

We remark that the above table covers 12 different fake projective planes up to biholomorphism. As mentioned earlier, the results for the first and the last rows have been obtained earlier in [GKMS] by a different method.

2. Line bundles on fake projective planes

2.1 We work over $\mathbb{C}$. Throughout this paper, we denote by C_m the cyclic group of order m, $\sim$ the linear equivalence, and $\equiv$ the numerical equivalence. Also, we denote by $C_7:C_3$ the unique (up to isomorphism) nonabelian finite group of order 21,

$$C_7:C_3=\langle x,y|x^3=y^3=1,xyx^{-1}=y^2\rangle.$$

2.2. Let M be a fake projective plane. First of all, we would like to list all fake projective planes with $K_M=3L$, where L is a generator of the torsion-free part of the Neron-Severi group.

Lemma 1. *Among the* 100 *fake projective planes,* 92 *of which satisfies the property that* $K_M=3L$*.*

Proof. Recall that a fake projective plane is a complex two ball quotient $B^2_{\mathbb{C}}/\Pi$ for an arithmetic group Π classified in [PY] and [CS]. From the argument of §10.2 of [PY], it is known that $K_M=3L$ if and only if Γ can be lifted to become a lattice in $\mathrm{SU}(2,1)$, and $K_M=3L$ if the second cohomology class of M has no three torsion. The latter fact is an immediate consequence of the Universal coefficient Theorem, see **2.3** below or Lemma 3.4 of [GKMS]. The section §10.2 also shows that Γ can be lifted to $\mathrm{SU}(2,1)$ if the number fields involved is not one of the types $\mathcal{C}_2$ or $\mathcal{C}_{18}$. There are 12 candidates for Π lying in $\mathcal{C}_2$ or $\mathcal{C}_{18}$. Out of these 12 examples, 3 of them do not have 3-torsion elements in $\mathrm{H}^2(M,\mathbb{Z})$ and hence the corresponding Π can be lifted to $\mathrm{SU}(2,1)$. Finally, it is listed in the file registerofgps.txt of the weblink of [CS], that the lattices can be lifted to $\mathrm{SU}(2,1)$ except for four cases in $\mathcal{C}_{18}$, corresponding to $(\mathcal{C}_{18},p=3,\{2\},D_3),(\mathcal{C}_{18},p=3,\{2\},(dD)_3),(\mathcal{C}_{18},p=3,\{2\},(d^2D)_3)$ and $(\mathcal{C}_{18},p=3,\{2I\})$ in the notation of the file, see also Table 2 in [Y2]. Since there are two non-biholomorphic conjugate complex structures on such surfaces, it leads to the result that 92 of the fake projective planes can be regarded as quotient of $B^2_{\mathbb{C}}$ by a lattice in $\mathrm{SU}(2,1)$.

□

2.3 For a smooth projective surface S, any holomorphic line bundle represents an element in the Neron-Severi group $i_*\mathrm{H}^2(S,\mathbb{Z})\cap\mathrm{H}^{1,1}(S)$, where $i:\mathbb{Z}\to\mathbb{C}$ is the inclusion map. Let us consider the torsion part $\mathrm{H}^2(S,\mathbb{Z})$, which gives rise to torsion line bundles. From the Universal Coefficient Theorem, we have

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_1(S,\mathbb{Z}),\mathbb{Z}) \to \mathrm{H}^2(S,\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_2(S,\mathbb{Z}),\mathbb{Z}) \to 0.$$

Since $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_2(S,\mathbb{Z}),\mathbb{Z}))$ is torsion free, for the sake of computation of torsion part of $i_*\mathrm{H}^2(S,\mathbb{Z}) \cap \mathrm{H}^{1,1}(M)$, it suffices for us to investigate $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{H}_1(S,\mathbb{Z}),\mathbb{Z})$. On the other hand, for any abelian group A, we know that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, A) \cong A/mA$. Hence p-torsions of $\mathrm{H}^2(S,\mathbb{Z})$ corresponds to p-torsions of $\mathrm{H}_1(S,\mathbb{Z})$.

Lemma 2. *For the fake projective plane M with no 3 torsion in $\mathrm{H}_1(M,\mathbb{Z})$, we may assume that $K_M = 3L$ for an $\mathrm{Aut}(M)$-invariant line bundle L. In particular, the space of sections $\mathrm{H}^0(M,kL)$, if non-zero, is an $\mathrm{Aut}(M)$-module.*

Proof. This first statement follows from Lemma 3.4 of [GKMS], see also **10.3, 10.4** of [PY]. From the above discussions, as there is no 3 torsion in $\mathrm{H}_1(M,\mathbb{Z})$ and hence in $\mathrm{H}^2(M,\mathbb{Z})$, we may write $E = 3E_1$ for a torsion line bundle E_1. We may simply let $L = L_1 + E_1$. Again, L is unique since there is no 3-torsion. Hence L is $\mathrm{Aut}(M)$-invariant. □

3. Holomorphic sections and group actions

3.1 From this point on, we assume that the automorphism group $\mathrm{Aut}(M)$ of M is non-trivial. We start with a simple statement, which has also been observed in [GKMS].

Lemma 3. *Let M be a fake projective space. Then $h^0(M,2L) \leqslant 2$.*

Proof. Consider the homomorphism

$$\mathrm{H}^0(M,2L) \times \mathrm{H}^0(M,2L) \xrightarrow{\alpha} \mathrm{H}^0(M,4L),$$

given by $\alpha(x,y) = x \times y$. This induces a mapping

$$\mathbb{P}(\mathrm{H}^0(M,2L)) \times \mathbb{P}(\mathrm{H}^0(M,2L)) \xrightarrow{\beta} \mathbb{P}(\mathrm{H}^0(M,4L))$$

The restriction of β to the first factor with fixed value in the second factor is clearly injective. Similarly when we reverse the roles of the first and the second factor. Apply now a classical result of Remmert-Ven der Van [RV], p.155, it follows that the image of β has dimension the same as dimension of the domain. It follows that $h^0(M,4L) \geqslant 2(h^0(M,2L)-1)$. Since $h^0(M,4L) = 3$ by Riemann-Roch formula and Kodaira vanishing theorem, it follows that $h^0(M,2L) \leqslant 2$.

□

3.2.

Lemma 4. *Let M be a fake projective plane with $K_M = 3L$, where L is invariant under $\mathrm{Aut}(M)$. Suppose that $h^0(M,2L) \neq 0$, then for any non-trivial subgroup $H < \mathrm{Aut}(M)$, there exists a section $\sigma \in \mathrm{H}^0(M,2L)$ such that the divisor Σ associated to σ is invariant under H. Here H acts non-trivially on Σ.*

Moreover, if Σ is not an irreducible and reduced curve, then one of the following holds:

(1) $\Sigma = \Sigma_1 + \Sigma_2$*, where* Σ_i*'s are irreducible and reduced curves. In particular,* Σ_1 *and* Σ_2 *intersects transversally at a smooth point.*
(2) $\Sigma = 2C$*, where* C *is an irreducible and reduced curve.*

Proof. If $h^0(M, 2L) = 1$, then there exists an effective divisor $\Sigma \sim 2L$ and is unique. Since $h^*L = L$, we conclude that $h^*\Sigma = \Sigma$, as asserted by the lemma.

Assume now that $h^0(M, 2L) = 2$. Let C_1, C_2 be two linearly independent sections of $\mathrm{H}^0(M, 2L)$. It follows that for all $a, b \in \mathbb{C}$, $aC_1 + bC_2$ is a global section of $\mathrm{H}^0(M, 2L)$ as well. Hence we get a complete linear system $aC_1 + bC_2$, with $[a : b] \in P^1_{\mathbb{C}}$. Under the action of $h \in H$, h^*C_i becomes a section of the form $aC_1 + bC_2$. Hence there is an induced action of H on $P^1_{\mathbb{C}}$. From Riemann-Hurwitz Formula, we know that $P^1_{\mathbb{C}}$ does not have non-trivial torsion free quotient. Hence there is at least one fixed point for the action of H. This corresponds to an effective divisor $\Sigma \sim 2L$ invariant under H.

We claim that H cannot act trivially on Σ. Assume on the contrary that it acts trivially on Σ. It follows that Σ is fixed pointwise by H. Since H is finite and Σ is complex dimension 1, we observe that Σ must be totally geodesic. To see this, consider a real geodesic curve $c(t), |t| < \epsilon$ on M with initial point $p \in \Sigma$ and initial tangent $\tau_p = c'(0) \in T_p\Sigma$. As both p and $c'(0)$ are fixed by H, the whole geodesic curve $c(t), |t| < \epsilon$ is fixed by H since the differential equation governing $c(t)$ is a second order ordinary equation and is determined by the initial conditions specified above. It follows that $c(t)$ actually lies on Σ. Since this is true for all points $p \in \Sigma$ and $\tau_p \in T_p\Sigma$, we conclude that Σ is totally geodesic. On the other hand, from the result of [PY], we know that the arithmetic lattice Π associated to M is arithmetic of second type. It follows that there is no totally geodesic curve on M, cf. Lemma 8 of [Y2]. The claim is proved.

Suppose that Σ is not integral. If $\Sigma = \sum_i m_i\Sigma_i$, where Σ_i's are irreducible and reduced, then $\Sigma_i \equiv n_iL$ for some $n_i \in \mathbb{Z}_{>0}$ as $\rho(X) = 1$ and L is a generator of the Neron-Severi group. Since $\Sigma \equiv 2L$, $\sum_i m_in_i = 2$. Hence either $\Sigma = \Sigma_1 + \Sigma_2$, or $\Sigma = 2C$. Moreover, if $\Sigma = \Sigma_1 + \Sigma_2$, then $\Sigma_1.\Sigma_2 = 1$ and they must intersect transversally exactly at one smooth point.

□

3.3 Now we apply Lefschetz Fixed Point Theorem to analyze the geometry of the H-invariant curve Σ guaranteed in Lemma 4. As we will see in Lemma 6, this action always has fixed points. We use the following lemma, cf. [P].

Lemma 5. *Let* C *be a compact Riemann surface. Let* $1 \neq g \in \mathrm{Aut}(C)$ *be an element of prime order* l *acting non-trivially on* C *with* n *fixed points. Denote* $(\ )^{\mathrm{inv}}$ *the eigenspace of eigenvalue* 1*. Then for* $\Delta = g(C) - \dim_{\mathbb{C}}\mathrm{H}^1(\mathcal{O}_C)^{\mathrm{inv}}$*, we have*

$$n = 2 - 2g(C) + \frac{2l}{l-1}\Delta.$$

Proof. We consider the holomorphic Lefschetz fixed point theorem,

$$\sum_{gp=p} \frac{1}{\det(1 - \mathcal{J}_p(g^k))} = \mathrm{tr}((g^k)^*|_{\mathrm{H}^0(C,\mathcal{O}_C)}) - \mathrm{tr}((g^k)^*|_{\mathrm{H}^1(C,\mathcal{O}_C)}),$$

where $\mathcal{J}_p(g^k)$ is the holomorphic Jacobian with respect to the action of g^k at a fixed point p.

Consider summing up $k = 1, \ldots, l-1$ of the above formula. For the complex $\langle g \rangle$-module $V = \mathrm{H}^1(C, \mathcal{O}_C)$ of an element $1 \neq g \in \mathrm{GL}_{\mathbb{C}}(V)$ with prime order l, we have

$$\sum_{k=1}^{l-1} \mathrm{tr}((g^k)^*|_{\mathrm{H}^1(C,\mathcal{O}_C)}) = (l-1)(g(C) - \Delta) - \Delta = (l-1)g(C) - l\Delta.$$

For the left hand side of Lefschetz formula, since C is one-dimensional, $\mathcal{J}_p(g^k) = \rho^k$, where ρ is an l-th root of unit. Hence each fixed point p contributes

$$\sum_{k=1}^{l-1} \frac{1}{1-\rho^k} = \frac{1}{2}(l-1).$$

After summation we get

$$\frac{n}{2}(l-1) = l - 1 + l\Delta - (l-1)g(C),$$

which simplifies to the prescribed formula. □

4. The case of $\mathrm{Aut}(M) = C_7 : C_3$

4.1. The goal of this section is apply our argument to the case of a fake projective plane M with $\mathrm{Aut}(M) = C_7 : C_3$, which gives an alternate approach to such cases dealt with in [GKMS].

Lemma 6. *Let M be a fake projective plane with $\mathrm{Aut}(M) = C_7 : C_3$ and consider the induced action of a subgroup $H = C_7 < \mathrm{Aut}(M)$ on $\mathrm{H}^0(M, 2L)$. If Σ is an H-invariant section of $\mathrm{H}^0(M, 2L)$ as in Lemma 4, then there is a fixed point on Σ.*

Proof. By Lemma 4, we consider three cases: $\Sigma = \Sigma_1 + \Sigma_2$, $\Sigma = 2C$, or Σ is irreducible and reduced.

If $\Sigma = \Sigma_1 + \Sigma_2$, then $\Sigma_1 \cap \Sigma_2 = \{p\}$ is a point. It follows that $p \in \Sigma$ must be a fixed point of H as any element of H carries an irreducible component of Σ to another irreducible component of Σ.

If $\Sigma = 2C$, then the arithmetic genus $g_a(C)$ of C is given by

$$2(g_a(C) - 1) = (K + L)L = 4,$$

from which we conclude that $g_a(C) = 3$. Also $3 = g_a(C) = g(C^\nu) + h^0(\delta)$, where $\nu : C^\nu \to C$ is the normalization map and $\delta = \nu_*\mathcal{O}_{C^\nu}/\mathcal{O}_C$ is a torsion sheaf supported on $\mathrm{Sing}(C)$. If H acts without fixed point on C, then H acts without fixed point on the normalization C^ν and the quotient C^ν/H is a compact Riemann surface of Euler-Poincáre number

$$\chi_{\mathrm{top}}(C^\nu/H) = \frac{-4 + 2h^0(\delta)}{|H|} = 2 - 2g(C^\nu/|H|).$$

Since $0 \le h^0(\delta) \le 3$, we get

$$0 < g(C^\nu/H) = 1 + \frac{2 - h^0(\delta)}{|H|} < 2.$$

Hence $g(C^\nu/H) = 1$, $h^0(\delta) = 2$, and $g(C^\nu) = 1$. In particular, an entire holomorphic map from C^ν lifts to the universal cover $\mathbb{C}$ which maps into M. This is impossible since the ball quotient M is hyperbolic.

Suppose now that Σ is irreducible and reduced. The arithmetic genus $g_a(\Sigma)$ of Σ is given by

$$2(g_a(\Sigma) - 1) = (K + 2L)(2L) = 10,$$

from which we conclude that $g_a(\Sigma) = 6$. If H acts without fixed point on Σ, then H acts without fixed point on Σ^ν, the normalization of Σ. Hence Σ^ν/H is a Riemann surface of Euler-Poincáre number

$$\chi_{\rm top}(\Sigma^\nu/H) = \frac{-10 + 2h^0(\delta)}{|H|} = 2 - 2g(\Sigma^\nu/H),$$

where again $\delta = \nu_*\mathcal{O}_{\Sigma^\nu}/\mathcal{O}_\Sigma$ is a torsion sheaf supported on $\mathrm{Sing}(\Sigma)$. We get

$$0 \le g(\Sigma^\nu/|H|) \le 1 + 5/|H| < 2.$$

Hence $0 \le g(\Sigma^\nu/H) \le 1$ which again contradicts to the fact that M is hyperbolic. □

4.2 We prove the first part of our Main Theorem which is done in [GKMS] by different method. We start with a lemma about plane curve singularities, which should be well-known but we can not find a good reference.

Lemma 7. *Let (Σ, o) be a germ of reduced curve with b irreducible branches $\Sigma_1, \ldots, \Sigma_b$. Denote by $C_i = \Sigma_i^\nu$ the normalization of each irreducible branches.*

(*a*) *For $\delta = \nu_*\mathcal{O}_{\Sigma^\nu}/\mathcal{O}_\Sigma$ and $\delta_i = \nu_*\mathcal{O}_{C_i}/\mathcal{O}_{\Sigma_i}$ supported over the singularity $o \in \Sigma$,*

$$h^0(\delta) \ge (b-1) + \sum_{i=1}^b h^0(\delta_i).$$

(*b*) *$h^0(\delta) = b - 1$ if and only if all the branches are smooth and*

$$\mathcal{O}_{\Sigma,o} \cong \frac{k[[t_1, \ldots, t_b]]}{(\{t_i t_j :\ i \ne j\})}.$$

(*c*) *If $h^0(\delta) = 1$, then either $b = 1$ and $o \in \Sigma$ is cuspidal or $b = 2$ and $o \in \Sigma$ is a normal crossing (analytically a node).*

Proof. The statements (a) follows straight forward from the exact sequences

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{O}_{\Sigma,o} & \to & \oplus\mathcal{O}_{\Sigma_i,o} & \to & \tilde{\delta} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_{\Sigma,o} & \to & \oplus\mathcal{O}_{C_i,o} \cong \oplus k[[t_i]] & \to & \delta & \to & 0 \\
 & & & & \downarrow & & & & \\
 & & & & \oplus\delta_i & & & &
\end{array}$$

and the snake lemma: $b-1 \le h^0(\tilde{\delta}) = h^0(\delta) - \sum_i h^0(\delta_i)$. Here $h^0(\tilde{\delta}) \ge b-1$ as it contains the k-vector space generated by $e_i/(1,\ldots,1)$, where e_i is the unit of $\mathcal{O}_{\Sigma_i,o}$ and $(1,\ldots,1)$ is the image of the unit of $\mathcal{O}_{\Sigma,o}$.

For (b), $h^0(\delta) = b-1$ if and only if $h^0(\delta_i)$'s are zero. It is clear from the above discussion that this happens exactly when $m_{\Sigma,o}$ maps surjectively onto (t_i) for each i. This is the same as saying that Σ is a normal crossing.

The statement (c) is a local computation. From the assumption, we know from (a) that $b \le 2$. If $b = 1$, then there is a sequence

$$0 \to \mathcal{O}_{\Sigma,o} \to k[[t]] \to \delta \to 0$$

with δ a zero dimensional sheaf of length one. As an k-algebra, it is only possible that $\mathcal{O}_{\Sigma,o} \cong k[[t^2, t^3]]$ and we get

$$\mathcal{O}_{\Sigma,o} \cong \frac{k[[x,y]]}{(x^2 - y^3)}.$$

If $b = 2$, then then there is a sequence

$$0 \to \mathcal{O}_{\Sigma,o} \to k[[t_1]] \oplus k[[t_2]] \to \delta \to 0.$$

If $h^0(\delta) = 1$, then it is only possible that after suitable change of coordinates

$$k[[t_1]] \oplus k[[t_2]] \supseteq \mathcal{O}_{\Sigma,o} = k[[t_1, 0), (0, t_2)]] \cong \frac{k[[x,y]]}{(xy)}.$$

□

Remark 1. *It is possible that $h^0(\tilde{\delta}) > b-1$. This happens if and only if $h^0(\delta) > \sum_i h^0(\delta_i)$. The difference comes from the gluing of Σ_i's along $o \in \Sigma$.*

Theorem 1. *Let M be a fake projective plane with $\mathrm{Aut}(M) = C_7 : C_3$ and $K_M = 3L$. Then the sequence $\mathcal{O}_M, -L, -2L$ forms an exceptional collection.*

Proof. From the definition of exceptional collection, it is easy to see that we only have to verify $h^0(M, 2L) = 0$. Suppose now that $h^0(M, 2L) \neq 0$.

Consider $H = C_7 < \mathrm{Aut}(X)$, the unique 7-Sylow subgroup, and by Lemma 2 the space of sections $\mathrm{H}^0(M, 2L)$ is H-invariant. There is an H-invariant section $\Sigma \in \mathrm{H}^0(M, 2L)$ by Lemma 4 and an H-fixed point by Lemma 6. Moreover, Σ is either irreducible and reduced, or $\Sigma = 2C$, or $\Sigma = \Sigma_1 + \Sigma_2$ is reducible.

Observe that $\mathrm{Fix}(\Sigma) = \mathrm{Fix}(M) \cap \Sigma$. In particular, $|\mathrm{Fix}(\Sigma)| \le 3$ by the work of [K]. For the induced action of H on Σ^ν, we denote by $x = \dim_{\mathbb{C}} \mathrm{H}^1(\mathcal{O}_{\Sigma^\nu})^{\mathrm{inv}}$ the dimension of H-invariant 1-forms and $n = |\mathrm{Fix}(\Sigma^\nu)|$ the number of H-fixed points on Σ^ν.

Case 1: Σ is irreducible and reduced. Here $g_a(\Sigma) = 6 = g(\Sigma^\nu) + h^0(\delta)$.

Assume first that $\Sigma = \Sigma^\nu$, then $g(\Sigma) = 6$ and $n \le 3$. For $l = 7$, Lemma 5 implies that $3n + 7x = 12$ and $(n, x) = (4, 0)$. This contradicts to the inequality $n \le 3$.

Assume now that $\Sigma \neq \Sigma^\nu$. Applying Lemma 5 to the lifted action of H on Σ^ν, the normalization of Σ, with $l = |H| = 7$, we get

$$3n + 7x + h^0(\delta) = 12.$$

Since fake projective planes are hyperbolic, $g(\Sigma^\nu) \geq 2$ and hence $1 \leq h^0(\delta) \leq 4$. We study case by case.

If $h^0(\delta) = 1$, then $3n + 7x = 11$ and there is no nonnegative integer solution.

If $h^0(\delta) = 2$, then $3n + 7x = 10$ and $(n, x) = (1, 1)$. From the holomorphic Lefschetz fixed point theorem, we have

$$\frac{1}{1-\eta} + \xi_1 + \xi_2 + \xi_3 = 0,$$

where $\eta, \xi_j \in (\mathbb{Z}/7\mathbb{Z})^\times$. It can be checked directly from Matlab that there is no solution to the above equation.

If $h^0(\delta) = 3$, $3n + 7x = 9$ and $(n, x) = (3, 0)$. From the holomorphic Lefschetz fixed point theorem, we have

$$\frac{1}{1-\eta_1} + \frac{1}{1-\eta_2} + \frac{1}{1-\eta_3} + \xi_1 + \xi_2 + \xi_3 = 1,$$

where $\eta_i, \xi_j \in (\mathbb{Z}/7\mathbb{Z})^\times$. It can be checked directly from Matlab that there is no solution to the above equation.

If $h^0(\delta) = 4$, then $3n + 7x = 8$ and there is no nonnegative integral solution.

Case 2: $\Sigma = \Sigma_1 + \Sigma_2$ is reducible with two irreducible components Σ_i's. Denote $\Sigma^\nu = C_1 \sqcup C_2$ the normalization of Σ. Here C_i's are irreducible components of Σ^ν and are the normalization of Σ_i's respectively.

For $g \in H = C_7$, if $gC_1 = C_2$, then $gC_2 = C_1$. As $|H|$ is odd, $C_1 = g^{|H|}C_1 = C_2$. This is impossible and hence H acts on C_i's. For $g_i = g(C_i)$,

$$6 = g_a(\Sigma) = g_1 + g_2 + h^0(\delta) - 1,$$

where again for $\nu : \Sigma^\nu \to \Sigma$ the normalization map, $\delta = \nu_* \mathcal{O}_{\Sigma^\nu}/\mathcal{O}_\Sigma$ is a torsion sheaf supporting on $\mathrm{Sing}(\Sigma) \neq \emptyset$. Since M is hyperbolic, $g(C_i) \geq 2$ and hence $1 \leq h^0(\delta) \leq 3$.

Denote n_i the number of H-fixed points on C_i. For $l = |H|$, by Lemma 5

$$n_i(l-1) + 2lx_i = 2(l-1) + 2g_i,$$

where $x_i = \dim_\mathbb{C} \mathrm{H}^1(\mathcal{O}_{C_i})^{\mathrm{inv}}$. Hence for $n = n_1 + n_2$ and $x = x_1 + x_2$,

$$n(l-1) + 2lx + 2h^0(\delta) = 4(l-1) + 14.$$

For $l = |H| = 7$, we get $3n + 7x + h^0(\delta) = 19$. If a singular point of Σ is not fixed, then $h^0(\delta) \geq 7$. But we have seen that $1 \leq h^0(\delta) \leq 3$. Hence all singularities of Σ are fixed. Denote by $p = \Sigma_1 \cap \Sigma_2$ the unique intersection point and write $\delta = \delta_p + \delta'$ for obvious reason, we have $h^0(\delta_p) \geq 1$.

If $h^0(\delta) = 1$, then $h^0(\delta_p) = 1$ and p is the unique singularity of $\Sigma = \Sigma_1 \cup \Sigma_2$. By Lemma 7, $p \in \Sigma$ is nodal and lifts to two fixed points on Σ^ν. Hence $n \geq 2$, $3n + 7x = 18$, and $(n, x) = (6, 0)$ is the only solution. But then apart from the two fixed points above p, there should be four more fixed points on Σ. This is a contradiction to $|\mathrm{Fix}(\Sigma)| \leq 3$.

If $h^0(\delta) = 2$, then either $h^0(\delta_p) = h^0(\delta') = 1$ or $h^0(\delta_p) = 2$. In the former case, either Σ has exactly two nodal singularities which lift to four fixed points on Σ^ν, or Σ has one node at p and a cusp at another point. Hence we have $n \geq 3$ for $3n + 7x = 17$ and there is no nonnegative integral solution.

In the later case where $h^0(\delta_p) = 2$, if $b = 3$ over $p \in \Sigma$, then $p \in \Sigma$ has embedded dimension 3 by Lemma 7 (b). This contradicts to the fact that $\Sigma \subseteq X$. Hence $b = 2$ and there are two fixed points over p. But now $n \geq 2$ and there is no nonnegative integral solution to $3n + 7x = 17$.

If $h^0(\delta) = 3$, then $3n+7x = 16$ and $(n, x) = (3, 1)$. Since $x_1+x_2 = 1$, $n_i \geq 1$, and $g_i \geq 2$, it is easy to see that there is no nonnegative integral solution to the system of linear equations

$$\begin{cases} 3n_1 - g_1 + 7x_1 = 6, \\ 3n_2 - g_2 + 7x_2 = 6, \\ n_1 + n_2 = 3. \end{cases}$$

Case 3: $\Sigma = 2C$ with C an irreducible and reduced curve.

Here $C \equiv L$ and $3 = g_a(C) = g(C^\nu) + h^0(\delta)$. Moreover, $g(C^\nu) \geq 2$ as M is hyperbolic. We consider two cases: $(g(C^\nu), h^0(\delta)) = (3, 0)$ or $(2, 1)$.

Suppose that $(g(C^\nu), h^0(\delta)) = (2, 1)$. For $l = 7$, $3n + 7x = 8$ by Lemma 5 applied to C. There is no integer solution.

Suppose that $(g(C^\nu), h^0(\delta)) = (3, 0)$ and hence $C = C^\nu$. Since $l = 7$, $3n + 7x = 9$ by Lemma 5 applied to C. It is only possible that $(n, x) = (3, 0)$ and there are three smooth fixed points on C. From the holomorphic Lefschetz fixed point theorem, we have

$$\frac{1}{1-\eta_1} + \frac{1}{1-\eta_2} + \frac{1}{1-\eta_3} + \xi_1 + \xi_2 + \xi_3 = 1,$$

where $\eta_i, \xi_j \in (\mathbb{Z}/7\mathbb{Z})^\times$. It can be checked directly from Matlab that there is no solution to the above equation.

We conclude that it is only possible $\mathrm{H}^0(M, 2L) = 0$ and hence the existence of the required exceptional collection. $\square$

5. The case of $\mathrm{Aut}(M) = C_3 \times C_3$

5.1 In this section, we prove the second part of the Main Theorem. We consider fake projective planes with $\mathrm{Aut}(M) = C_3 \times C_3$. We refer the readers to the list in §1.

We have shown in Lemma 1 and Lemma 2 that there is an ample line bundle L with $K_M = 3L$ and L is $\mathrm{Aut}(M)$-invariant. Hence $\mathrm{H}^0(M.2L)$ is an $\mathrm{Aut}(M)$-module and we aim to show that $h^0(M, 2L) = 0$ as in the last section.

Recall that from Lemma 3, $h^0(M, 2L) \leqslant 2$. This can be improved for specific automorphism groups such as $C_3 \times C_3$. We first make the following simple observation.

Lemma 8. *There is no non-trivial faithful action of $C_3 \times C_3$ fixing a point on a connected smooth complex surface.*

Proof. From Cartan's Theorem about biholomorphism of an open set in $\mathbb{C}^2$, we know that a biholomorphism acts as a linear map in some neighborhood of the fixed point. This is determined by eigenvalues and eigenvectors, cf. [BPHV], III5. Now as the action of the two factors of $\mathrm{Aut}(M)$ commutes, the eigenvectors are the same.

It follows that actually the orbits of the product has the same effect as the action of one copy of C_3. In other words, the action is the same as the action of a copy of C_3 in a neighborhood of the fixed point.

□

5.2

Lemma 9. *In the case that* $\mathrm{Aut}(M) = C_3 \times C_3$, $h^0(M, 2L) \leqslant 1$.

Proof. Assume for the sake of proof by contradiction that $h^0(M, 2L) = 2$, so that $\mathrm{H}^0(M, 2L)$ is consisting of sections $as_1 + bs_2$ with $s_1, s_2 \in \mathrm{H}^0(M, 2L)$. From definition, $s_1 \cdot s_2 = 4$. Since L is invariant, we know that $\mathrm{Aut}(M)$ acts on $\mathrm{H}^0(M, 2L)$. We may define a metric on the sections of $\mathrm{H}^0(M, 2L)$ naturally induced by the invariant Bergman (Killing) metric on M, so that the L^2 pairing with respect to the invariant metric is invariant under automorphism. In other words, it acts as unitary transformations on the space of sections $as_1 + bs_2 \in \mathrm{H}^0(M, 2L)$. In particular, it acts holomorphically on $\mathbb{P}^1_{\mathbb{C}} := \{(a, b)|as_1 + bs_2 \in \mathrm{H}^0(M, 2L)\}/\mathbb{C}^*$.

Consider now a subgroup C_3 acting on $\mathbb{P}^1_{\mathbb{C}}$. We claim that there are at least two fixed points. This follows directly from Lefschetz fixed point formula applied to $\mathbb{P}^1_{\mathbb{C}}$. Hence there are at least two fixed sections of C_3 in $\mathrm{H}^0(M, 2L)$. We may assume that s_1, s_2 are fixed sections of $\mathrm{H}^0(M, 2L)$ (the choice of s_1 and s_2 depends on the particular C_3 subgroup chosen). Since they intersect at four points up to multiplicity, there are two cases:

(i) Intersect at two fixed points F_0 and F_1 with multiplicity 2 each;
(ii) Intersect at a fixed point F_0 of C_3, and three other points of M which form an orbit under C_3.

Let us first rule out Case (i). Consider now the action of another copy of C_3 in $\mathrm{Aut}(M)$, named as H_2. There are again (at least) two fixed sections D_1, D_2 of H_2. In this case, since all the sections of $\mathrm{H}^0(M, 2L)$ are linear combinations of s_1, s_2 and meet at F_0 and F_1, we conclude that D_1 and D_2 pass through F_0 and F_1 as well. Since C_1 and C_2 can also be generated by D_1 and D_2, F_0 and F_1 are the only two intersection points for D_1 and D_2. As the order of H_2 is 3, this implies that both F_0 and F_1 are fixed points of H_2 (under H_2, either form an orbit or is a fixed point). However, this implies that F_0 is fixed by all the C_3 subgroups of $\mathrm{Aut}(M)$ and hence by $\mathrm{Aut}(M) = C_3 \times C_3$. This is however impossible as explained in Lemma 7.

Consider now Case (ii). In such case, s_1 and s_2 meet transversally at all the four intersection points. As H_1 acts on both s_1 and s_2, it acts on the four intersection points as well. The only situation that can happen is that one is fixed and the other three are cyclic images of the copy of C_3. This happens for all C_3 subgroups of $\mathrm{Aut}(M)$, since all sections of $H^0(M, 2L)$ meet at these four points. Hence in $\mathrm{Aut}(M) = C_3 \times C_3$, let us name the first factor to be H_1 generated by an element γ, and the second H_2, generated by σ. Note that the actions of H_1 and H_2 commute with each other. Let the four points be x_1, x_2, x_3, x_4. After renaming of indices, we may assume that

$$\gamma x_1 = x_1, \gamma x_2 = x_3, \gamma x_3 = x_4, \gamma x_4 = x_2,$$

and

$$\sigma x_2 = x_2,\ x_1 \xrightarrow{\sigma} x_3 \xrightarrow{\sigma} x_4 \xrightarrow{\sigma} x_1, \text{or } x_1 \xrightarrow{\sigma} x_4 \xrightarrow{\sigma} x_3 \xrightarrow{\sigma} x_1.$$

Now we check that

$$\sigma\gamma x_2 = \sigma x_3 = x_i$$

with $x_i \in \{x_1, x_4\}$, but

$$\gamma\sigma x_2 = \gamma x_2 = x_3.$$

We reach a contradiction that the actions of σ and γ do not commute. □

5.3

Lemma 10. *Let M be a fake projective plane with* $\mathrm{Aut}(M) = C_3 \times C_3$. *Then there exists a non-trivial p-torsion line bundle E for some prime $p \neq 2$ that is* $\mathrm{Aut}(M)$-*invariant.*

Proof. We know from the first paragraph of **2.3** that p-torsion line bundles correspond to torsions of $\mathrm{H}_1(M, \mathbb{Z})$. Hence it suffices to find an $\mathrm{Aut}(M)$-invariant p-torsion. For classes $(\mathcal{C}_2, p = 2, \emptyset)$ and $(\mathcal{C}_2, p = 2, \{3\})$, we pick any non-trivial 7-torsion. For class $(\mathcal{C}_{18}, p = 3, \emptyset)$, we pick any non-trivial 13-torsion. Since any C_3-representation on C_7 or C_{13} can only be trivial. These torsion line bundles must be $\mathrm{Aut}(M)$-invariant. □

5.4 We are ready to prove the second part of the Main Theorem.

Theorem 2. *Let M be a fake projective plane with* $\mathrm{Aut}(M) = C_3 \times C_3$ *and $K_M = 3L$. Either the sequence*

$$\mathcal{O}_M, -L, -2L$$

forms an exceptional collection. Or there is a line bundle $L' = L + E_p$, where E_p is a non-trivial p-torsion line bundle with $p \neq 2$, such that the sequence

$$\mathcal{O}_M, -L', -2L'$$

forms an exceptional collection of M.

Proof. For $L' = L + E_p$ a numerical cubic root of K_M, the sequence

$$\mathcal{O}_M, -L', -2L'$$

forms an exceptional collection if and only if

$$h^0(M, 2L') = h^2(M, L') = h^2(M, 2L') = 0.$$

Here if $h^0(M, 2L') = 0$, then $h^0(M, L') = 0$. If furthermore $h^2(M, L') = h^2(M, 2L') = 0$, then $h^1(M, L') = h^1(M, 2L') = 0$ from Riemann-Roch formula and the fact that $q(M) = p_g(M) = 0$. Hence $\mathrm{Hom}(E_j, E_i[k]) = 0$ for $j \geq i$ and $k \in \mathbb{Z}$ except $i = j$ and $k = 0$ where in this case it is one dimensional as required.

Note that if $E_p = 0$, then the criterion still holds and it is sufficient to show that $h^0(M, 2L) = 0$. In this case, the sequence $\mathcal{O}_M, -L, -2L$ forms an exceptional collection.

We can assume that $h^0(M, 2L) \neq 0$. From Lemma 3 and Lemma 8, we can also assume that $h^0(M, 2L) = 1$. We want to show that $h^0(M, 2L') = h^0(M, 2L + 2E_p)$, $h^2(M, L') = h^0(M, 2L - E_p)$, and $h^2(M, 2L') = h^0(M, L - 2E_p)$ are all zero. The last vanishing follows from $h^0(M, 2(L - 2E_p)) = 0$. In particular, it suffices to show the vanishing of $\mathrm{H}^0(M, 2L + E)$ for E a non-trivial p-torsion line bundle with $p \neq 2$.

Choose E_p as in Lemma 9. Hence E is a non-trivial p-torsion line bundle with $p \neq 2$ and is $\mathrm{Aut}(M)$-invariant. We claim that the $\mathrm{Aut}(M)$-module $\mathrm{H}^0(M, 2L + E)$ can only be the zero space.

Let $\Sigma \in \mathrm{H}^0(M, 2L)$ be the unique section which is $\mathrm{Aut}(M)$-invariant. By the same argument as in Lemma 4, if $\mathrm{H}^0(M, 2L+E) \neq 0$, then there is also an $\mathrm{Aut}(M)$-invariant section $\Sigma' \in \mathrm{H}^0(M, 2L + E)$. Here we note that by the same argument as in Lemma 3 and Lemma 8, we can assume that $h^0(M, 2L + E) = 1$ and hence Σ' is also invariant under $\mathrm{Aut}(M)$.

We have $\Sigma \neq \Sigma'$ and $\Sigma.\Sigma' = 4$. There are two cases of intersection as in Lemma 8 by considering $H_1 = C_3 = \langle\gamma\rangle < \mathrm{Aut}(M)$ acting on $\Sigma \cap \Sigma'$:

- (i) Intersect at two H_1-fixed points F_0 and F_1 with multiplicity 2 each;
- (ii) Intersect at an H_1-fixed point F_0 of C_3, and three other points of M which form an orbit of H_1.

Now consider the action of another subgroup $H_2 = C_3 = \langle\sigma\rangle$ on $\Sigma \cap \Sigma'$. Case (i) is impossible since F_0 and F_1 must be fixed by H_2 and hence by the whole $\mathrm{Aut}(M)$. But we have seen in Lemma 7 that there is no point of $\mathbb{C}^2$ with the stabilizer of any group action to be $C_3 \times C_3$. For case (ii), the same argument as in Lemma 8 shows a contradiction. □

5.4 The Main Theorem is the combination of Theorem 1 and Theorem 2.

Acknowledgment This work is partially done during the first author's visit at National Center of Theoretical Science (NCTS/TPE) and National Taiwan University at Taiwan, and the second authors visit of the Institute of Mathematics of the University of Hong Kong. The authors thank the warm hospitality of the institutes.

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Mathematics Department, Purdue University, West Lafayette, IN 47907 USA
E-mail address: lai37@math.purdue.edu

Mathematics Department, Purdue University, West Lafayette, IN 47907 USA
E-mail address: yeung@math.purdue.edu