AUGMENTED WEIL-PETERSSON METRICS ON MODULI SPACES OF POLARIZED RICCI-FLAT KÄHLER MANIFOLDS AND ORBIFOLDS

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Abstract. *We show that the base complex manifold of an effectively parametrized family of compact polarized Ricci-flat Kähler orbifolds, and in particular manifolds, admits a smooth augmented Weil-Petersson metric whose holomorphic sectional curvature is bounded above by a negative constant. As a consequence, we conclude that such base manifold is Kobayashi hyperbolic.*

Keywords: Augmented Weil-Petersson metric, moduli space, Ricci-flat Kähler manifolds, Kähler orbifolds.

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1. **Introduction**

An important and extensively investigated object in the study of the moduli space $\mathcal{M}_g$ (and the Teichmüller space $\mathcal{T}_g$) of compact Riemann surfaces of genus $g \geq 1$ is the Weil-Petersson metric. In particular, when $g \geq 2$, Ahlfors ([Ah1], [Ah2]) showed that the Weil-Petersson metric on $\mathcal{T}_g$ is a Kähler metric whose Ricci and holomorphic sectional curvatures are negative. Royden [R] later proved that the holomorphic sectional curvature of the Weil-Petersson metric is bounded away from zero. Subsequently Wolpert [Wo] showed that the Weil-Petersson metric is of holomorphic sectional curvature bounded above by $-\frac{1}{2\pi(g-1)}$. An immediate consequence of Royden's or Wolpert's result is that $\mathcal{T}_g$ is Kobayashi hyperbolic. Similar results also hold in the case when $g = 1$, since $\mathcal{T}_1$ (endowed with the Weil-Petersson metric) is known to be biholomorphically isometric to the upper half plane $\mathbb{H}$ in the complex plane $\mathbb{C}$ (endowed with the Poincaré metric of constant negative sectional curvature). It is interesting and natural to ask whether analogous results hold for the moduli spaces of higher dimensional manifolds (or more generally, those of orbifolds), and in particular,

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whether (and how) one can achieve negative curvature for such moduli spaces within the context of Weil-Petersson geometry.

In this direction, Siu [Siu] made the first breakthrough and gave a computation of the curvature of the Weil-Petersson metric on (the smooth points of) the moduli space of compact Kähler-Einstein manifolds of negative Ricci curvature (or equivalently, canonically polarized manifolds). Based on Siu's approach, Nannicini [Na] computed the curvature of the Weil-Petersson metric on the moduli space of compact polarized Ricci-flat Kähler manifolds (or equivalently, polarized Kähler manifolds of zero first Chern class). In [Sch2], Schumacher also obtained a simplified formula for the two cases under the additional assumption that the Kodaira-Spencer map is surjective at a smooth point of the moduli space under consideration. Unlike the case of Riemann surfaces, the curvature tensors of the Weil-Petersson metrics in both higher dimensional cases mentioned above (and as obtained in [Siu], [Na], [Sch2]) contain terms of different signs, and no conclusion can be made on the sign of the holomorphic sectional curvature except in some restricted cases, say when each fiber manifold M satisfies the condition $H^2(M, \bigwedge^2 TM) = 0$. Recently by modifying suitably the Weil-Petersson metric, To and Yeung [TY] constructed on the base complex manifold of any effectively parametrized family of compact canonically polarized manifolds a Finsler metric (which will be called an *augmented Weil-Petersson metric* in this article) whose holomorphic sectional curvature is bounded above by a negative constant. As an immediate consequence, one can apply Schwarz lemma to conclude that such base complex manifold is Kobayashi hyperbolic.

The main goal of this article is to study the Kähler Ricci-flat case, which is the higher dimensional analogue of family of elliptic curves mentioned earlier. We state our first result as follows:

Theorem 1. *Let $\pi : \mathcal{X} \to S$ be an effectively parametrized holomorphic family of compact polarized Kähler manifolds of zero first Chern class over a complex manifold S. Then S admits a C^∞ $Aut(\pi)$-invariant augmented Weil-Petersson metric whose holomorphic sectional curvature is bounded above by a negative constant.*

We refer the reader to Section 2 for the precise definitions of the various terms in Theorem 1. Our approach is rather robust, and allows us to consider the more general situation of a family of compact polarized Ricci-flat Kähler orbifolds. We state our main result which generalizes Theorem 1 to such situation:

Theorem 1'. *Let $\pi : \mathcal{X} \to S$ be an effectively parametrized holomorphic family of compact polarized Ricci-flat Kähler orbifolds over a complex manifold S.*

Then S admits a C^∞ augmented Weil-Petersson metric whose holomorphic sectional curvature is bounded above by a negative constant.

The definitions needed for the more general setting of Theorem 1' will be given in Section 5. We remark that while Theorem 1' covers the case of Theorem 1, we state them separetely to facilitate our subsequent discussion as well as for motivational purpose. As in [TY], Theorem 1 and Theorem 1' lead immediately to the following:

Corollary 1. *Let $\pi : \mathcal{X} \to S$ be as in Theorem 1 or Theorem 1'. Then S is Kobayashi hyperbolic.*

The proof of Theorem 1' follows from suitable modifications from that of Theorem 1, and both are parallel to the Ricci-negative case treated in [TY]. For simplicity, we describe here briefly our approach for the proof of Theorem 1. Roughly speaking, we start with the curvature expression of the usual Weil-Petersson metric h_1 in [Na] (see also Section 2), which consists of a good term which is negative and a bad term which is non-negative. The bad term can be expressed as a ratio h_2/h_1, where h_2 is (the restriction of) some generalized Weil-Petersson pseudo-metric on the symmetric product $S^2(TS)$. This process is repeated. For each $1 \leq \ell \leq n$, one constructs some generalized Weil-Petersson pseudo-metric h_ℓ on $S^\ell(TS)$ (associated to an induced Kodaira-Spencer map $\rho_\ell : S^\ell(T_tS) \to H^n(M_t, \wedge^\ell TM_t)$, $t \in S$) and obtain curvature estimate of h_ℓ consisting of a good term involving $h_\ell/h_{\ell-1}$ and a bad term involving $h_{\ell+1}/h_\ell$. Here $n = \dim_{\mathbb{C}} M_t$ denotes the dimension of the fiber manifold M_t. Then the augmented Weil-Petersson metric h in Theorem 1 is constructed as a suitable finite linear combination of the $h_\ell^{1/\ell}$'s. The required curvature estimate for h is then derived from those of the h_ℓ's by a telescopic argument.

In both the results of [TY] for families of canonically polarized manifolds and Theorem 1 (or more generally Theorem 1') in this paper, we achieved the goal of proving hyperbolicity using generalized or augmented Weil-Petersson metrics. A natural question is whether the use of the augmented metric is essential or just a technical convenience; in other words, one may ask whether one can achieve the same goal using merely the Weil-Petersson metric itself in the sense that the bad term in the curvature formula of the Weil-Petersson metric might perhaps somehow be controlled by the good terms. It turns out that a modification of the Weil-Petersson metric could not be avoided. In fact, the holomorphic sectional curvature of the Weil-Petersson metric itself on the moduli space of compact polarized Ricci-flat Kähler manifolds may actually be positive at some points and negative at other points of the moduli space.

Candelas et al [CDGP] showed that such property is possessed by the one-dimensional moduli space of Calabi-Yau threefolds which are mirror manifolds of the quintic hypersurfaces in $\mathbb{CP}^4$ (see [CDGP, p. 51, Fig. 10]). Hence the example shows that, at least in the Ricci-flat case, one must modify the Weil-Petersson metric in order to obtain negative curvature from the perspective of Weil-Petersson geometry.

At this point, we would like to remark on an alternative approach to the problem. Let $\pi : \mathcal{X} \to S$ be as in Theorem 1. In the special case of families of polarized Calabi-Yau manifolds (or slightly more generally, when the canonical line bundle of each fiber manifold is holomorphically trivial), S actually admits a Kähler metric (called the Hodge metric) with holomorphic sectional curvature bounded above by a negative constant. The Hodge metric is due to [Lu1], and its construction is based on Hodge theory and depends on Griffiths' results [Gri1] on the curvature properties of the invariant metrics of the classifying spaces for polarized Hodge structures (see also [Gri2]). As such, at least in this special case, one can give an alternative Hodge-theoretic proof of Corollary 1 using the Hodge metric in place of the augmented Weil-Petersson metric. Nonetheless, this alternative approach does not generalize to cover the general orbifold case in Theorem 1', and there appear to be some subtleties for this alternative approach to apply to the case treated in Theorem 1 (cf. Corollary 2, Remark 4 and Remark 5 in Section 6).

We remark that in general, the augmented Weil-Petersson metric in Theorem 1 (or Theorem 1') is not unique, and its construction actually gives rise to a continuous family of new Finsler metrics of negative holomorphic sectional curvature bounded away from zero. In fact, for the moduli space of Calabi-Yau threefolds considered by Candelas et al [CDGP] mentioned earlier, it is easy to check that at least some of the augmented Weil-Petersson metrics are not constant multiples of the Hodge metric (see Remark 2 in Section 4 and Remark 3 in Section 6).

We also mention here that for the case of moduli space of polarized Calabi-Yau manifolds of dimension n, Lu and Sun [LS] considered "partial Hodge metrics" of the form $g_{WP} + c \cdot \mathrm{Ric}(g_{WP})$ for appropriate positive constant c, where g_{WP} is the Weil-Petersson metric and $\mathrm{Ric}(g_{WP})$ denotes its Ricci tensor. In general, it is not known whether the partial Hodge metric has negative holomorphic sectional curvature or not, except in the cases when $n = 3$ and when $n = 4$. In those two cases, the partial Hodge metric (with some appropriate choice of the constant c) coincides with the Hodge metric (see [Lu2] and [LS]).

For a succint presentation of our main ideas, we will first consider only the smooth case (as given in Theorem 1). Then in the subsequent treatment of the general orbifold case (as given in Theorem 1'), we will minimize repeating the arguments from the smooth case by indicating only the necessary modifications, whenever appropriate. As such, the organization of this paper is as follows. In Section 2, we give some background materials and introduce some notations. In Section 3, we introduce the generalized Weil-Petersson pseudometrics and computed their curvature. In Section 4, we finish the proof of Theorem 1. In Section 5, we treat the orbifold case and give the proof of Theorem 1'. In Section 6, we give a brief review of Lu's Hodge metric for the case of polarized families of Calabi-Yau manifolds and discuss the difficulty in trying to generalize this alternative approach to the general Ricci-flat manifold case or the more general orbifold case.

The origin of this work can be traced with the authors' indebtedness to a conversation of late Professor Viehweg with the second author in 2006, during which Professor Viehweg mentioned that the argument of [VZ] does not appear to generalize to the case of polarized Ricci-flat Kähler manifolds, and asked if the result there also holds in such case. The authors would like to thank Professor Yum-Tong Siu for his inspirations and suggestions leading to the approach adopted in this paper. The authors would also like to thank Professor Ngaiming Mok for his comments and clarifications.

2. **Background materials and generalized Weil-Petersson pseudo-metrics**

Let M be a compact complex manifold of zero first Chern class, i.e., one has $c_1(TM)_{\mathbb{R}} = 0 \in H^2(M, \mathbb{R})$. Let $\eta \in H^{1,1}(M, \mathbb{R})$ be a Kähler class on M, i.e., η can be represented by a Kähler form on M. Then (M, η) is said to be a *polarized Kähler manifold of zero first Chern class*. By a result of Yau [Yau], there exists a unique Ricci-flat Kähler metric g (with the associated Kähler form ω) on M, whose Kähler class is η, i.e., $[\omega] = \eta \in H^{1,1}(M, \mathbb{R})$.

Let $\pi : \mathcal{X} \to S$ be a holomorphic family of compact complex manifolds of zero first Chern class over a base complex manifolds S, i.e., $\pi : \mathcal{X} \to S$ is a surjective holomorphic map of maximal rank between two complex manifolds $\mathcal{X}$ and S, and each fiber $M_t := \pi^{-1}(t)$, $t \in S$, is a compact complex manifold of zero first Chern class. Let $\lambda \in R^1\pi_*\Omega_{\mathcal{X}|S}$, and let $\lambda_t := \lambda\big|_{M_t}$ for $t \in S$. Then $(\pi : \mathcal{X} \to S,\ \lambda)$ is said to be a *holomorphic family of polarized Kähler manifolds of zero first Chern class*, if, in addition, (i) each λ_t is a Kähler class on M_t, and (ii) under the natural map from $R^1\pi_*\Omega_{\mathcal{X}|S}$ to $R^2\pi_*\mathbb{R}$ arising from variation of Hodge structure, the image of λ is a horizontal section of the local system $R^2\pi_*\mathbb{R}$ (or equivalently, for any $t_o \in S$, any open neighborhood U of t_o in S and any underlying smooth family of diffeomorphisms $i_t : M_{t_o} \to M_t$, $t \in U$, such that i_{t_o} is the identity map on M_{t_o}, one has $i_t^*\lambda_t = \lambda_{t_o}$ in $H^2(M_{t_o}, \mathbb{R})$). For each $t \in S$, let $g(t)$ and $\omega(t)$ denote the unique Ricci-flat Kähler metric and its Kähler form on M_t. Then by [Sch1, Proposition 2.3], one knows that such a family admits a d-closed $(1,1)$-form ω on $\mathcal{X}$ satisfying

$$\omega\big|_{TM_t} = \omega(t) \quad \text{for each } t \in S. \tag{2.1}$$

We remark that in the special case (which we do not assume here) when the polarization λ is given by the first Chern class of a holomorphic line bundle over $\mathcal{X}$ which restricts to an ample line bundle over each M_t, and in such case, the conditions (i) and (ii) above are automatically satisfied. Also, when no confusion arises, the underlying polarization class λ will be dropped from our notation for the family (as in Theorem 1). As usual, the family $\pi : \mathcal{X} \to S$ is said to be *effectively parametrized* if the Kodaira-Spencer map $\rho_t : T_tS \to H^1(M_t, TM_t)$ is injective for each $t \in S$.

Next we recall some notions in the Finsler geometry of complex manifolds. A *Finsler pseudo-metric* h on a complex manifold S is simply a continuous function $h : TS \to \mathbb{R}$ such that $h(u) \geq 0$ for all $u \in TS$ and $h(cu) = |c|h(u)$ for all $u \in TS$ and $c \in \mathbb{C}$. If, in addition, $h(u) > 0$ for all $0 \neq u \in TS$, then we say that h is a *Finsler metric* on S. A Finsler pseudo-metric h is said to be C^∞ (resp. C^ℓ for a non-negative integer ℓ) if for any open subset $U \subset S$ and any non-vanishing C^∞ section u_t of $TS\big|_U$, $h(u_t)$ is a C^∞ (resp. C^ℓ) function on U. For a C^2 Finsler metric h on S, a point $t \in S$ and a non-zero tangent vector $u \in T_tS$, the holomorphic sectional curvature $K(u)$ of h in the direction u is simply given by

$$K(u) = \sup_R K(R, h\big|_R)(t), \tag{2.2}$$

where the supremum is taken over all local one-dimensional complex submanifolds R of S satisfying $t \in R$ and $T_tR = \mathbb{C}u$, and $K(R, h\big|_R)(t)$ is the sectional curvature of (the Riemannian metric) $(R, h\big|_R)$ at t (cf. (4.9)). We say that the holomorphic sectional curvature of the Finsler metric h on S is bounded above by a negative constant if there exists a constant $C > 0$ such that $K(u) < -C$ for all $0 \neq u \in TS$.

For the rest of this section, we let $(\pi : \mathcal{X} \to S,\ \lambda)$ be an effectively parametrized holomorphic family of compact n-dimensional polarized Kähler manifolds of zero first Chern class over an m-dimensional complex manifold S. First we set up some notation. We will use $(z, t) = (z^1, \cdots, z^n, t^1, \cdots, t^m)$ to denote local holomorphic coordinate functions on some coordinate open subset of $\mathcal{X}$, so that π corresponds to the coordinate projection map $(z, t) \to t$, and $t = (t^1, \cdots, t^m)$ also forms local holomorphic coordinate functions on some coordinate open subset of S. We will index components of tensors on M_t in the holomorphic tangential directions by Greek alphabets α, β, etc (with the range

$1, 2, \cdots, n$), while those in the complexified tangential directions are indexed by lower case Latin letters a, b, c, d, etc (with the range $1, 2, \cdots, n, \bar{1}, \bar{2}, \cdots, \bar{n}$). On the other hand, the components of tensors along the base directions will be indexed by the letters i, j (with the range $1, 2, \cdots, m$), etc. We also adopt the Einstein summation notation for indices along the fibers. We denote $\partial_\alpha := \frac{\partial}{\partial z^\alpha}$ and $\partial_{\bar{\alpha}} := \frac{\partial}{\partial \bar{z}^\alpha}$ for $\alpha = 1, \cdots, n$, and $\partial_i := \frac{\partial}{\partial t^i}$ for $i = 1, \cdots, m$, etc. The Ricci tensor of $g(t)$ is locally given by $R_{\alpha\bar{\beta}}(t) = -\partial_\alpha \partial_{\bar{\beta}} \log(\det(g_{\gamma\bar{\delta}}(t)))$, and the Ricci-flat condition means that $R_{\alpha\bar{\beta}}(t) = 0$ on each M_t. When no confusion arises, we will drop the parameter t, and we simply write $R_{\alpha\bar{\beta}}$ for $R_{\alpha\bar{\beta}}(t)$, etc. In local coordinates, we also write $\omega = \sqrt{-1} g_{I\bar{J}}(z, t) dw^I \wedge d\bar{w}^J$, where w can be z or t and the indices I, J can be i or α, etc. In particular, one has $g_{\alpha\bar{\beta}} = g_{\alpha\bar{\beta}}(t)$ along each fiber M_t.

Next we recall the 'horizontal lifting' of vector fields as defined by Schumacher in [Sch2], which is a special type of 'canonical lifting' in the sense of Siu in [Siu]. For $t \in S$ and a local tangent vector field u (of type $(1, 0)$) on an open subset U of S, the *horizontal lifting* v_u of u is the unique smooth vector field v_u (of type $(1, 0)$) on $\pi^{-1}(U)$ such that $\pi_* v_u = u$ and $v_u(z, t)$ is orthogonal to $T_{(z,t)} M_t$ (with respect to ω) for each $(z, t) \in \pi^{-1}(U)$. By [Sch2, p. 342, Proposition 1.1], one knows that $\Phi(u(t)) := \bar{\partial} v_u\big|_{M_t} \in \mathcal{A}^{0,1}(M_t)$ is the unique harmonic Kodaira-Spencer representative of $\rho_t(u(t))$, i.e., $\rho_t(u(t)) = \big[\Phi(u(t))\big]$ in $H^1(M_t, TM_t)$ for each $t \in U$. When $u = \partial/\partial t^i$ is a coordinate vector field, we will simply denote $v_i := v_{\partial/\partial t^i}$ and $\Phi_i := \Phi(\partial/\partial t^i)$. Write $\Phi_i = (\Phi_i)^\alpha_{\bar{\beta}} \partial_\alpha \otimes d\bar{z}^\beta$. It is easy to see that v_i and the $(\Phi_i)^\alpha_{\bar{\beta}}$'s are given locally by

$$v_i = \partial_i + v_i^\alpha \partial_\alpha, \quad \text{where } v_i^\alpha := -g^{\bar{\beta}\alpha} g_{i\bar{\beta}}, \text{ and} \tag{2.3}$$

$$(\Phi_i)^\alpha_{\bar{\beta}} = \partial_{\bar{\beta}} v_i^\alpha = -\partial_{\bar{\beta}}(g^{\bar{\gamma}\alpha} g_{i\bar{\gamma}}). \tag{2.4}$$

(see [Sch2, p. 342, equation (1.2)]). Here $g^{\bar{\beta}\alpha}$ denotes the components of the inverse of $g_{\alpha\bar{\beta}}$. For a given tensor T of covariant degree 1 and of contravariant degree 1, we recall that the components (along the fiber direction) of its Lie derivative $\mathcal{L}_{v_i} T$ with respect to v_i are given locally by

$$(\mathcal{L}_{v_i} T)^b_a = \partial_i(T^b_a) + T^b_c \partial_a v_i^c - T^c_a \partial_c v_i^b \tag{2.5}$$

(see e.g. [Siu, p. 268]), and similar formula holds for tensors of higher degree. We recall that the Weil-Petersson metric $h^{(WP)} = \sum_{i,j=1}^m h^{(WP)}_{i\bar{j}} dt^i \otimes d\bar{t}^j$ on S is defined by

$$h^{(WP)}_{i\bar{j}}(t) := \int_{M_t} \langle \Phi_i, \Phi_j \rangle \frac{\omega^n}{n!}, \quad \text{where } \langle \Phi_i, \Phi_j \rangle := (\Phi_i)^\gamma_{\bar{\alpha}} \overline{(\Phi_j)^\delta_{\bar{\beta}}} g_{\gamma\bar{\delta}} g^{\bar{\alpha}\beta} \tag{2.6}$$

denotes the pointwise Hermitian inner product on tensors with respect to ω. The injectivity of ρ_t means that $h^{(WP)}$ is positive definite on each $T_t S$. It follows from Koiso's result [Koi] that $h^{(WP)}$ is Kähler. We denote by V the volume of M_t with respect to $\omega(t)$, which does not depend on $t \in S$ because ω is d-closed on $\mathcal{X}$. For any smooth tensor Ψ on M_t, we denote by $H(\Psi)$ the harmonic projection of Ψ with respect to $\omega(t)$. Let $R^{(WP)}$ denote the curvature tensor of $h^{(WP)}$. By Nannicini's result [Na, p. 425], the components of $R^{(WP)}$ with respect to normal coordinates (of $h^{(WP)}$) at a point $t \in S$ are given by

(2.7)
$$\begin{aligned} R^{(WP)}_{i\bar{j}k\bar{\ell}}(t) = -\frac{1}{V}(h^{(WP)}_{i\bar{j}}h^{(WP)}_{k\bar{l}} + h^{(WP)}_{i\bar{l}}h^{(WP)}_{k\bar{j}}) - \int_{M_t} \langle H(\mathcal{L}_{v_i}\Phi_k), H(\mathcal{L}_{v_j}\Phi_\ell)\rangle \frac{\omega^n}{n!} \\ + \int_{M_t} \langle H(\Phi_i \otimes \Phi_k), H(\Phi_j \otimes \Phi_\ell)\rangle \frac{\omega^n}{n!}. \end{aligned}$$

Here $\Phi_i \otimes \Phi_k$ is as in (2.8) below, and by normal coordinates of $h^{(WP)}$ at the point $t \in S$, we mean $h^{(WP)}_{i\bar{j}}(t) = \delta_{ij}$, and $\partial_k h^{(WP)}_{i\bar{j}}(t) = \partial_{\bar{k}} h^{(WP)}_{i\bar{j}}(t) = 0$.

Now we construct some generalized Weil-Petersson pseudo-metrics on S similar to those in the Ricci-negative case in [TY]. For integers $p, q, r, s \geq 0$, $t \in S$, $\Phi \in \mathcal{A}^{0,p}(\wedge^r TM_t)$ and $\Psi \in \mathcal{A}^{0,q}(\wedge^s TM_t)$, we denote by $\Phi \otimes \Psi \in \mathcal{A}^{0,p+q}(\wedge^{r+s}TM_t)$ the $(\wedge^{r+s}TM_t)$-valued $(0, p+q)$-form obtained by taking wedge product on the level of forms as well as that of tangent vectors. When $p = r$ and $q = s$, one easily sees that $\Phi \otimes \Psi = \Psi \otimes \Phi$. If Φ and Ψ are $\overline{\partial}$-closed, then $\Phi \otimes \Psi$ is also $\overline{\partial}$-closed. If, in addition, either Φ or Ψ is $\overline{\partial}$-exact, then $\Phi \otimes \Psi$ is $\overline{\partial}$-exact. In particular, the operator $\otimes$ induces a homomorphism on the associated cohomology groups, which we denote by the same symbol, i.e., we have

$$[\Phi] \otimes [\Psi] := [\Phi \otimes \Psi] \in H^{0,p+q}(\wedge^{r+s}TM_t) \tag{2.8}$$

for any classes $[\Phi] \in H^{0,p}(\wedge^r TM_t)$ and $[\Psi] \in H^{0,q}(\wedge^s TM_t)$ represented by Φ and Ψ respectively. (See [TY, Section 3] for the local expression for $\otimes$ and its properties mentioned above.) For a fixed integer ℓ satisfying $1 \leqslant \ell \leqslant n$, let $\Phi, \Psi \in \mathcal{A}^{0,\ell}(\wedge^\ell TM_t)$. Their pointwise inner product is given by

$$\langle \Phi, \Psi \rangle := \frac{1}{(\ell!)^2} \Phi^{\alpha_1 \cdots \alpha_\ell}_{\overline{\beta}_1 \cdots \overline{\beta}_\ell} \overline{\Psi^{\alpha'_1 \cdots \alpha'_\ell}_{\overline{\beta}'_1 \cdots \overline{\beta}'_\ell}} g_{\alpha_1 \overline{\alpha}'_1} \cdots g_{\alpha_\ell \overline{\alpha}'_\ell} g^{\beta_1 \overline{\beta}'_1} \cdots g^{\beta_\ell \overline{\beta}'_\ell}, \tag{2.9}$$

and their L^2-inner product on M_t is given by

$$(\Phi, \Psi) = \int_{M_t} \langle \Phi, \Psi \rangle \frac{\omega^n}{n!}. \tag{2.10}$$

Here the $\Phi^{\alpha_1 \cdots \alpha_\ell}_{\overline{\beta}_1 \cdots \overline{\beta}_\ell}$'s denote the tensor components of Φ, etc. We denote by $\|\Phi\|_2 := \sqrt{(\Phi, \Phi)}$ the fiberwise L^2-norm of Φ. Then for each $t \in S$ and $u_1, \ldots, u_\ell, u'_1, \ldots, u'_\ell \in T_t S$, we generalize (2.6) and define, in terms of (2.10),

$$
\begin{aligned}
(2.11) \qquad & (u_1 \otimes \cdots \otimes u_\ell, u_1' \otimes \cdots \otimes u_\ell')_{WP} \\
&:= (H(\Phi(u_1) \owedge \cdots \owedge \Phi(u_\ell)), H(\Phi(u_1') \owedge \cdots \owedge \Phi(u_\ell'))).
\end{aligned}
$$

Here each $\Phi(u_i)$ is the harmonic Kodaira-Spencer representative of $\rho_t(u_i)$. It is easy to see that (2.11) extends to a positive semi-definite Hermitian bilinear form on $\otimes^\ell T_t S$, which varies smoothly in t. We simply call it the *ℓ-th generalized Weil-Petersson pseudo-metric on $\otimes^\ell TS$*. The associated *ℓ-th generalized Weil-Petersson pseudo-metric on TS* is given by

$$
(2.12) \qquad \|u\|_{WP,\ell} := (\underbrace{u \otimes \cdots \otimes u}_{\ell-\text{times}}, \underbrace{u \otimes \cdots \otimes u}_{\ell-\text{times}})_{WP}^{\frac{1}{2\ell}} \quad \text{for } u \in T_t S,\ t \in S.
$$

Finally we define an *augmented Weil-Petersson metric* on S to be any Finsler metric h_{aWP} of the form

$$
(2.13) \qquad h_{aWP}(u) = \Big(\sum_{\ell=1}^{n} a_\ell \|u\|_{WP,\ell}^{2N} \Big)^{\frac{1}{2N}} \quad \text{for } u \in T_t S \text{ and } t \in S
$$

for some fixed numbers $a_1, \cdots, a_n > 0$ and fixed positive integer N (independent of t and u). Since $\| \ \|_{WP,1}$ is non-degenerate on S, it follows that each h_{aWP} is also non-degenerate on S.

Remark 1. *For any given pair of automorphisms $(F, f) \in Aut(\mathcal{X}) \times Aut(\mathcal{S})$ satisfying $f \circ \pi = \pi \circ F$ and preserving the polarization λ (i.e. $F^*\lambda = \lambda$), one easily sees from the uniqueness of the Ricci-flat Kähler metric $g(t)$ in the Kähler class λ_t of each M_t that $(F\big|_{M_t})^* g(f(t)) = g(t)$ for all $t \in S$. Then it follows readily that each Finsler pseudo-metric $\| \ \|_{WP,\ell}$ is $Aut(\pi)$-invariant in the sense that $f^*\| \ \|_{WP,\ell} = \| \ \|_{WP,\ell}$ for all pairs (F, f) as described above. As such, every augmented Weil-Petersson metric h_{aWP} is also $Aut(\pi)$-invariant, i.e., $f^* h_{aWP} = h_{aWP}$ for all pairs (F, f) as above. Furthermore, similar to* [TY, Lemma 16]*, one also easily checks that each h_{aWP} is C^∞.*

3. Curvature of generalized Weil-Petersson pseudo-metrics

Let $\pi : \mathcal{X} \to S$ be as in Theorem 1. In this section, we are going to obtain estimates for the holomorphic sectional curvatures of the restrictions of the Finsler pseudo-metrics $\| \ \|_{WP,\ell}$'s to local one-dimensional complex submanifolds of S (at those points where the restrictions are non-degenerate), which will lead to estimate for that of the augmented Weil-Petersson metric in Section 4. As most of the curvature computations are similar to the Ricci-negative case in [TY], we will refer the reader to [TY] (and follow the notation there as well as that in Section 2) whenever possible, and work out only the necessary changes in detail.

As in Section 2, we fix a coordinate open subset $U \subset S$ with coordinate functions $t = (t^1, \ldots, t^m)$. For each $t \in S$ and each coordinate tangent vector $\frac{\partial}{\partial t^i}$, we recall the horizontal lifting v_i and the harmonic representative Φ_i of $\rho_t(\frac{\partial}{\partial t^i})$ on M_t as given in (2.3) and (2.4). Fix an integer ℓ satisfying $1 \leqslant \ell \leqslant n$, and let $J = (j_1, \ldots, j_\ell)$ be an ℓ-tuple of integers satisfying $1 \leqslant j_d \leqslant m$ for each $1 \leqslant d \leqslant \ell$. We denote by

$$\Psi_J := H(\Phi_{j_1} \oslash \cdots \oslash \Phi_{j_\ell}) \in \mathcal{A}^{0,\ell}(\wedge^\ell TM_t) \tag{3.1}$$

the harmonic projection of $\Phi_{j_1} \oslash \cdots \oslash \Phi_{j_\ell}$. As t varies, we still denote the resulting family of tensors by Ψ_J (suppressing its dependence on t), when no confusion arises. We are going to compute $\partial_i \partial_{\bar{i}} \log \|\Psi_J\|_2^2$ (as a function on U) wherever $\Psi_J \not\equiv 0$ on M_t. For this purpose, we will need to consider families of tensors on the fibers (or in short, relative tensors) arising from restrictions of tensors on $\mathcal{X}$ to the fibers. We adopt the semi-colon notation to denote covariant derivatives of tensors on M_t, so that $(\Phi_i)^\beta_{\overline{\alpha};\gamma} := \nabla_\gamma (\Phi_i)^\beta_{\overline{\alpha}}$ $(= (\nabla_{\frac{\partial}{\partial z^\gamma}} \Phi_i)^\beta_{\overline{\alpha}})$, etc. We also denote $(\Phi_i)_{\overline{\alpha},\overline{\beta}} = g_{\gamma\overline{\beta}} (\Phi_i)^\gamma_{\overline{\alpha}}$, etc. We recall the following lemma in [TY, Lemma 1], which also holds in the present Ricci-flat case:

Lemma 1. *(i)* $[v_i, \partial_{\overline{\alpha}}] = -(\Phi_i)^\beta_{\overline{\alpha}} \partial_\beta$.
(ii) For a smooth (n,n)-form Υ on $\mathcal{X}$, one has

$$\frac{\partial}{\partial t^i} \int_{M_t} \Upsilon = \int_{M_t} \mathcal{L}_{v_i} \Upsilon \quad \textit{and} \quad \frac{\partial}{\partial \overline{t}^i} \int_{M_t} \Upsilon = \int_{M_t} \mathcal{L}_{\overline{v_i}} \Upsilon.$$

(iii) $[v_i, \overline{v_j}] = g^{\overline{\gamma}\alpha} \partial_{\overline{\gamma}} (g_{v_i \overline{v_j}}) \partial_\alpha - g^{\overline{\beta}\gamma} \partial_\gamma (g_{v_i \overline{v_j}}) \partial_{\overline{\beta}}$.
(iv) $(\Phi_i)_{\overline{\alpha},\overline{\beta}} = (\Phi_i)_{\overline{\beta},\overline{\alpha}}$ *for all* α, β.
(v) $\mathcal{L}_{v_i}(g_{\alpha\overline{\beta}} dz^\alpha \wedge d\overline{z}^\beta) = (\Phi_i)_{\overline{\beta},\overline{\gamma}} d\overline{z}^\beta \wedge d\overline{z}^\gamma = 0$. *In particular, one has* $\mathcal{L}_{v_i}(\omega^n) = 0$ *(as relative tensor).*
Here $[\cdot,\cdot]$ denote the Lie bracket of two vector fields.

Let $T^{\mathbb{C}} M_t = TM_t \otimes_{\mathbb{R}} \mathbb{C}$ denote the complexified tangent bundle of M_t, and for $\ell, \ell' \geq 0$, consider the space $\mathcal{A}^\ell(\wedge^{\ell'} T^{\mathbb{C}} M_t)$ with decomposition $\mathcal{A}^\ell(\wedge^{\ell'} T^{\mathbb{C}} M_t) = \oplus_{q+p=\ell, r+s=\ell'} \mathcal{A}^{q,p}(\wedge^r TM_t \wedge \overline{\wedge^s TM_t})$ and corresponding Weil operator $\mathcal{C}_W$ given by scalar multiplication by $(\sqrt{-1})^{q-p+r-s}$ on each summand $\mathcal{A}^{q,p}(\wedge^r TM_t \wedge \overline{\wedge^s TM_t})$. As usual, we denote the (positive definite) L^2-inner product and the corresponding L^2-norm on $\mathcal{A}^\ell(\wedge^{\ell'} T^{\mathbb{C}} M_t)$ with respect to $\omega(t)$ by $(\ ,\ )$ and $\|\ \|_2$ respectively. Then it is well-known that there is a pointwise Hermitian bilinear pairing $\langle\ ,\ \rangle$ (of mixed signature) on $\mathcal{A}^\ell(\wedge^{\ell'} T^{\mathbb{C}} M_t)$ such that for all

$\Upsilon, \Upsilon' \in \mathcal{A}^{\ell}(\wedge^{\ell'} T^{\mathbb{C}} M_t)$, one has

$$\mathcal{L}_{v_i}\langle \Upsilon, \Upsilon' \rangle = \langle \mathcal{L}_{v_i}\Upsilon, \Upsilon' \rangle + \langle \Upsilon, \mathcal{L}_{\overline{v_i}}\Upsilon' \rangle, \quad \text{and} \tag{3.2}$$

$$(\Upsilon, \Upsilon') = \int_{M_t} \langle \mathcal{C}_W(\Upsilon), \Upsilon' \rangle \frac{\omega^n}{n!}; \tag{3.3}$$

moreover, $\mathcal{C}_W$ restricts to the identity map on $\mathcal{A}^{0,\ell}(\wedge^{\ell} TM_t)$, and (3.3) agrees with (2.10) (cf. e.g. [TY, Section 3]).

Let Ψ_J be as in (3.1) . First one easily checks that

$$\partial_i \overline{\partial_i} \log \|\Psi_J\|_2^2 = \frac{\partial_i \partial_{\bar{i}} \|\Psi_J\|_2^2}{\|\Psi_J\|_2^2} - \frac{(\partial_i \|\Psi_J\|_2^2)(\partial_{\bar{i}} \|\Psi_J\|_2^2)}{\|\Psi_J\|_2^4}. \tag{3.4}$$

We note that [TY, Lemma 3] also holds in the present Ricci-flat case, so that the component of $\mathcal{L}_{\overline{v_i}}\Psi_J$ in $\mathcal{A}^{0,\ell}(\wedge^{\ell} TM_t)$ is $\overline{\partial}$-exact on M_t. Together with the harmonicity of Ψ_J, it follows that

$$(\mathcal{L}_{\overline{v_i}}\Psi_J,\, \Psi_J) = 0 \tag{3.5}$$

as a function on the base manifold. Together with a direct computation using Lemma 1, (3.2) and (3.3), one has, as in [TY, equations (4.7)-(4.9)],

$$\partial_i \|\Psi_J\|_2^2 = (\mathcal{L}_{v_i}\Psi_J, \Psi_J). \tag{3.6}$$

By taking ∂_i of (3.5), taking $\partial_{\bar{i}}$ of (3.6), and using the identity $\mathcal{L}_{\overline{v_i}}\mathcal{L}_{v_i} = \mathcal{L}_{v_i}\mathcal{L}_{\overline{v_i}} + \mathcal{L}_{[\overline{v_i}, v_i]}$, one gets, as in [TY, equations (4.9)-(4.12)],

$$\partial_i \partial_{\bar{i}} \|\Psi_J\|_2^2 = I + II + III, \quad \text{where} \tag{3.7}$$

$$\begin{aligned} I &:= -(\mathcal{L}_{\overline{v_i}}\Psi_J, \mathcal{L}_{\overline{v_i}}\Psi_J), \\ II &:= (\mathcal{L}_{[\overline{v_i}, v_i]}\Psi_J, \Psi_J), \quad \text{and} \\ III &:= (\mathcal{L}_{v_i}\Psi_J, \mathcal{L}_{v_i}\Psi_J). \end{aligned} \tag{3.8}$$

We are going to compute the terms I, II and III in the next three propositions. First we consider I. As in [TY, equations (5.6) and (5.10)], we let Let $\overline{\Phi_i} \cdot \Psi_J \in \mathcal{A}^{0,\ell-1}(\wedge^{\ell-1} TM_t)$, $\overline{\Phi_i} \searrow \Psi_J \in \mathcal{A}^{1,\ell-1}(\wedge^{\ell} TM_t)$ and $\overline{\Phi_i} \nearrow \Psi_J \in \mathcal{A}^{0,\ell}(\wedge^{\ell-1} TM_t \wedge \overline{TM_t})$ be the relative tensors with components given by

$$(\overline{\Phi_i} \cdot \Psi_J)^{\alpha_1 \cdots \alpha_{\ell-1}}_{\overline{\beta_1} \cdots \overline{\beta_{\ell-1}}} := \overline{(\Phi_i)^{\sigma}_{\overline{\gamma}}} \cdot (\Psi_J)^{\gamma \alpha_1 \cdots \alpha_{\ell-1}}_{\overline{\sigma}\overline{\beta_1} \cdots \overline{\beta_{\ell-1}}}, \tag{3.9}$$

$$\begin{aligned} (\overline{\Phi_i} \searrow \Psi_J)^{\alpha_1 \cdots \alpha_\ell}_{\delta \overline{\beta_1} \cdots \overline{\beta_{\ell-1}}} &:= \overline{(\Phi_i)^{\sigma}_{\overline{\delta}}} (\Psi_J)^{\alpha_1 \cdots \alpha_\ell}_{\overline{\sigma}\overline{\beta_1} \cdots \overline{\beta_{\ell-1}}} \quad \text{and} \\ (\overline{\Phi_i} \nearrow \Psi_J)^{\alpha_1 \cdots \alpha_{\ell-1}\overline{\gamma}}_{\overline{\beta_1} \cdots \overline{\beta_\ell}} &:= \overline{(\Phi_i)^{\gamma}_{\overline{\sigma}}} (\Psi_J)^{\alpha_1 \cdots \alpha_{\ell-1}\sigma}_{\overline{\beta_1} \cdots \overline{\beta_\ell}}. \end{aligned} \tag{3.10}$$

For a relative tensor $\Upsilon \in \oplus_{p,q,r,s} \mathcal{A}^{q,p}(\wedge^r TM_t \wedge \overline{\wedge^s TM_t})$, we denote by $\Upsilon^{(q,p)}_{(r,s)}$ the component of Υ in $\mathcal{A}^{q,p}(\wedge^r TM_t \wedge \overline{\wedge^s TM_t})$. Moreover, for $\Upsilon \in \mathcal{A}^{0,p}(\wedge^r TM_t)$, we denote $\overline{D_2}^*\Upsilon \in \mathcal{A}^{0,p}(\wedge^{r-1}TM_t)$, given by

$$(\overline{D_2}^*\Upsilon)^{\alpha_1\cdots\alpha_{r-1}}_{\overline{\beta_1}\cdots\overline{\beta_p}} = -\nabla_\sigma\Upsilon^{\sigma\alpha_1\cdots\alpha_{r-1}}_{\overline{\beta_1}\cdots\overline{\beta_p}}.$$

(cf. [Siu, p.288] and [TY, Section 5]). As usual, we denote by $\Box := \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ the $\bar\partial$-Laplacian on M_t with respect to $\omega(t)$, and denote its associated Green's operator by G.

Lemma 2. *Let Φ_i and Ψ_J be as in (3.1).*
(i) There exists $K \in \mathcal{A}^{0,\ell-1}(\wedge^\ell TM_t)$ such that $\overline{\partial}K = (\mathcal{L}_{\overline{v_i}}\Psi_J)^{(0,\ell)}_{(\ell,0)}$.
(ii) For any $\Upsilon \in \mathcal{A}^{0,\ell-1}(\wedge^{\ell-1}TM_t)$, we have $\langle\overline{\Phi_i}\cdot\Psi_J, \Upsilon\rangle = \langle\Psi_J, \Phi_i \oslash \Upsilon\rangle$.
(iii) We have $\overline{\partial}^*(\overline{\Phi_i}\cdot\Psi_J) = 0$.
(iv) The tensor $\overline{D_2}^*((\mathcal{L}_{\overline{v}_i}\Psi_J)^{(0,\ell)}_{(\ell,0)})$ is $\overline{\partial}$-exact. Explicitly, we have

$$\nabla_\sigma(\mathcal{L}_{\overline{v}_i}\Psi_J)^{\sigma\alpha_1\cdots\alpha_{\ell-1}}_{\overline{\beta}_1\cdots\overline{\beta}_\ell} = (\overline{\partial}(\overline{\Phi_i}\cdot\Psi_J))^{\alpha_1\cdots\alpha_{\ell-1}}_{\overline{\beta}_1\cdots\overline{\beta}_\ell}. \tag{3.11}$$

(v) Let K be as in (i) above. Suppose that $\overline{\partial}^*K = 0$. Then

$$\overline{D_2}^*K = -\Box G(\overline{\Phi_i}\cdot\Psi_J),$$

Proof. The proofs of (i)-(iv) are the same as those in Lemma 3, Lemma 4, Lemma 5 and Lemma 6 of [TY] respectively. The proof of (v) follows mutatis mutandis from that of [TY, Lemma 7], and we just remark that the arguments leading to [TY, equation (5.20)] in the proof of [TY, Lemma 7] show that in our present case, $\overline{D_2}^*K$ is $\overline{\partial}^*$-exact, and there exists a harmonic tensor Q satisfying $Q = \overline{\Phi_i}\cdot\Psi_J + \overline{D_2}^*K$, which readily imply (v). We also remark that as in [TY], the proof of (v) depends on (iii) and (iv), while that of (iv) depends on (ii) and (iii). $\square$

Parallel to [TY, Proposition 1], we compute I in (3.8) as follows:

Proposition 1. *We have*

$$\begin{aligned}(\mathcal{L}_{\overline{v_i}}\Psi_J, \mathcal{L}_{\overline{v_i}}\Psi_J) &= (\Box G(\overline{\Phi_i}\cdot\Psi_J), \overline{\Phi_i}\cdot\Psi_J)\\ &\quad - (\overline{\Phi_i}\searrow\Psi_J, \overline{\Phi_i}\searrow\Psi_J) - (\overline{\Phi_i}\nearrow\Psi_J, \overline{\Phi_i}\nearrow\Psi_J).\end{aligned}$$

Proof. As in [TY, equation (5.11)], one easily checks that

$$\begin{aligned}(\mathcal{L}_{\overline{v_i}}\Psi_J, \mathcal{L}_{\overline{v_i}}\Psi_J) &= ((\mathcal{L}_{\overline{v_i}}\Psi_J)^{(0,\ell)}_{(\ell,0)}, (\mathcal{L}_{\overline{v_i}}\Psi_J)^{(0,\ell)}_{(\ell,0)})\\ &\quad - (\overline{\Phi_i}\searrow\Psi_J, \overline{\Phi_i}\searrow\Psi_J) - (\overline{\Phi_i}\nearrow\Psi_J, \overline{\Phi_i}\nearrow\Psi_J).\end{aligned} \tag{3.12}$$

Then following the proof of [TY, Proposition 1] with [TY, Lemma 7] replaced by Lemma 2(v), one has

$$((\mathcal{L}_{\overline{v_i}}\Psi_J)^{(0,\ell)}_{(\ell,0)}, (\mathcal{L}_{\overline{v_i}}\Psi_J)^{(0,\ell)}_{(\ell,0)}) = (\Box G(\overline{\Phi_i}\cdot\Psi_J), \overline{\Phi_i}\cdot\Psi_J),$$

which together with (3.12), lead to the proposition. $\square$

Parallel to [TY, Proposition 2], we also compute II in (3.8) as follows:

Proposition 2.

$$(\mathcal{L}_{[\overline{v_i},v_i]}\Psi_J,\Psi_J) \;=\; -(\Box G\langle\Phi_i,\Phi_i\rangle,\langle\Psi_J,\Psi_J\rangle).$$

Proof. In the present Ricci-flat case, the proof of [TY, Proposition 2] still gives

$$(\mathcal{L}_{[\overline{v_i},v_i]}\Psi_J,\Psi_J) = -(\Box\langle v_i,v_i\rangle,\langle\Psi_J,\Psi_J\rangle). \tag{3.13}$$

On the other hand, using identity $\Box^2(\langle v_i,v_j\rangle) = \Box\langle\Phi_i,\Phi_j\rangle$ given in [Sch2, equation (2.9)], we have

$$\begin{aligned}(\Box\langle v_i,v_i\rangle,\langle\Psi_J,\Psi_J\rangle) &= ((H+G\Box)\Box\langle v_i,v_i\rangle,\langle\Psi_J,\Psi_J\rangle)\\ &= (\Box^2\langle v_i,v_i\rangle, G\langle\Psi_J,\Psi_J\rangle) \quad (\text{since } H\Box=0)\\ &= (\Box\langle\Phi_i,\Phi_j\rangle, G\langle\Psi_J,\Psi_J\rangle)\\ &= (\Box G\langle\Phi_i,\Phi_j\rangle,\langle\Psi_J,\Psi_J\rangle),\end{aligned} \tag{3.14}$$

which gives the proposition. ☐

Next we proceed to compute III. Similar to [Siu, p. 288] and as in [TY, Section 7], for $1\leqslant\ell\leqslant n$, we denote by $X^{(\ell)}$ the space of (relative) tensors $\Xi\in\mathcal{A}(\overline{\otimes^\ell T^*M_t}\otimes\overline{\otimes^\ell T^*M_t})$ with components $\Xi_{\overline{\alpha_1}\cdots\overline{\alpha_\ell},\overline{\beta_1}\cdots\overline{\beta_\ell}}$ satisfying the following three properties:

(P-i) $\Xi_{\overline{\alpha_1}\cdots\overline{\alpha_\ell},\overline{\beta_1}\cdots\overline{\beta_\ell}}$ is skew-symmetric in any pair of indices α_i,α_j for $i<j$;
(P-ii) $\Xi_{\overline{\alpha_1}\cdots\overline{\alpha_\ell},\overline{\beta_1}\cdots\overline{\beta_\ell}}$ is symmetric in the two ℓ-tuples of indices $(\overline{\alpha_1},\cdots,\overline{\alpha_\ell})$ and $(\overline{\beta_1},\cdots,\overline{\beta_\ell})$, and
(P-iii) for given indices $\alpha_1,\cdots,\alpha_{\ell-1}$, and $\beta_1,\cdots,\beta_{\ell+1}$, one has

$$\sum_{\nu=1}^{\ell+1}(-1)^\nu\Xi_{\overline{\alpha_1}\cdots\overline{\alpha_{\ell-1}}\overline{\beta_\nu},\overline{\beta_1}\cdots\widehat{\overline{\beta_\nu}}\cdots\overline{\beta_{\ell+1}}}=0,$$

where $\widehat{\overline{\beta_\nu}}$ means that the index $\overline{\beta_\nu}$ is omitted.

As in [Siu, p. 289] and [TY, Section 7], for $s=1,2$, we let $\overline{D_s}$ denote the operator $X^{(\ell)}$ given by taking $\overline{\partial}$ to the s-th ℓ-tuple of skew-symmetric indices, and we let $\overline{D_s}^*$ denote the adjoint operator of $\overline{D_s}$. Also, we denote $\Box_s=\overline{D_s}^*\overline{D_s}+\overline{D_s}\overline{D_s}^*$, and we denote by H_s the harmonic projection operator on $X^{(\ell)}$ with respect to $\Box_s$. The Green's operator on $X^{(\ell)}$ with respect to $\Box_s$ is denoted by G_s.

Lemma 3. *For any $\Xi\in X^{(\ell)}$, we have*

(a) $\overline{D_1D_2}\Xi=\overline{D_2D_1}\Xi$,
(b) $\overline{D_1}^*\overline{D_2}\Xi=\overline{D_2}\overline{D_1}^*\Xi$,
(c) $\overline{D_1}^*\overline{D_2}^*\Xi=\overline{D_2}^*\overline{D_1}^*\Xi$,
(d) $\overline{D_1}\overline{D_2}^*\Xi=\overline{D_2}^*\overline{D_1}\Xi$,
(e) $\Box_1\Xi\in X^{(\ell)}$ *and* $H_1(\Xi)\in X^{(\ell)}$,

(f) $\Box_1\Xi = \Box_2\Xi$, $H_1(\Xi) = H_2(\Xi)$, $G_1\Xi = G_2\Xi$, *and*
(g) if $\overline{D_1}\Xi = 0$, *then* $G_1\overline{D_2}^*\Xi = \overline{D_2}^* G_2\Xi$.

Proof. The proofs of the above properties (a) to (f) of $X^{(\ell)}$ (resp. (g)) follow mutatis mutandis from those in [Siu, p. 289-292] (resp. [Na, p.422-424]), which treated the case when $\ell = 2$. We will leave the details to the reader. ☐

Let Φ_i, Ψ_J (with $|J| = \ell$) be as in (3.1). By lowering indices of these objects, we obtain corresponding covariant tensors, which will be denoted by the same symbols (when no confusion arises). For example, Ψ_J also denotes the covariant tensor with components given by

$$(\Psi_J)_{\overline{\alpha_1}\cdots\overline{\alpha_\ell},\overline{\beta_1}\cdots\overline{\beta_\ell}} = g_{\gamma_1\overline{\beta_1}}\cdots g_{\gamma_\ell\overline{\beta_\ell}}(\Psi_J)^{\gamma_1\cdots\gamma_\ell}_{\overline{\alpha_1}\cdots\overline{\alpha_\ell}}.$$

We will skip the proof of the following simple lemma, which is the same as that in [TY, Lemma 10].

Lemma 4. *For each* $1 \leqslant \ell \leqslant n$, *we have* $\Psi_J \in X^{(\ell)}$ *and* $\Phi_i \owedge \Psi_J \in X^{(\ell+1)}$.

We will also skip the following lemma, whose statement and proof are the same as in [TY, Lamma 11] (and similar to that given in [Siu, pp. 286-288]).

Lemma 5. *We have*
(i) $\overline{D_2}^*(\Phi_i \owedge \Psi_J) = \overline{D_1}(\mathcal{L}_{v_i}\Psi_J)$,
(ii) $\overline{\partial}(\Phi_i \owedge \Psi_J) = 0$, *and*
(iii) $\overline{\partial}^*(\mathcal{L}_{v_i}\Psi_J) = 0$.

Parallel to [TY, Proposition 3], we give the computation of III in (3.8) as follows:

Proposition 3. *We have*

$$(\mathcal{L}_{v_i}\Psi_J, \mathcal{L}_{v_i}\Psi_J) = (H(\mathcal{L}_{v_i}\Psi_J), \mathcal{L}_{v_i}\Psi_J) + (\Box G(\Phi_i \owedge \Psi_J), \Phi_i \owedge \Psi_J).$$

Proof. The proof is similar to that of [TY, Proposition 3], involving generalizing the arguments in [Siu, p. 292-293]. First we have

$$(\mathcal{L}_{v_i}\Psi_J, \mathcal{L}_{v_i}\Psi_J) = (H(\mathcal{L}_{v_i}\Psi_J), \mathcal{L}_{v_i}\Psi_J) + (G\Box(\mathcal{L}_{v_i}\Psi_J), \mathcal{L}_{v_i}\Psi_J). \tag{3.15}$$

Now we consider the last term of (3.15). Upon lowering indices, it is given by

$$\begin{aligned}
& (G_1\Box_1(\mathcal{L}_{v_i}\Psi_J), \mathcal{L}_{v_i}\Psi_J) \\
= \;& ((G_1\overline{D_1}^*\overline{D_1}(\mathcal{L}_{v_i}\Psi_J), \mathcal{L}_{v_i}\Psi_J) \\
& \quad (\text{since } \overline{D_1}^*(\mathcal{L}_{v_i}\Psi_J) = 0 \text{ by Lemma 5(iii)}) \\
= \;& (\overline{D_1}^* G_1 \overline{D_2}^*(\Phi_i \oslash \Psi_J), \mathcal{L}_{v_i}\Psi_J) \quad (\text{by Lemma 5(i)}) \\
= \;& (G_1 \overline{D_2}^*(\Phi_i \oslash \Psi_J), \overline{D_1}(\mathcal{L}_{v_i}\Psi_J)) \\
= \;& (\overline{D_2}^* G_2(\Phi_i \oslash \Psi_J), \overline{D_2}^*(\Phi_i \oslash \Psi_J)) \\
& \quad (\text{by Lemma 4, Lemma 5(i), (ii) and Lemma 3(g)}) \\
= \;& (\overline{D_2 D_2}^* G_2(\Phi_i \oslash \Psi_J), \Phi_i \oslash \Psi_J) \\
= \;& (\Box_2 G_2(\Phi_i \oslash \Psi_J), \Phi_i \oslash \Psi_J) \quad (\text{since } G_2\overline{D_2} = \overline{D_2}G_2 \\
& \qquad \text{and } \overline{D_2}(\Phi_i \oslash \Psi_J) = 0 \text{ (by Lemma 5(ii)))}.
\end{aligned}$$

Upon raising indices and using (3.15), one obtains Proposition 3 readily.

Next we recall from [TY] the following pointwise identity:

Lemma 6. *(*[TY, Lemma 12]*) One has*

$$\begin{aligned}
(\Phi_i \oslash \Psi_J, \Phi_i \oslash \Psi_J) \;=\; & (\overline{\Phi_i} \cdot \Psi_J, \overline{\Phi_i} \cdot \Psi_J) + (\langle \Phi_i, \Phi_i \rangle, \langle \Psi_J, \Psi_J \rangle) \\
& -(\overline{\Phi_i} \searrow \Psi_J, \overline{\Phi_i} \searrow \Psi_J) - (\overline{\Phi_i} \nearrow \Psi_J, \overline{\Phi_i} \nearrow \Psi_J).
\end{aligned}$$

Similar to ([TY, Proposition 4]), we have

Proposition 4. *We have*

$$\begin{aligned}
\partial_i \overline{\partial_i} \log \|\Psi_J\|_2^2 = \frac{1}{\|\Psi_J\|_2^2} \big(& H(\overline{\Phi_i} \cdot \Psi_J), \overline{\Phi_i} \cdot \Psi_J) + (H(\langle \Phi_i, \Phi_i \rangle), \langle \Psi_J, \Psi_J \rangle) \\
& + ((H(\mathcal{L}_{v_i}\Psi_J), \mathcal{L}_{v_i}\Psi_J) - \big|(\mathcal{L}_{v_i}\Psi_J, \frac{\Psi_J}{\|\Psi_J\|_2})\big|^2 \\
& - (H(\Phi_i \oslash \Psi_J), H(\Phi_i \oslash \Psi_J))\big).
\end{aligned}$$

Proof. Similar to ([TY, Proposition 4]), the proposition follows immediately by combining (3.4), (3.6), (3.7), (3.8), Proposition 1, Proposition 2, Proposition 3 and Lemma 6. ☐

Proposition 5. *We have*

$$\begin{aligned}
\partial_i \overline{\partial_i} \log \|\Psi_J\|_2^2 \geqslant \frac{1}{\|\Psi_J\|_2^2} \big(& (H(\overline{\Phi_i} \cdot \Psi_J), \overline{\Phi_i} \cdot \Psi_J) + ((H(\langle \Phi_i, \Phi_i \rangle), \langle \Psi_J, \Psi_J \rangle) \\
& - (H(\Phi_i \oslash \Psi_J), H(\Phi_i \oslash \Psi_J))\big).
\end{aligned}$$

Proof. Similar to [TY, Proposition 5], by considering the spectral decomposition of $\mathcal{L}_{v_i}\Psi_J$ with respect to $\Box$ and using the fact that Ψ_J is harmonic, one clearly has

$$((H(\mathcal{L}_{v_i}\Psi_J), \mathcal{L}_{v_i}\Psi_J) = \|H(\mathcal{L}_{v_i}\Psi_J)\|_2^2 \geqslant \big|(\mathcal{L}_{v_i}\Psi_J, \frac{\Psi_J}{\|\Psi_J\|_2})\big|^2. \tag{3.16}$$

By combining (3.16) and Proposition 4, one obtains Proposition 5 easily. $\Box$

Now we state the main result in this section which is to be used to construct the augmented Weil-Petersson metric in the next section. For a positive integer ℓ, we define the relative tensor

$$H^{(\ell)} := H(\underbrace{\Phi_i \oslash \cdots \oslash \Phi_i}_{\ell\text{-times}}), \tag{3.17}$$

so that $H^{(\ell)} = \Psi_J$ with J given by the ℓ-tuple $(i, i, \cdots, i)$. Note that $H^{(\ell)}$ actually depends on i, but for simplicity, this is suppressed in the notation. We also adopt the convention that $H^{(0)}$ is the constant function 1. Parallel to [TY, Proposition 6], we have

Proposition 6. *Let $i, \ell, H^{(\ell)}$ be as in (3.17). Suppose $\|H^{(\ell)}\|_2 > 0$ (so that $\|H^{(\ell-1)}\|_2 > 0$ (cf. (3.20)). Then we have*

$$\partial_i\overline{\partial_i}\log\|H^{(\ell)}\|_2^2 \geq \frac{\|H^{(\ell)}\|_2^2}{\|H^{(\ell-1)}\|_2^2} - \frac{\|H^{(\ell+1)}\|_2^2}{\|H^{(\ell)}\|_2^2}. \tag{3.18}$$

Proof. To deduce (3.18) from Proposition 5 (with Ψ_J there given by $H^{(\ell)}$), one first observes that $H(\langle\Phi_i, \Phi_i\rangle)$ is a positive constant function (say, with constant value $c > 0$). Then the second last term of Proposition 5 satisfies

$$((H(\langle\Phi_i, \Phi_i\rangle), \langle H^{(\ell)}, H^{(\ell)}\rangle) = c \cdot \|H^{(\ell)}\|_2^2 > 0. \tag{3.19}$$

Next for $\ell \geq 0$, we recall from [TY, Lemma 13] the following two equalities, whose proofs also hold in the present case:

$$H(\Phi_i \oslash H^{(\ell-1)}) = H^{(\ell)}, \quad \text{and} \tag{3.20}$$

$$(\overline{\Phi_i} \cdot H^{(\ell)}, H^{(\ell-1)}) = \|H^{(\ell)}\|_2^2. \tag{3.21}$$

Then by considering the spectral decomposition of $\overline{\Phi_i} \cdot H^{(\ell)}$ with respect to $\Box$, one has, from (3.21),

$$(H(\overline{\Phi_i} \cdot H^{(\ell)}), \overline{\Phi_i} \cdot H^{(\ell)}) \geq \big|(\overline{\Phi_i} \cdot H^{(\ell)}, \frac{H^{(\ell-1)}}{\|H^{(\ell-1)}\|_2})\big|^2 = \frac{\|H^{(\ell)}\|_2^4}{\|H^{(\ell-1)}\|_2^2}. \tag{3.22}$$

By combining Proposition 5, (3.19), (3.20) and (3.22), one obtains Proposition 6 readiy. $\Box$

4. Curvature of the augmented Weil-Petersson metric

Let $\pi : \mathcal{X} \to S$ be an effectively parametrized family of polarized Ricci-flat Kähler manifolds as in Theorem 1. As before, we let $M_t = \pi^{-1}(M_t)$ for $t \in S$, and denote $n = \dim_{\mathbb{C}} M_t$ and $m = \dim_{\mathbb{C}} S$. Without loss of generality, we assume that $n \geq 2$. Following the arguments in [TY, Section 9], we are going to construct an augmented Weil-Petersson metric on S, whose holomorphic sectional curvature is bounded above by a negative constant.

First we let $N = n!$, and let V be the volume of (X_t, ω_t) (which is independent of t) as in (2.7). We consider the following two sequences of positive numbers $\{C_\ell\}_{1\leq \ell \leq n}$ and $\{a_\ell\}_{1\leq \ell\leq n}$ given by

$$C_1 : = \min\big\{1, \frac{1}{V}\big\}, \quad C_\ell = \frac{C_{\ell-1}}{3} = \frac{C_1}{3^{\ell-1}}, \quad 2 \leq \ell \leq n, \tag{4.1}$$

$$a_1 : = 1, \quad a_\ell = \big(\frac{3a_{\ell-1}}{C_1}\big)^N = \big(\frac{3}{C_1}\big)^{\frac{N(N^{\ell-1}-1)}{N-1}}, \quad 2 \leq \ell \leq n. \tag{4.2}$$

First we recall from [TY, Lemma 14] (with the constant A there replaced by V) the following inequality:

Lemma 7. *(*[TY, Lemma 14]*) Let $N \geq n \geq 2$, V and $\{C_\ell\}_{1\leq\ell\leq n}$ and $\{a_\ell\}_{1\leq\ell\leq n}$ be as above, and let κ be an integer satisfying $1 \leq \kappa \leq n$. Then for all real numbers $x_1, \cdots, x_\kappa > 0$, we have*

$$\frac{a_1 x_1^{N+1}}{V} + \sum_{\ell=2}^{\kappa} \big(\frac{a_\ell}{\ell} \cdot \frac{x_\ell^{N+\ell}}{x_{\ell-1}^{\ell-1}} - \frac{a_{\ell-1}}{\ell-1} \cdot x_{\ell-1}^{N-\ell+1} x_\ell^\ell\big) \geq C_\kappa \cdot \sum_{\ell=1}^{\kappa} x_\ell^{N+1}. \tag{4.3}$$

(When $\kappa = 1$, the first summation in (4.3) is understood to be zero.)

With the above choice of the a_ℓ's, we define an augmented Weil-Petersson metric h_{aWP} on S to be the Finsler metric given by

$$h_{aWP}(u) = \Big(\sum_{\ell=1}^{n} a_\ell \|u\|^{2N}_{WP,\ell}\Big)^{\frac{1}{2N}} \quad \text{for } u \in T_tS \text{ and } t \in S. \tag{4.4}$$

Here $\| \ \|_{WP,\ell}$ is as defined in (2.12). Next we recall the following well-known simple lemma:

Lemma 8. *(*[Sch2, Lemma 8] *or* [TY, Lemma 15]*) Let U be a complex manifold, and ϕ_ℓ, $1 \leq \ell \leq r$, be positive C^2 functions on U. Then*

$$\sqrt{-1}\partial\overline{\partial}\log(\sum_{\ell=1}^{r} \phi_\ell) \geq \frac{\sum_{\ell=1}^{r} \phi_\ell \sqrt{-1}\partial\overline{\partial}\log\phi_\ell}{\sum_{j=1}^{r}\phi_j}. \tag{4.5}$$

Let $u \in TS$ and ℓ be an integer satisfying $1 \le \ell \le n$. Similar to (3.17), we denote

$$H^{(\ell)}(u) := H(\underbrace{\Phi(u) \oslash \cdots \oslash \Phi(u)}_{\ell\text{–times}}), \tag{4.6}$$

where $\Phi(u)$ is the harmonic representative of $\rho_t(u)$ as in Section 2. This gives rise to a function $r : \mathbb{P}TS \to \mathbb{Z}$ given by

$$r([u]) := \max\{\ell \,\big|\, H^{(\ell)}(u) \neq 0\} \quad \text{for } 0 \neq u \in TS, \tag{4.7}$$

where $[u]$ denotes the class of u in $\mathbb{P}TS$. Since ρ_t is injective for each $t \in S$, it follows that $1 \le r([u]) \le n$ for each $[u] \in \mathbb{P}TS$. Now we let R be a local one-dimensional complex submanifold of S. Then it is easy to see that r induces a function $r_R : R \to \mathbb{Z}$ given by

$$r_R(t) := r([u_t]) \quad \text{for } t \in R, \tag{4.8}$$

where u_t is any non-zero vector in T_tR. Let κ be an integer satisfying $1 \le \kappa \le n$. Following [TY, Section 9], we say that a point $t_o \in R$ is a *κ-stable point of R* if there exists an open neighborhood $\mathcal{U}_{t_o}$ of t_o in R such that $r_R(t) = \kappa$ for all $t \in \mathcal{U}_{t_o}$. We also recall that for a C^∞ Finsler metric h on S, the sectional curvature $K(R, h\big|_R)(t_o)$ of $h\big|_R$ at a point $t_o \in R$ is given by

$$K(R, h\big|_R)(t_o) = -\frac{\partial_t \partial_{\bar{t}} \log((h(\frac{\partial}{\partial t}))^2)}{(h(\frac{\partial}{\partial t}))^2}\Big|_{t=t_o}, \tag{4.9}$$

where t denotes a local holomorphic coordinate function on some open subset of R containing t_o.

Proposition 7. *Let h_{aWP} be as in (4.4). Let R be a local one-dimensional complex submanifold of S, and let $t_o \in R$ be a κ-stable point of R for some integer $1 \le \kappa \le n$. Then we have*

$$K(R, h_{aWP}\big|_R)(t_o) \le -\frac{C_\kappa}{\kappa^{\frac{1}{N}} a_\kappa^{1+\frac{1}{N}}},$$

where a_κ and C_κ are as in (4.2) and (4.1).

Proof. The proof follows from a calculation exactly as that in [TY, Proposition 7]. For convenience of the reader, we will sketch the calculation here and refer the reader to [TY, Proposition 7] for details. First since t_o is a κ-stable point of R, there exists an open coordinate neighborhood $\mathcal{U}$ of t_o in R (with coordinate tangent vector $\partial/\partial t$) such that the terms on the right hand side of (4.4) corresponding to $\ell = \kappa + 1$ to $\ell = n$ are all identically zero on U, so that one may write (4.4) as

$$(4.10)\quad h_{aWP}(\frac{\partial}{\partial t}) = \Big(\sum_{\ell=1}^{\kappa} a_\ell \|H^{(\ell)}\|_2^{\frac{2N}{\ell}}\Big)^{\frac{1}{2N}} \quad \text{on } \mathcal{U}, \quad \text{where } H^{(\ell)} := H^{(\ell)}(\frac{\partial}{\partial t})$$

(cf. (4.6)). Together with Lemma 8 and Proposition 6, one gets

$$(4.11)\quad \partial_t\partial_{\bar t}\log((h(\frac{\partial}{\partial t}))^2) \geq \frac{\sum_{\ell=1}^{\kappa} \frac{a_\ell}{\ell}\cdot \|H^{(\ell)}\|_2^{\frac{2N}{\ell}}\cdot \big(\frac{\|H^{(\ell)}\|_2^2}{\|H^{(\ell-1)}\|_2^2} - \frac{\|H^{(\ell+1)}\|_2^2}{\|H^{(\ell)}\|_2^2}\big)}{\sum_{\ell=1}^{\kappa} a_\ell \|H^{(\ell)}\|_2^{\frac{2N}{\ell}}} =: \frac{B}{C}.$$

By rearranging the terms of B telescopically (so that the first expression of the ℓ-th term is grouped with the second expression of the $(\ell-1)$-th term) and using that fact that $\|H^{(\kappa+1)}\|_2 = 0$, one deduces readily from Lemma 7 (with x_ℓ given here by $\|H^{(\ell)}\|_2^{\frac{2}{\ell}}$) that

$$(4.12)\qquad B \geq C_\kappa \cdot \sum_{\ell=1}^{\kappa} \|H^{(\ell)}\|_2^{\frac{2(N+1)}{\ell}},$$

From (4.10), (4.11) and using the fact that $a_\ell \geq a_{\ell-1}$, one has

$$(4.13)\quad C\cdot (h(\frac{\partial}{\partial t}))^2 \leq \Big(\sum_{\ell=1}^{\kappa} a_\kappa \|H^{(\ell)}\|_2^{\frac{2N}{\ell}}\Big)^{\frac{N+1}{N}} \leq \big(\kappa a_\kappa^{N+1}\big)^{\frac{1}{N}} \cdot \sum_{\ell=1}^{\kappa} \|H^{(\ell)}\|_2^{\frac{2(N+1)}{\ell}},$$

where the last inequality follows from Hölder inequality. By combining (4.9), (4.11), (4.12) and (4.13), one obtains the proposition readily. $\square$

We are ready to give the proof of Theorem 1 as follows:

Proof of Theorem 1. Let $\pi : \mathcal{X} \to S$ be as in Theorem 1, and let $n := \dim M_t$. Let h_{aWP} be as in (4.4). As mentioned in Remark 1, h_{aWP} is $\mathrm{Aut}(\pi)$-invariant and C^∞. Take a point $t \in S$, and let R be a local one-dimensional complex submanifold of S passing through t (i.e. $t \in R$). As in [TY, Lemma 17], the set

$$(4.14)\qquad \mathcal{Q}_R := \{t \in R \,\big|\, t \text{ is a } \kappa\text{-stable point of } R \text{ for some } 1 \leq \kappa \leq n\}$$

is easily seen to be a dense subset of R (with respect to the usual topology). Thus there exists a sequence of points $\{t_j\}_{j=1}^\infty$ in $\mathcal{Q}_R$ such that $\lim_{j\to\infty} t_j = t$ in R. In particular, each t_j is a κ_j-stable point of R for some integer κ_j satisfying $1 \leq \kappa_j \leq n$. By Proposition 7, we have, for each j,

$$(4.15)\qquad K(R, h_{aWP}\big|_R)(t_j) \leq -\frac{C_{\kappa_j}}{\kappa_j^{\frac{1}{N}} a_{\kappa_j}^{1+\frac{1}{N}}} \leq -\frac{C_n}{n^{\frac{1}{N}} a_n^{1+\frac{1}{N}}},$$

where the last inequality follows from the facts that C_κ decreases with κ while a_κ increases with κ. Together with the fact that $h_{aWP}\big|_R$ is C^∞, one concludes readily that (4.15) also holds at t (i.e., with t_j there replaced by t). Together with (2.2), it follows that the holomorphic sectional curvature of h_{aWP} on S is bounded above by a negative constant. $\square$

Remark 2. *We remark that in general, the augmented Weil-Petersson metric in Theorem 1 is not unique, and its construction actually gives rise to a continuous family of Finsler metrics of negative holomorphic sectional curvature bounded away from zero. One way to see this is as follows: for any ϵ satisfying $0 \le \epsilon < 1$, if one replaces the constant $a_1 = 1$ by $1-\epsilon$ in (4.4) while keeping the other a_ℓ's unchanged, one gets a family of augmented Weil-Petersson metrics parametrized by ϵ given by*

(4.16)
$$h_{aWP,\epsilon}(u) = \Big((1-\epsilon)\|u\|_{WP,1}^{2N} + \sum_{\ell=2}^{n} a_\ell \|u\|_{WP,\ell}^{2N}\Big)^{\frac{1}{2N}} \quad \text{for } u \in T_tS \text{ and } t \in S.$$

It is easy to see that there exists $\epsilon_o > 0$ such that for each ϵ satisfying $0 \le \epsilon < \epsilon_o$, the holomorphic sectional curvature of $h_{aWP,\epsilon}$ is bounded above by some negative constant. Next we consider the one-dimensional moduli space $\mathcal{M}$ of Calabi-Yau threefolds which are mirror manifolds of the quintic hypersurfaces in $\mathbb{CP}^4$. As mentioned in Section 1, Candelas et al [CDGP] *showed that the holomorphic sectional curvature of the Weil-Petersson metric (i.e. $\| \ \|_{WP,1}$) on $\mathcal{M}$ is positive at some points of $\mathcal{M}$ and negative at other points of $\mathcal{M}$. Using this fact and the negativity of the holomorphic sectional curvature of the $h_{aWP,\epsilon}$'s, one easily sees that $h_{aWP,\epsilon}$ is not a constant multiple of $h_{aWP,\epsilon'}$ on $\mathcal{M}$ (and we simply say that they are inequivalent) whenever $0 \le \epsilon < \epsilon' < \epsilon_o$ (in fact, for $0 \le \epsilon < \epsilon'$, the equation $h_{aWP,\epsilon} = c \cdot h_{aWP,\epsilon'}$ for some constant $c > 0$ implies readily that both $h_{aWP,\epsilon}$ and $h_{aWP,\epsilon'}$ are constant multiples of $\| \ \|_{WP,1}$). It follows that $\mathcal{M}$ admits a continuous family of pairwise inequivalent augmented Weil-Petersson metrics.*

5. **The general case of families of polarized Ricci-flat Kähler orbifolds**

In this section, we are going to consider the general case of a family of polarized Ricci-flat Kähler orbifolds and give the proof of Theorem 1'. When the arguments or calculations given in the previous sections also work in the present orbifold case, we will often avoid repeating them here and only indicate the necessary modifications. First we recall some definitions.

An n-dimensional *complex orbifold* is a complex analytic space M of complex dimension n together with a basis of open subsets $\{\widehat{U}_\alpha\}_{\alpha\in\mathcal{A}}$ covering M such that for each open set $\widehat{U}_\alpha \in \mathcal{A}$, there exist an associated open set $U_\alpha \subset \mathbb{C}^n$, a finite subgroup $\Gamma_\alpha \subset \mathrm{Aut}\,(U_\alpha)$ (here $\mathrm{Aut}\,(U_\alpha)$ denotes the group of self-biholomorphisms on U_α) and a holomorphic map $p_\alpha : U_\alpha \to \widehat{U}_\alpha$ which is Γ_α-invariant (i.e., $p_\alpha \circ \gamma = p_\alpha$ for all $\gamma \in \Gamma_\alpha$) and induces a biholomorphism between $\widehat{U}_\alpha$ and U_α/Γ_α (so that we may write $\widehat{U}_\alpha = U_\alpha/\Gamma_\alpha$); furthermore, if $\widehat{U}_\alpha \subset \widehat{U}_\beta$ (with $\alpha, \beta \in \mathcal{A}$), then there exist a group homomorphism $\tau_{\alpha\beta} : \Gamma_\alpha \to \Gamma_\beta$ and an injective holomorphic map $\phi_{\alpha\beta} : U_\alpha \to U_\beta$ such that

$$p_\beta \circ \phi_{\alpha\beta} = p_\alpha, \text{ and } \tau_{\alpha\beta}(\gamma) \circ \phi_{\alpha\beta} = \phi_{\alpha\beta} \circ \gamma \text{ for all } \gamma \in \Gamma_\alpha. \tag{5.1}$$

(A complex orbifold is also called a *complex V-manifold* à la Satake and Baily (cf. [Ba1], [Ba2], [Sa]).) Each local holomorphic covering projection map $p_\alpha : U_\alpha \to \widehat{U}_\alpha = U_\alpha/\Gamma_\alpha$ is known as an *orbifold chart* of M, and the collection orbifold charts is called an orbifold atlas of M. The *orbifold singular set* M^s of M is the subset $\{x \in M \,|\, \gamma(y) = y \text{ for some } y \in p_\alpha^{-1}(x),\ e \neq \gamma \in \Gamma_\alpha \text{ with } \alpha \in \mathcal{A}\}$ (here e denotes the identity element of Γ_α). Note that M^s is a complex analytic subvariety of M and $M \setminus M^s$ lies in the smooth part M° of (the underlying complex space) M, but it may happen that $M \setminus M^s \subsetneqq M^\circ$.

A differential form η on a complex orbifold M (with an orbifold atlas $\{p_\alpha : U_\alpha \to \widehat{U}_\alpha = U_\alpha/\Gamma_\alpha\}_{\alpha\in\mathcal{A}}$) is a collection of differential forms $\{\eta_\alpha\}_{\alpha\in\mathcal{A}}$, where each η_α is a differential form on U_α invariant under Γ_α (i.e., $\gamma^*\eta_\alpha = \eta_\alpha$ for all $\gamma \in \Gamma_\alpha$), and $\phi^*_{\alpha\beta}\eta_\beta = \eta_\alpha$ whenever $\widehat{U}_\alpha \subset \widehat{U}_\beta$. The differential η on M is said to possess a certain property (such as being an (r, s)-form, or being d-closed) if each η_α possesses such property on U_α. (Alternatively, a differential form (with a certain property) on the complex orbifold M can also be defined as a differential form η on $M \setminus M^s$ such that $p^*_\alpha\eta$ extends to a differential form (with the same property) on U_α for each $\alpha \in \mathcal{A}$.) In particular, a differential fom ω on M is said to be a Kähler form if ω_α is a Kähler form on U_α for each $\alpha \in \mathcal{A}$. A *(Ricci-flat) Kähler orbifold* is a complex orbifold M equipped with a (Ricci-flat) Kähler form ω. We remark that when M is smooth and M^s is a smooth divisor of M, then ω gives an example of a conical Ricci-flat Kähler form on the pair (M, M^s) (see [Br] for the definition and existence results of conical Ricci-flat Kähler forms).

An orbifold vector bundle E over a complex orbifold M (with an orbifold atlas $\{p_\alpha : U_\alpha \to \widehat{U}_\alpha = U_\alpha/\Gamma_\alpha\}_{\alpha\in\mathcal{A}}$) is a collection of vector bundles $\{E_\alpha\}_{\alpha\in\mathcal{A}}$, where E_α is a vector bundle over U_α (with the projection map denoted by $q_\alpha : E_\alpha \to U_\alpha$) for each $\alpha \in \mathcal{A}$, and for each $\alpha \in \mathcal{A}$, there is an associated group homomorphism $\nu_\alpha : \Gamma_\alpha \to \mathrm{Aut}(E_\alpha)$ (here $\mathrm{Aut}(E_\alpha)$ denotes the group of vector bundle automorphisms of E_α) such that $q_\alpha \circ \nu_\alpha(\gamma) = \gamma \circ q_\alpha$ for all $\gamma \in \Gamma_\alpha$; furthermore, if $\widehat{U}_\alpha \subset \widehat{U}_\beta$ for some $\alpha, \beta \in \mathcal{A}$, then there exists a bundle map $\rho_{\alpha\beta} : E_\alpha \to E_\beta$ such that $(\nu_\beta \circ \tau_{\alpha\beta})(\gamma) \circ \rho_{\alpha\beta} = \rho_{\alpha\beta} \circ \nu_\alpha(\gamma)$ for all $\gamma \in \Gamma_\alpha$ (here $\tau_{\alpha\beta}$ is as in (5.1)), and one has $\rho_{\alpha\beta} \circ \rho_{\beta\delta} = \rho_{\alpha\delta}$ if $\widehat{U}_\alpha \subset \widehat{U}_\beta \subset \widehat{U}_\delta$. Note that E descends to a vector bundle on $M \setminus M^s$ (in the usual sense). A typical example of an orbifold vector bundle is the orbifold tangent bundle TM given by the collection of tangent bundles $\{TU_\alpha\}_{\alpha\in\mathcal{A}}$ with the action of $\gamma \in \Gamma_\alpha$ given by the jacobian of γ, i.e., $\nu_\alpha(\gamma) = \gamma_*$. Hermitian metrics on orbifold vector bundles over complex orbifolds are defined in the obvious way.

Let $\varphi = \{\varphi_\alpha\}_{\alpha\in\mathcal{A}}$ be an (n,n)-form on an n-dimensional complex orbifold M with an orbifold atlas $\{p_\alpha : U_\alpha \to \widehat{U}_\alpha = U_\alpha/\Gamma_\alpha\}_{\alpha\in\mathcal{A}}$. For each $\alpha \in \mathcal{A}$, the integral of φ over $\widehat{U}_\alpha$ is given by $\displaystyle\int_{\widehat{U}_\alpha} \varphi := \frac{1}{|\Gamma_\alpha|}\int_{U_\alpha} \varphi_\alpha$, where $|\Gamma_\alpha|$ denotes the order of the group Γ_α. Then the integral $\displaystyle\int_M \varphi$ is defined by using a partition of unity in the obvious manner. We remark that by well-known results of Baily (cf. [Ba1], [Ba2]), many natural differential operators on manifolds (such as the de Rham operator d (and its decomposition $d = \partial + \overline{\partial}$) and associated objects (such as de Rham and Dolbeault cohomology classes) generalize to orbifolds in the obvious manner, and many standard results on these operators (such as Stokes' Theorem) also hold for orbifolds. In particular, Hodge decomposition theorem also holds for orbifold vector bundles over compact Kähler orbifolds; and under such setting, the harmonic projection operator and Green's operator make sense in the obvious manner (see [Ba1, Section 2 and Section 7]).

Let $\pi : \mathcal{X} \to S$ be a holomorphic map from an $(n+m)$-dimensional complex orbifold $\mathcal{X}$ (with an orbifold atlas $\{p_\alpha : \mathcal{U}_\alpha \to \widehat{\mathcal{U}}_\alpha = \mathcal{U}_\alpha/\Gamma_\alpha\}_{\alpha\in\mathcal{A}}$) to an m-dimensional complex manifold S. Then $\pi : \mathcal{X} \to S$ is said to form a holomorphic family of complex orbifolds over S if (i) π is surjective and of maximal rank (here π is of maximal rank means that for each $\alpha \in \mathcal{A}$, $\pi\circ p_\alpha$ is of maximal rank at all points of $\mathcal{U}_\alpha$; and (ii) for each $\alpha \in \mathcal{A}$, $\mathcal{U}_\alpha$ is of the form $U_\alpha \times W_\alpha$, where U_α is an open subset of $\mathbb{C}^n$ and $W_\alpha \subset \mathbb{C}^m$ is an open coordinate subset of S, so that $\pi \circ p_\alpha$ is given by the coordinate projection map onto the second factor W_α; furthermore, each $\gamma \in \Gamma_\alpha (\subset \mathrm{Aut}(\mathcal{U}_\alpha))$ is of the form $\begin{pmatrix} \gamma' & 0 \\ 0 & \mathrm{e}_m \end{pmatrix}$ for some $\gamma' \in \mathrm{Aut}(U_\alpha)$, where e_m denotes the identity map on W_α. (Note that under the identification $\gamma \longleftrightarrow \gamma'$, we may regard Γ_α as a subgroup of $\mathrm{Aut}(U_\alpha)$, and we may write

$$\widehat{\mathcal{U}}_\alpha = \mathcal{U}_\alpha/\Gamma_\alpha = (U_\alpha \times W_\alpha)/\Gamma_\alpha = (U_\alpha/\Gamma_\alpha) \times W_\alpha = \widehat{U}_\alpha \times W_\alpha. \tag{5.2}$$

Note that with the above identification, we may write $p_\alpha : \mathcal{U}_\alpha \to \widehat{\mathcal{U}}_\alpha$ as $(p'_\alpha, e_m) : U_\alpha \times W_\alpha \to \widehat{U}_\alpha \times W_\alpha$, where $p'_\alpha : U_\alpha \to \widehat{U}_\alpha$ is the projection map under the induced action of Γ_α on U_α. It is easy to see that under these two conditions, for each $t \in S$, the fiber $M_t := \pi^{-1}(t)$ is a complex orbifold with an orbifold atlas given by $\{p'_\alpha : U_\alpha \to \widehat{U}_\alpha = U_\alpha/\Gamma_\alpha\}_{\alpha\in\mathcal{A}})$. Note that the fibers M_t's are all homeomorphic to each other. Finally a holomorphic family of complex orbifolds $\pi : \mathcal{X} \to S$ over a complex manifold S is said to form a holomorphic family of compact polarized Ricci-flat Kähler orbifolds if the fiber M_t is compact for each $t \in S$, and the complex orbifold $\mathcal{X}$ is endowed with a d-closed $(1,1)$-form ω such that its restriction $\omega_t := \omega\big|_{M_t}$ to each fiber M_t, $t \in S$, is a Ricci-flat Kähler form on the complex orbifold M_t.

We proceed to consider the deformation theory of compact complex orbifolds. Let $\pi : \mathcal{X} \to S$ be a holomorphic family of compact complex orbifolds over a complex manifold S (satisfying conditions (i) and (ii) in the above paragraph). We fix a point $t_o \in S$, and let W be an open coordinate neighborhood of t_o in S. Then shrinking W and replacing $\mathcal{A}$ by a subset if necessary, we may assume that the restricted family $\pi\big|_{\pi^{-1}(W)} : \pi^{-1}(W) \to W$ admits an orbifold atlas $\{p_\alpha : \mathcal{U}_\alpha \to \widehat{\mathcal{U}}_\alpha = \mathcal{U}_\alpha/\Gamma_\alpha\}_{\alpha\in\mathcal{A}}$ and the fiber $M_{t_o} = \pi^{-1}(t_o)$ admits a corresponding orbifold atlas $\{p'_\alpha : U_\alpha \to \widehat{U}_\alpha = U_\alpha/\Gamma_\alpha\}_{\alpha\in\mathcal{A}}$, where for each $\alpha \in \mathcal{A}$, we may write, as in (5.2),

$$\widehat{\mathcal{U}}_\alpha = \mathcal{U}_\alpha/\Gamma_\alpha = (U_\alpha \times W)/\Gamma_\alpha = (U_\alpha/\Gamma_\alpha) \times W = \widehat{U}_\alpha \times W, \tag{5.3}$$

and one has an associated decomposition of p_α given by $p_\alpha = (p'_\alpha, e_W)$. Here e_W denotes the identity map on W. Now we take a homeomorphism $\Xi : M_{t_o}\times W \to \pi^{-1}(W)$ such that $\Xi\big|_{M_{t_o}\times\{t_o\}}$ is the identity map on M_{t_o}, $\pi \circ \Xi(x,t) = t$ for all $t \in W$ and $x \in M_{t_o}$, and for $\alpha, \beta \in \mathcal{A}$ such that $\Xi(\widehat{U}_\alpha \times W) \subset \widehat{\mathcal{U}}_\beta$, one has a lifting of $\Xi\big|_{\widehat{U}_\alpha\times W}$ to a diffeomorphism $\Xi_{\alpha\beta} : U_\alpha \times W \to \mathcal{U}_\beta$ from $U_\alpha \times W$ into $\mathcal{U}_\beta$ such that $\tau_{\alpha\beta}(\gamma)(\Xi_{\alpha\beta}(z,t)) = \Xi_{\alpha\beta}(\gamma(z), t)$ for all $(z,t) \in U_\alpha \times W$ and $\gamma \in \Gamma_\alpha$ (here $\tau_{\alpha\beta}$ is as in (5.1)). Such a Ξ corresponds to a lifting of a vector field u on W to a vector field v_u on $(\mathcal{X} \setminus \mathcal{X}^s) \cap \pi^{-1}(W)$ (so that $\pi_* v_u = u$) such that for each orbifold chart $p_\alpha : \mathcal{U}_\alpha \to \widehat{\mathcal{U}}_\alpha = \mathcal{U}_\alpha/\Gamma_\alpha$ with $\pi(\widehat{\mathcal{U}}_\alpha) \subset W$, v_u lifts (and then extends) to a Γ_α-invariant vector field $v_{u,\alpha}$ on $\mathcal{U}_\alpha$ (so that $(p_\alpha)_* v_{u,\alpha} = v_u$ on $(\mathcal{X} \setminus \mathcal{X}^s) \cap \pi^{-1}(W)$ and $\gamma_* v_{u,\alpha} = v_{u,\alpha}$ for all $\gamma \in \Gamma_\alpha$).

As in the smooth case and with the identification given in (5.3), by taking $\overline{\partial}$ of $v_{u,\alpha}$ along the fiber directions on $U_\alpha \times \{t_o\}$ (which is identified with U_α), one gets a Γ_α-invariant $\overline{\partial}$-closed TU_α-valued $(0,1)$ form on U_α. As such, we get a $\overline{\partial}$-closed TM_{t_o}-valued $(0,1)$-form on M_{t_o}, which is simply denoted by $\overline{\partial} v_u\big|_{M_{t_o}}$. As in the smooth case, it is easy to see that if $\overline{\partial} v'_u\big|_{M_{t_o}}$ is another $\overline{\partial}$-closed TM_{t_o}-valued $(0,1)$-form on M_{t_o} corresponding to another lifting of u, then $\overline{\partial} v_u\big|_{M_{t_o}} - \overline{\partial} v'_u\big|_{M_{t_o}}$ is $\overline{\partial}$-exact on M_{t_o}. Thus, we have a well-defined Kodaira-Spencer map $\rho : T_{t_o}S \to H^{0,1}(M_{t_o}, TM_{t_o})$ given by $\rho_{t_o}(v) = [\overline{\partial} v_u\big|_{M_{t_o}}]$, where $[\overline{\partial} v_u\big|_{M_{t_o}}]$ denotes the Dolbeault cohomology class of $\overline{\partial} v_u\big|_{M_{t_o}}$. As in the smooth case, a holomorphic family $\pi : \mathcal{X} \to S$ of complex orbifolds is said to be effectively parametrized if the Kodaira spencer map ρ_t is injective for each $t \in S$.

For the remainder of the section and as in Theorem 1', we let $\pi : \mathcal{X} \to S$ be an effectively parametrized holomorphic family of n-dimensional compact polarized Ricci-flat Kähler orbifolds over an m-dimensional complex manifold S. Let ω be the associated d-closed $(1,1)$-form on $\mathcal{X}$ such that its restriction $\omega_t := \omega\big|_{M_t}$ is a Ricci-flat Kähler form on the complex orbifold M_t for each $t \in S$, and let g be the associated metric tensor of type $(1,1)$ on $\mathcal{X}$. Let $\{p_\alpha : \mathcal{U}_\alpha \to \widehat{\mathcal{U}}_\alpha = \mathcal{U}_\alpha/\Gamma_\alpha\}_{\alpha\in\mathcal{A}}$ be an orbifold atlas of $\mathcal{X}$ which gives rise to an associated orbifold atlas $\{p'_\alpha : U_\alpha \to \widehat{U}_\alpha = U_\alpha/\Gamma_\alpha\}_{\alpha\in\mathcal{A}}$ on M_t for each $t \in S$ as given in (5.2). For a local tangent vector field u of type $(1,0)$ on an open subset W of S, one obtains the horizontal lifting v_u of u on $(\mathcal{X} \setminus \mathcal{X}^s) \cap \pi^{-1}(W)$ with respect to ω as described in Section 2 for the smooth case. Alternatively, for each orbifold chart $p_\alpha : \mathcal{U}_\alpha \to \widehat{\mathcal{U}}_\alpha = \mathcal{U}_\alpha/\Gamma_\alpha$ such that $\widehat{U}_\alpha \subset \pi^{-1}(W)$, recall that ω lifts to a Γ_α-invariant form ω_α on $\mathcal{U}_\alpha$. Then one can consider the horizontal lifting $v_{u,\alpha}$ of u to $\mathcal{U}_\alpha$ with respect to ω_α (and the projection map $\pi \circ p_\alpha$). It is easy to see that each $v_{u,\alpha}$ is Γ_α-invariant, and $(p_\alpha)_* v_{u,\alpha} = v_u$ on $\widehat{\mathcal{U}}_\alpha \cap (\mathcal{X} \setminus \mathcal{X}^s)$. As in the smooth case in Section 2, one easily sees that $\Phi(u) := \overline{\partial} v_u\big|_{M_t}$ is the unique harmonic Kodaira-Spencer representative of $\rho_t(u)$ for $t \in W \subset S$. When W is a coordinate open subset of S with coordinates given by $t = (t_1, \cdots t_m)$ and $u = \partial/\partial t_i$ is a coordinate vector field, it is easy to see that its horizontal lifting v_i of $\partial/\partial t_i$ and the correspnding harmonic Kodaira-Spencer representative $\Phi_i := \overline{\partial} v_i\big|_{M_t}$ are such that their Γ_α-invariant liftings on the orbifold charts $\mathcal{U}_\alpha$'s (and thus also v_i and Φ_i on $\widehat{\mathcal{U}}_\alpha \cap (\mathcal{X} \setminus \mathcal{X}^s)$) are given locally by the same expressions as in (2.3) and (2.4) respectively. Then the Weil-Petersson metric $h^{(WP)}$ on S is given by the same expression as in (2.6). For integers $p, q, r, s \geq 0$, $t \in S$, $\Phi \in \mathcal{A}^{0,p}(\wedge^r TM_t)$ and $\Psi \in \mathcal{A}^{0,q}(\wedge^s TM_t)$, one defines $\Phi \owedge \Psi \in \mathcal{A}^{0,p+q}(\wedge^{r+s} TM_t)$ in the obvious manner, namely by first lifting Φ, Ψ to corresponding bundle-valued forms Φ_α, Ψ_α on each U_α (for each orbifold chart $p'_\alpha : U_\alpha \to \widehat{U}_\alpha = U_\alpha/\Gamma_\alpha$) and considering the Γ_α-invariant bundle-valued form $\Phi_\alpha \owedge \Psi_\alpha$ on U_α (as described in Section 2). Then for each $1 \leq \ell \leq n$, one can define the ℓ-th generalized Weil-Petersson pseudo-metric on S as given in (2.11) and (2.12). Finally one can define augmented Weil-Petersson metrics on S as given in (2.13).

Proof of Theorem 1'. Let $\pi : \mathcal{X} \to S$ be an effectively parametrized holomorphic family of n-dimensional compact polarized Ricci-flat Kähler orbifolds over a complex manifold S. Using the finite set of positive numbers $\{a_\ell\}_{1\leq\ell\leq n}$ as given in (4.1) and (4.2), we define the associated augmented Weil-Petersson metric h_{aWP} on S as given in (4.4) (and as discussed above). Then by following mutatis mutandis the arguments and computations in Section 3 and Section 4, one sees that h_{aWP} is a C^∞ Finsler metric whose holomorphic sectional curvature is bounded above by a negative constant. We only indicate the new ingredients and the modifications needed for the proof in the orbifold case. First the Lie derivative $\mathcal{L}_{v_i}T$ of a relative tensor T with respect to the horizontal lifting v_i of some coordinate tangent vector field $\partial/\partial t_i$ on some coordinate open subset of S makes sense as a relative tensor in the obvious manner, namely by first lifting v_i and T to corresponding Γ_α-invariant objects on each $\mathcal{U}_\alpha$ (for each orbifold chart $p_\alpha : \mathcal{U}_\alpha \to \widehat{\mathcal{U}}_\alpha = \mathcal{U}_\alpha/\Gamma_\alpha$) and considering the corresponding Lie derivative there, which is easily seen to be Γ_α-invariant. Similar remark holds for the operations on tensors (and the tensors themselvers) that appear in Section 3 and Section 4, so that they also make sense (and have the same properties) for the case of orbifolds. Also, at various places in Section 3 and Section 4 where integration by part arguments are involved, the usual Stokes' theorem for manifolds will be replaced here by Stokes' theorem for orbifolds (see e.g. [Ba1, p. 866]). As mentioned earlier, one knows from [Ba1, Section 2 and Section 7] that the Hodge decomposition theorem also holds for orbifold vector bundles over compact Kähler orbifolds; and under such setting, the harmonic projection operator and Green's operator make sense in the obvious manner. These will replace the Hodge decomposition theorem and the harmonic projection operator and Green's operator (for manifolds) used in various parts of Section 3 and Section 4. The rest of the arguments in Section 3 and Section 4 prevail verbatim in the present orbifold case. □

Proof of Corollary 1. Let $\pi : \mathcal{X} \to S$ be as in Theorem 1 or Theorem 1'. By Theorem 1 and Theorem 1', S admits an augmented Weil-Petersson metric h_{aWP} whose holomorphic sectional curvature is bounded above by a negative constant. Together with standard arguments involving the usual Ahlfors lemma, the existence of the Finsler metric h_{aWP} on S with the above curvature property implies readily that S is Kobayashi hyperbolic (cf. e.g. [Kob, p. 112, Theorem 3.7.1]). □

6. **Alternative approach from period mappings in the smooth case**

In this section, we discuss in the smooth case an alternative approach of constructing a Kähler metric (called the Hodge metric) on the base manifold S with holomorphic sectional curvature bounded above by a negative constant. This alternative approach is more classical and could be found in [Lu1] (see also [Gri1], [Gri2], [Ti]), and it is based on Hodge-theoretic considerations. The alternative approach will work for at least the case of families of Calabi-Yau manifolds (or slightly more generally, when the canonical line bundle of each fiber manifold is holomorphically trivial), although it does not appear to generalize readily to the general Ricci-flat manifold (or orbifold) case. For convenience of the reader, we recall briefly Lu's approach as follows:

Let $\pi : \mathcal{X} \to S$ be an effectively parametrized holomorphic family of compact n-dimensional polarized Kähler manifolds of zero first Chern class over a complex manifold S, and such that $K_{M_{t_o}} = \mathcal{O}_{M_{t_o}}$ for some $t_o \in S$. From the deformation invariance of the Hodge number $h^{n,0}$ and the fact that a holomorphic n-form on an n-dimensional compact Ricci-flat Kähler manifold is automatically parallel with respect to the Levi-Civita connection, it follows readily that $K_{M_t} = \mathcal{O}_{M_t}$ for all $t \in S$. For each $t \in S$, let $P^n(M_t, \mathbb{C}) := \{\eta \in H^n(M, \mathbb{C}) \,\big|\, \eta \wedge \omega_t = 0\}$ denotes the primitive cohomology classes in $H^n(M_t, \mathbb{C})$. By considering the Hodge decomposition of $H^n(M_t, \mathbb{C})$, one obtains a well-defined holomorphic period mapping $p : S \to D$ from S to the classifying space D of certain polarized Hodge structures given by

$$t \to \{H^{p,q}(M_t) \cap P^n(M_t, \mathbb{C})\}_{p+q=n}. \tag{6.1}$$

By a result of Griffiths ([Gri1, Proposition 3.6]), when each K_{M_t} is trivial, the period mapping p is an immersion (this follows from the fact that if one takes a nonzero Kodaira-Spencer class $\eta \in H^1(M_s, TM_t)$ and a nonzero n-form $\Omega_t \in H^{n,0}(M_t)$, then the interior product $\eta \lrcorner \Omega_t$ is a nonzero element in $H^{n-1,1}(M_t)$). The classifying space D is a homogeneous complex manifold admitting a certain invariant two form ω_D. In [Lu1], Lu showed that $p^*\omega_D$ is a Kähler form whose holomorphic sectional curvature is bounded above by a negative constant. This Kähler metric was called the *Hodge metric* in [Lu1]. We remark that in his proof, Lu needed to use the fact that $p(S)$ lies in certain 'horizontal slice' of D due to Griffiths transversality, and he also made essential use of Griffiths' results [Gri1] on the curvature properties of ω_D (see also [Gri2]). In summary, Lu obtained the following result:

Theorem 2. *(*[Lu1]*) Let $\pi : \mathcal{X} \to S$ be an effectively parametrized holomorphic family of compact polarized Kähler manifolds of zero first Chern class over a complex manifold S. Suppose that $K_{M_{t_o}} = \mathcal{O}_{M_{t_o}}$ for some $t_o \in S$ (and hence $K_{M_t} = \mathcal{O}_{M_t}$ for all $t \in S$). Then S admits a well-defined Hodge metric whose holomorphic sectional curvature is bounded above by a negative constant.*

Remark 3. *Let $\mathcal{M}$ and the $h_{aWP,\epsilon}$'s on $\mathcal{M}$ be as in Remark 2. Since the $h_{aWP,\epsilon}$'s are pairwise inequivalent, it follows that apart from one possible exception, each of the $h_{aWP,\epsilon}$'s is not a constant multiple of the Hodge metric on $\mathcal{M}$.*

An immediate consequence of Theorem 2 is the following

Corollary 2. *Let $\pi : \mathcal{X} \to S$ be as in Theorem 1. Suppose the family $\pi : \mathcal{X} \to S$ is diffeomorphically trivial, in the sense that there exists a diffeomorphism*

$f : \mathcal{X} \to M_{t_o} \times S$ such that $\pi = pr_2 \circ f$, where $pr_2 : M_{t_o} \times S \to S$ denotes the projection onto the second factor, and $M_{t_o} = \pi^{-1}(t_o)$ for some $t_o \in S$. Then S admits a Kähler metric whose holomorphic sectional curvature is bounded above by a negative constant.

Proof. First we note that the diffeomorphism $f : \mathcal{X} \to M_{t_o} \times S$ induces the following isomorphism of fundamental groups:

$$\pi_1(\mathcal{X}) \cong \pi_1(M_{t_o}) \times \pi_1(S). \tag{6.2}$$

It follows from a result of Beauville [Bea] that there exists a finite cover M'_{t_o} of the fiber M_{t_o} corresponding to some subgroup $G \subset \pi_1(M_{t_o})$ of finite index such that $K_{M'_{t_o}}$ is holomorphically trivial. Then via the isomophism in (6.2), one may regard $G \times \pi_1(S)$ as a subgroup of $\pi_1(\mathcal{X})$ of finite index. Then one gets an associated finite cover $\mathcal{X}'$ of $\mathcal{X}$. Denote the associated covering projection map by $q : \mathcal{X}' \to \mathcal{X}$, and let $\pi' = \pi \circ q : \mathcal{X}' \to S$. Then it is easy to see that $\pi' : \mathcal{X}' \to S$ forms an effectively parametrized holomorphic family of compact polarized Kähler manifolds of zero first Chern class over S (with the polarization provided by $q^*\lambda$, where λ denotes the polarization of the family $\pi : \mathcal{X} \to S$), and one has $(\pi')^{-1}(t_o) = M'_{t_o}$. Hence one may apply Theorem 2 to the family $\pi' : \mathcal{X}' \to S$ to yield the desired conclusion on S. □

Remark 4. *Let $\pi : \mathcal{X} \to S$ be as in Theorem 1. For any $t_o \in S$, one easily sees that there exists some open neighborhood U of t_o in S such that the restricted family $\pi\big|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ over U is diffeomorphically trivial, shrinking U if necessary. As such, one may apply Corollary 2 to get a Kähler metric h_U on U with holomorphic sectional curvature bounded above by some negative constant. However, the 'local Hodge metrics' h_U's are not uniquely defined, as they depend on the choices of the local finite coverings, and it is not clear that the h_U's will patch together to form a well-defined Kähler metric on S.*

Remark 5. *The Hodge theoretic approach described in this section does not apply to the general orbifold case treated in Theorem 1', since in that case (and in addition to Remark 4), each fiber M_t is only an orbifold, and the orbifold charts of M_t may not lead to a finite (ramified or unramified) cover M'_t of M_t such that M'_t is a compact Kähler manifold with trivial canonical line bundle.*

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