# EXOTIC STRUCTURES ARISING FROM FAKE PROJECTIVE PLANES

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**Abstract** *We present examples of exotic structures on $pP^2_{\mathbb{C}}\#q\overline{P^2_{\mathbb{C}}}$ for some small $p$ and $q$ obtained naturally from quotients of fake projective planes in complex geometry, which are classified by the work of Prasad-Yeung [PY] and Cartwright-Steger [CS1].*

## §1 Statements of results.

**(1.1)** An exotic structure on a differentiable manifold is a differentiable structure homeomorphic but not diffeomorphic to the given one. A fundamental question in differentiable topology is the existence and non-existence of exotic differentiable structures on manifolds. In particular, one is interested in manifolds with relatively simple topology. The first such example was constructed by Milnor [Mi] for exotic spheres.

The first example in dimension 4 was found by Donaldson in [D1]. Recently there have been a lot of activities related to exotic structures on compact four dimensional manifolds, including the work of Szabo [S], Jongil Park [P], Stipsicz and Szabo [SS], Fintushel and Stern [FS], Akhmedov [A], Akhmedov and B. Doug Park [AP]. Most of the results are obtained through topological surgery.

**(1.2)** In this article we present some exotic structures obtained from a different direction. We start with some projective algebraic manifolds equipped with some natural finite group action, consider their quotient manifolds, resolve the singularities and take some coverings if necessary. It is a direct consequence of the work of Prasad and Yeung [PY], and Cartwright and Steger [CS1], on their classification of fake projective planes. Our main purpose is to list all the possible exotic structures of relatively small Euler Poincaré characteristics obtained from these surfaces. It is our hope that the somewhat concrete constructions presented here and their relations to explicitly described arithmetic lattices may make the examples applicable for other purposes.

Fake projective planes are surfaces with the same Betti numbers as the complex projective plane $P^2_{\mathbb{C}}$ but not biholomorpic to $P^2_{\mathbb{C}}$. It is known that such surfaces are actually complex hyperbolic space-forms (cf. [Y], §2). In the study of Prasad and Yeung [PY], the authors observed that most (though not all) of the examples constructed are naturally equipped with a finite group action, such that the quotient is not a smooth complex surface. For those fake projective planes with fundamental group which is a proper subgroup of a maximal arithmetic lattice, the fake projective plane covers a complex two ball quotient with isolated singularity named as a

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maximal ball quotient. The set of all possible maximal ball quotients associated with fake projective planes are listed in [PY], where it is also shown that the maximal ball quotient can be obtained as the quotient of some fake projective plane by a finite subgroup corresponding to a proper congruence subgroup of the maximal arithmetic lattice. However, we should mention that there are fake projective planes whose fundamental groups are just the maximal arithmetic lattices and hence there are no finite group actions according to [PY].

In the specific example constructed by Mumford earlier in [Mu], it was known from [I] that the example covers a complex two ball quotient with trivial fundamental group. Hence for those quotients of the complex two ball by a maximal arithmetic group containing a fake projective plane in its class in the notion of [PY], one asks the question of whether the topological fundamental groups of the quotients, which are necessarily singular, are finite. The answer is affirmative following the work of Cartwright and Steger [CS1]. This is the starting point of the argument in this paper. The results of this paper follows by applying different tools in geometric topology to the concrete complex ball quotients mentioned above. It relies crucially on the computation in [CS1] (see also the weblink there) of the topological fundamental groups of the quotient by the maximal arithmetic groups studied in [PY].

**(1.3)** Related to the definition in §1.1, we define the notion of a complex exotic structure. Let $M$ be a complex manifold. A complex manifold $N$ homeomorphic but not biholomorphic to $M$ is called a complex exotic structure on $M$. It is an interesting question to see if there are complex exotic structure for a given manifold $M$.

**(1.4)** The main result of this article is the following.

**Theorem 1.** *A suitable unramified covering of the resolution of singularities of quotients of fake projective planes gives rise to algebraic surfaces which are homeomorphic but not diffeomorphic to $pP^2_{\mathbb{C}}\#q\overline{P}^2_{\mathbb{C}}$ for the following pairs of $(p,q)$.*

$$\begin{aligned}(p,q) \quad = \quad & (1,6),(1,8),(1,9),(3,13),(3,17),(3,19),(5,20),\\ & (7,27),(11,41),(13,48),(15,55),(25,90),(27,97).\end{aligned}$$

*Hence the above give rise to exotic structures on $pP^2_{\mathbb{C}}\#q\overline{P}^2_{\mathbb{C}}$ with values of $(p,q)$ stated above. Moreover, in the case of $p=1$, these give rise to complex exotic structures on $P^2_{\mathbb{C}}$ blown up at $q$ points.*

**(1.5)** Here is the organization of the article. In §2, we collect some well-known results in geometric topology which is to be used in the proof. In §3, we tabulate the classification of fake projective planes from [PY] and [CS1]. The examples listed in the table provides the basic structures to work with. The actual construction and proof were given in §4.

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## §2. Preliminaries from geometric topology.

**(2.1)** We recall several topological facts about a simply connected oriented compact four dimensional real manifolds $M$. The cup product provides a unimodular bilinear quadratic form $Q_M$ to the lattice $H^2(M,\mathbb{Z})$. $Q_M$ is said to be even if $Q_M(x,x)$ is even for all $x \in H^2(M,\mathbb{Z})$. Otherwise it is said to be odd.

To understand the homeomorphic structure, we need the following two facts. The first one is a well-known result of Freedman [F] (cf. [BHPV], page 379)

**(2.2) Theorem** ([F]) *Let $M$ be a simply connected topological manifold of real dimension 4.*
*(a). If $Q_M$ is odd, $M$ is homeomorphic to a connected sum $pP^2_{\mathbb{C}}\#q\overline{P}^2_{\mathbb{C}}$ of the projective plane $P^2_{\mathbb{C}}$ and $\overline{P}^2_{\mathbb{C}}$, the surface obtained by reversing the orientation of $P^2_{\mathbb{C}}$.*
*(b). If $Q_M$ is even, $Q_M = aH + b(\pm E_8)$, where $H$ is the intersection form for he quadric $P^1_{\mathbb{C}} \times P^1_{\mathbb{C}}$. Furthermore, if $M$ is projective algebraic, $M$ is homeomorphic to a connected sum $p(P^1_{\mathbb{C}} \times P^1_{\mathbb{C}})\#qK$, where $K$ is a K3 surface with orientation possibly reversed.*

The second one is a result of Rochlin [R], see also [BHPV], page 378-379.

**(2.3) Theorem** ([R]) *Suppose $M$ is a differential manifold and $Q_M$ is even. Then the index $\tau_M$ of $M$ is divisible by 16.*

We need the following two results to distinguish the differentiable structures. The first one is the following theorem of Donaldson [D2] (cf. [BHPV], page 391).

**(2.4) Theorem** ([D]) *Suppose an algebraic surface $M$ is diffeomorphic to $M_1\#M_2$ with $b_2(M_i) > 0$ for $i = 1, 2$. Then the intersection form of $M_1$ or $M_2$ is $\mathbb{Z}$ equivalent to $< -1 > \oplus \cdots \oplus < -1 >$. Hence an orientable differentiable manifold $pP^2_{\mathbb{C}}\#q\overline{P}^2_{\mathbb{C}}$ carries a complex structure only if $p = 1$.*

The second one has been a conjecture of Van der ven that Kodaira dimension of an algebraic surface is invariant under diffeomorphism. This was proved in Friedman-Qin [FQ]. We refer the readers also to the expository in [BHPV].

**(2.5) Theorem** ([FQ]) *The Kodaira dimension of a complex surface is invariant under diffeomorphism.*

## §3. Classification of fake projective planes.

**(3.1)** The classification of fake projective planes is now complete following the work of Prasad-Yeung [PY] and Cartwright-Steger [CS1]. Since the present article depends on the explicit examples constructed in [PY] and [CS1], we recall the naming system of the examples obtained.

Note that a fake projective plane $M$ is the quotient of the complex hyperbolic two space $B^2_{\mathbb{C}}$ by a lattice $\Pi$ in $PU(2,1)$. $M$ is determined by $\Pi$ as a Riemannian manifold equipped with the Poincaré metric. But each such manifold supports two distinct complex structures, one conjugate to each other, cf. [KK]. These well-known facts can be found in [Ye]. Hence to list all fake projective planes, it suffices to list all possible $\Pi$.

In the following we recall the notations from [PY]. We refer the readers to [PY] for all the unexplained notations. Let $k$ be a totally real number field. Let $\ell$ be a totally imaginary quadratic extension of $k$. Let $\mathcal{D}$ be a division algebra with center $\ell$ of degree 3 equipped with an involution $\sigma$ of second kind, such that for the hermitian form $h_0$ on $\mathcal{D}$ defined by $h_0(x,y)=\sigma(x)y$, the group $\mathrm{SU}(h_0)$ is isotropic at $v_o$, and is anisotropic at every other real place of $k$. For $x\in\mathcal{D}^\times$, let $\mathrm{Int}(x)$ denote the automorphism $z\mapsto xzx^{-1}$ of $\mathcal{D}$. Let $\mathcal{D}^\sigma=\{z\in\mathcal{D}\,|\,\sigma(z)=z\}$. Observe that for all $x\in\mathcal{D}^\sigma$, $\mathrm{Int}(x)\cdot\sigma$ is again an involution of $\mathcal{D}$ of the second kind, and any involution of $\mathcal{D}$ of the second kind is of this form. Now for $x\in\mathcal{D}^\sigma$, given an hermitian form $h'$ on $\mathcal{D}$ with respect to the involution $\mathrm{Int}(x)\cdot\sigma$, the form $h=x^{-1}h'$ is a hermitian form on $\mathcal{D}$ with respect to $\sigma$, and $\mathrm{SU}(h')=\mathrm{SU}(h)$. Therefore it suffices to work just with the involution $\sigma$, and to consider all hermitian forms $h$ on $\mathcal{D}$, with respect to $\sigma$, of determinant 1, such that the group $\mathrm{SU}(h)$ is isotropic at $v_o$, and is anisotropic at all other real places of $k$. Let $h$ be such a hermitian form. Then $h(x,y)=\sigma(x)ay$, for some $a\in\mathcal{D}^\sigma$. The determinant of $h$ is $\mathrm{Nrd}(a)$ modulo $N_{\ell/k}(\ell^\times)$. As the elements of $N_{\ell/k}(\ell^\times)$ are positive at all real places of $k$, we see that the signatures of $h$ and $h_0$ are equal at every real place of $k$, which leads to the isometry between the hermitian forms $h$ and $h_0$. Hence, $\mathrm{SU}(h)$ is $k$-isomorphic to $\mathrm{SU}(h_0)$. Thus $\mathcal{D}$ determines a unique $k$-form $G$ of $\mathrm{SU}(2,1)$, up to a $k$-isomorphism, namely $\mathrm{SU}(h_0)$, with the desired behavior at the real places of $k$. The group $G(k)$ of $k$-rational points of this $G$ is

$$G(k)=\{z\in\mathcal{D}^\times\mid z\sigma(z)=1\text{ and }\mathrm{Nrd}(z)=1\}.$$

Let $P=(P_v)_{v\in V_f}$ be a coherent collection of parahoric subgroups $P_v$ for each place $v\in V_f$, the set of all finite places of $k$, chosen as in [PY] (see also the Addendum). Then there are twenty-eight distinct set $\{k,\ell,G,(P_v)_{v\in V_f}\}$ which can support fake projective planes in the following sense. Each of these twenty-eight classes determines a unique principal arithmetic subgroup $\Lambda$ ($=G(k)\cap\prod_{v\in V_f}P_v$), whose normalizer in $\overline{G}(k_{v_o})$ is denoted by $\overline{\Gamma}$. Each $\Lambda$ determines a class of fake projective planes with fundamental group given by a lattice $\Pi$ of $PU(2,1)$, where $\Pi$ is an element in

$$A_\Lambda=\{\Pi<\overline{\Gamma}:[\overline{\Gamma}:\Pi]=\frac{3}{\chi(\overline{\Gamma})},|\Pi/[\Pi,\Pi]|<\infty,\text{and }\Pi\text{ is torsion-free}\}$$

**(3.2)** In [PY], it was shown that each of these classes is non-empty, and there can at most be five more $\Lambda$ with $[D:\ell]=1$ supporting any fake projective planes. These cases are eliminated by Cartwright and Steger in [CS1]. It is also shown in [PY] that the values of $[\overline{\Gamma}:\Pi]$ in the twenty-eight cases can only take the values of $1,3,9$ or 21. Cartwright and Steger determine in [CS1] the set $A_\Lambda$ by enumerating all subgroups of the appropriate indices. This is achieved by writing down a set of generators and relations for each $\Gamma$. This also allows them to determine $\pi_1(B^2_{\mathbb{C}}/\overline{\Gamma})$.

Among those fake projective planes $M$ with $\pi_1(M)\cong\Pi$ being a normal subgroup in $\overline{\Gamma}$, such as those found in [PY], the quotient group $H=\overline{\Gamma}/\Pi$ is a finite group with order in $\{1,3,9,21\}$ acting on $M$. It is not difficult to see that the possible order of a finite group $H$ acting on any fake projective plane $M$ lies in $\{1,3,7,9,21\}$. In the process of listing all fake projective planes, Cartwright and Steger also listed all possible finite quotients of fake projective planes in [CS1].

The above discussions are summarized in the following theorem.

**(3.3) Theorem** ([PY], [CS1]) *(a). There are one hundred fake projective planes up to biholomorphism.*
*(b). These are classified into* 28 *classes. Within each class there exists a maximal arithmetic group* $\overline{\Gamma}$.
*(c). The order of a finite group* $H$ *acting on a fake projective plane takes a value in* $\{1, 3, 7, 9, 21\}$.

**(3.4)** In the following, we tabulate fake projective planes and their finite quotients. For each class $A_\Lambda$ considered in **(3.1)**, we use the notation of [PY] and represent it by the pair of number fields $(k, \ell)$ and $\mathcal{T}$, the set of non-Archimedean places $v$ of $k$ which are unramified in $\ell$ and $P_v$ is not a hyperspecial parahoric subgroup of $G(k_v)$. We arrange the results in two tables, the first one with $k = \mathbb{Q}$, and the second one with $\deg_{\mathbb{Q}} k \geqslant 2$ and hence $\deg_{\mathbb{Q}} k = 2$ according to [PY].

In the following two columns, the first follows convention of [PY]. The second column shows that there are altogether 28 classes of fake projective planes as explained in [PY]. The third column shows that altogether there are 50 classes of fake projective planes when they are regarded as locally symmetric spaces identified up to isometry, as shown in [CS1]. The naming here follows the terminology of Cartwright and Steger in [CS2], which is going to appear as an extended version of [CS1]. The most important entry for our purpose in this paper is the fifth column, which is a summary of some scattered information among the computer files of Carwright and Steger in the website (http://www.maths.usyd.edu.au/u/donaldc/) provided in [CS1]. Note again that there are precisely two fake projective planes as complex surface within teach isometry class, corresponding to holomorphic and anti-holomorphic structures (cf. [KK]).

In the cases that $\pi_1(M/H) = 1$, these are the surfaces that we proposed to call 'Cartwright-Steger surfaces' in the addendum to [PY] and [Y].

In the following tables, the entry $N$ means that it is not applicable, in other words, there is no non-trivial natural finite group action on the surfaces involved. In the last column, $\mathbb{Z}_n$ is the cyclic group of order $n$, $Q_8$ is the quaternionic group of order 8, $D_8$ is the dihedral group of order 8, and $S_3$ is the symmetric group of three elements.

**Table (I):** $k = \mathbb{Q}$

| $(k,\ell,\mathcal{T})$ | class | $M$ | $\lvert H\rvert$ | $M/H$ | $\pi_1(M/H)$ | exotic |
|---|---|---|---|---|---|---|
| $(\mathbb{Q},\mathbb{Q}(\sqrt{-1}),\{5\})$ | $(a=1,p=5,\emptyset)$ | $(a=1,p=5,\emptyset,D_3)$ | 3 | $(a=1,p=5,\emptyset)$ | $\mathbb{Z}_2\times\mathbb{Z}_4$ | $15P_{\mathbb{C}}^2\#55\overline{P}_{\mathbb{C}}^2$ |
| | $(a=1,p=5,\{2\})$ | $(a=1,p=5,\emptyset,\{2\},D_3)$ | 3 | $(a=1,p=5,\{2\})$ | $\mathbb{Z}_4$ | $7P_{\mathbb{C}}^2\#27\overline{P}_{\mathbb{C}}^2$ |
| $(\mathbb{Q},\mathbb{Q}(\sqrt{-1}),\{2,5\})$ | $(a=1,p=5,\{2I\})$ | $(a=1,p=5,\{2I\})$ | $N$ | $N$ | $N$ | $N$ |
| $(\mathbb{Q},\mathbb{Q}(\sqrt{-2}),\{3\})$ | $(a=2,p=3,\emptyset)$ | $(a=2,p=3,\emptyset,D_3)$ | 3 | $(a=2,p=3,\emptyset)$ | $\mathbb{Z}_2\times\mathbb{Z}_2$ | $7P_{\mathbb{C}}^2\#27\overline{P}_{\mathbb{C}}^2$ |
| | $(a=2,p=3,\{2\})$ | $(a=2,p=3,\emptyset,\{2\},D_3))$ | 3 | $(a=2,p=3,\{2\})$ | $\mathbb{Z}_4$ | $7P_{\mathbb{C}}^2\#27\overline{P}_{\mathbb{C}}^2$ |
| $(\mathbb{Q},\mathbb{Q}(\sqrt{-2}),\{2,3\})$ | $(a=2,p=3,\{2I\})$ | $(a=2,p=3,\{2I\})$ | $N$ | $N$ | $N$ | $N$ |
| $(\mathbb{Q},\mathbb{Q}(\sqrt{-7}),\{2\})$ | $(a=7,p=2,\emptyset)$ | $(a=7,p=2,\emptyset,D_3,2_7)$ | 3 | $(a=7,p=2,\emptyset,2_7)$ | $D_8$ | $15P_{\mathbb{C}}^2\#55\overline{P}_{\mathbb{C}}^2$ |
| | | | 7 | $(a=7,p=2,\emptyset,D_3)$ | $\mathbb{Z}_2$ | $3P_{\mathbb{C}}^2\#19\overline{P}_{\mathbb{C}}^2$ |
| | | | 21 | $(a=7,p=2,\emptyset)$ | $\mathbb{Z}_2$ | $3P_{\mathbb{C}}^2\#19\overline{P}_{\mathbb{C}}^2$ |
| | | $(a=7,p=2,\emptyset,7_{21})$ | 21 | $(a=7,p=2,\emptyset)$ | $\mathbb{Z}_2$ | $3P_{\mathbb{C}}^2\#19\overline{P}_{\mathbb{C}}^2$ |
| | | $(a=7,p=2,\emptyset,D_3,X_7)$ | 21 | $(a=7,p=2,\emptyset)$ | $\mathbb{Z}_2$ | $3P_{\mathbb{C}}^2\#19\overline{P}_{\mathbb{C}}^2$ |
| | $(a=7,p=2,\{7\})$ | $(a=7,p=2,\{7\},D_3,2_7)$ | 3 | $(a=7,p=2,\{7\},2_7)$ | $\mathbb{Z}_3$ | $5P_{\mathbb{C}}^2\#20\overline{P}_{\mathbb{C}}^2$ |
| | | | 7 | $(a=7,p=2,\{7\},D_3)$ | $\{1\}$ | $P_{\mathbb{C}}^2\#9\overline{P}_{\mathbb{C}}^2$ |
| | | | 7 | $(a=7,p=2,\{7\})$ | $\{1\}$ | $P_{\mathbb{C}}^2\#9\overline{P}_{\mathbb{C}}^2$ |
| | | $(a=7,p=2,\{7\},D_3,7_7)$ | 3 | $(a=7,p=2,\{7\},7_7)$ | $\mathbb{Z}_2$ | $3P_{\mathbb{C}}^2\#13\overline{P}_{\mathbb{C}}^2$ |
| | | $(a=7,p=2,\{7\},D_3,7_7')$ | 3 | $(a=7,p=2,\{7\},7_7')$ | $\mathbb{Z}_2\times\mathbb{Z}_2$ | $7P_{\mathbb{C}}^2\#27\overline{P}_{\mathbb{C}}^2$ |
| | | $(a=7,p=2,\{7\},7_{21})$ | $N$ | $N$ | $N$ | $N$ |
| $(\mathbb{Q},\mathbb{Q}(\sqrt{-7}),\{2,3\})$ | $(a=7,p=2,\{3\})$ | $(a=7,p=2,\{3\},D_3)$ | 3 | $(a=7,p=2,\{3\})$ | $\mathbb{Z}_2\times\mathbb{Z}_4$ | $15P_{\mathbb{C}}^2\#55\overline{P}_{\mathbb{C}}^2$ |
| | | $(a=7,p=2,\{3\},3_3)$ | 3 | $(a=7,p=2,\{3\})$ | $\mathbb{Z}_2\times\mathbb{Z}_4$ | $15P_{\mathbb{C}}^2\#55\overline{P}_{\mathbb{C}}^2$ |
| | $(a=7,p=2,\{3,7\})$ | $(a=7,p=2,\{3,7\},D_3)$ | 3 | $(a=7,p=2,\{3,7\}$ | $\mathbb{Z}_4$ | $7P_{\mathbb{C}}^2\#27\overline{P}_{\mathbb{C}}^2$ |
| | | $(a=7,p=2,\{3,7\},3_3)$ | 3 | $(a=7,p=2,\{3,7\}$ | $\mathbb{Z}_4$ | $7P_{\mathbb{C}}^2\#27\overline{P}_{\mathbb{C}}^2$ |
| $(\mathbb{Q},\mathbb{Q}(\sqrt{-7}),\{2,5\})$ | $(a=7,p=2,\{5\})$ | $(a=7,p=2,\{5\})$ | $N$ | $N$ | $N$ | $N$ |
| | $(a=7,p=2,\{5,7\})$ | $(a=7,p=2,\{5,7\})$ | $N$ | $N$ | $N$ | $N$ |
| $(\mathbb{Q},\mathbb{Q}(\sqrt{-15}),\{2\})$ | $(a=15,p=2,\emptyset)$ | $(a=15,p=2,\emptyset,D_3)$ | 3 | $(a=15,p=2,\emptyset)$ | $\mathbb{Z}_2\times\mathbb{Z}_2$ | $7P_{\mathbb{C}}^2\#27\overline{P}_{\mathbb{C}}^2$ |
| | | $(a=15,p=2,\emptyset,3_3)$ | 3 | $(a=15,p=2,\emptyset)$ | $\mathbb{Z}_2\times\mathbb{Z}_2$ | $7P_{\mathbb{C}}^2\#27\overline{P}_{\mathbb{C}}^2$ |
| | $(a=15,p=2,\{3\})$ | $(a=15,p=2,\{3\},D_3)$ | 3 | $(a=15,p=2,\{3\})$ | $\mathbb{Z}_2\times\mathbb{Z}_3$ | $11P_{\mathbb{C}}^2\#41\overline{P}_{\mathbb{C}}^2$ |
| | | $(a=15,p=2,\{3\},3_3)$ | 3 | $(a=15,p=2,\{3\})$ | $\mathbb{Z}_2\times\mathbb{Z}_3$ | $11P_{\mathbb{C}}^2\#41\overline{P}_{\mathbb{C}}^2$ |
| | | $(a=15,p=2,\{3\},3_3,(D3)_3)$ | 3 | $(a=15,p=2,\{3\})$ | $\mathbb{Z}_2\times\mathbb{Z}_3$ | $11P_{\mathbb{C}}^2\#41\overline{P}_{\mathbb{C}}^2$ |
| | $(a=15,p=2,\{5\})$ | $(a=15,p=2,\{5\},D_3)$ | 3 | $(a=15,p=2,\{5\})$ | $\mathbb{Z}_2$ | $3P_{\mathbb{C}}^2\#13\overline{P}_{\mathbb{C}}^2$ |
| | | $(a=15,p=2,\{5\},3_3)$ | $N$ | $N$ | $N$ | $N$ |
| | $(a=15,p=2,\{3,5\})$ | $(a=15,p=2,\{3,5\},D_3)$ | 3 | $(a=15,p=2,\{3,5\})$ | $\mathbb{Z}_3$ | $5P_{\mathbb{C}}^2\#20\overline{P}_{\mathbb{C}}^2$ |
| | | $(a=15,p=2,\{3,5\},3_3)$ | 3 | $(a=15,p=2,\{3,5\})$ | $\mathbb{Z}_3$ | $5P_{\mathbb{C}}^2\#20\overline{P}_{\mathbb{C}}^2$ |
| | | $(a=15,p=2,\{3,5\},(D3)_3)$ | 3 | $(a=15,p=2,\{3,5\})$ | $\mathbb{Z}_3$ | $5P_{\mathbb{C}}^2\#20\overline{P}_{\mathbb{C}}^2$ |
| $(\mathbb{Q},\mathbb{Q}(\sqrt{-23}),\{2\})$ | $(a=23,p=2,\emptyset)$ | $(a=23,p=2,\emptyset)$ | $N$ | $N$ | $N$ | $N$ |
| | $(a=23,p=2,\{23\})$ | $(a=23,p=2,\{23\})$ | $N$ | $N$ | $N$ | $N$ |

**Table (II):** $\deg_{\mathbb{Q}} k = 2$

| $(k,\ell,\mathcal{T})$ | class | $M$ | $\lvert H\rvert$ | $M/H$ | $\pi_1(M/H)$ | exotic |
|---|---|---|---|---|---|---|
| $(\mathcal{C}_2,\{v_2\})$ | $(\mathcal{C}_2,p=2,\emptyset)$ | $(\mathcal{C}_2,p=2,\emptyset,d_3,D_3)$ | 3 | $(\mathcal{C}_2,p=2,\emptyset,D_3)$ | $\mathbb{Z}_2\times\mathbb{Z}_7$ | $27P^2_{\mathbb{C}}\#97\overline{P}^2_{\mathbb{C}}$ |
| | | | 3 | $(\mathcal{C}_2,p=2,\emptyset,d_3)$ | $S_3$ | $11P^2_{\mathbb{C}}\#41\overline{P}^2_{\mathbb{C}}$ |
| | | | 3 | $(\mathcal{C}_2,p=2,\emptyset,(dD)_3)$ | $\mathbb{Z}_2$ | $3P^2_{\mathbb{C}}\#13\overline{P}^2_{\mathbb{C}}$ |
| | | | 3 | $(\mathcal{C}_2,p=2,\emptyset,(d^2D)_3)$ | $\mathbb{Z}_2$ | $3P^2_{\mathbb{C}}\#13\overline{P}^2_{\mathbb{C}}$ |
| | | | 9 | $(\mathcal{C}_2,p=2,\emptyset)$ | $\mathbb{Z}_2$ | $3P^2_{\mathbb{C}}\#17\overline{P}^2_{\mathbb{C}}$ |
| | | $(\mathcal{C}_2,p=2,\emptyset,D_3,X_3)$ | 3 | $(\mathcal{C}_2,p=2,\emptyset,X_3)$ | $\mathbb{Z}_2\times\mathbb{Z}_3$ | $11P^2_{\mathbb{C}}\#41\overline{P}^2_{\mathbb{C}}$ |
| | | $(\mathcal{C}_2,p=2,\emptyset,(dD)_3,X_3)$ | 3 | $(\mathcal{C}_2,p=2,\emptyset,X_3)$ | $\mathbb{Z}_2\times\mathbb{Z}_3$ | $11P^2_{\mathbb{C}}\#41\overline{P}^2_{\mathbb{C}}$ |
| | | $(\mathcal{C}_2,p=2,\emptyset,(d^2D)_3,X_3)$ | 3 | $(\mathcal{C}_2,p=2,\emptyset,X_3)$ | $\mathbb{Z}_2\times\mathbb{Z}_3$ | $11P^2_{\mathbb{C}}\#41\overline{P}^2_{\mathbb{C}}$ |
| | | $(\mathcal{C}_2,p=2,\emptyset,d_3,X_3')$ | $N$ | $N$ | $N$ | $N$ |
| | | $(\mathcal{C}_2,p=2,\emptyset,X_9)$ | $N$ | $N$ | $N$ | $N$ |
| | $(\mathcal{C}_2,p=2,\{3\})$ | $(\mathcal{C}_2,p=2,\{3\},d_3,D_3)$ | 3 | $(\mathcal{C}_2,p=2,\{3\},D_3)$ | $\mathbb{Z}_7$ | $13P^2_{\mathbb{C}}\#48\overline{P}^2_{\mathbb{C}}$ |
| | | | 3 | $(\mathcal{C}_2,p=2,\{3\},d_3)$ | $\{1\}$ | $P^2_{\mathbb{C}}\#6\overline{P}^2_{\mathbb{C}}$ |
| | | | 3 | $(\mathcal{C}_2,p=2,\{3\},(dD)_3)$ | $\{1\}$ | $P^2_{\mathbb{C}}\#6\overline{P}^2_{\mathbb{C}}$ |
| | | | 3 | $(\mathcal{C}_2,p=2,\{3\},(d^2D)_3)$ | $\{1\}$ | $P^2_{\mathbb{C}}\#6\overline{P}^2_{\mathbb{C}}$ |
| | | | 9 | $(\mathcal{C}_2,p=2,\{3\})$ | $\{1\}$ | $P^2_{\mathbb{C}}\#8\overline{P}^2_{\mathbb{C}}$ |
| $(\mathcal{C}_{10},\{v_2\})$ | $(\mathcal{C}_{10},p=2,\emptyset)$ | $(\mathcal{C}_{10},p=2,\emptyset,D_3)$ | 3 | $(\mathcal{C}_{10},p=2,\emptyset)$ | $\mathbb{Z}_2$ | $3P^2_{\mathbb{C}}\#13\overline{P}^2_{\mathbb{C}}$ |
| | $(\mathcal{C}_{10},p=2,\{17-\})$ | $(\mathcal{C}_{10},p=2,\{17-\},D_3)$ | 3 | $(\mathcal{C}_{10},p=2,\{17-\})$ | $\mathbb{Z}_2$ | $3P^2_{\mathbb{C}}\#13\overline{P}^2_{\mathbb{C}}$ |
| $(\mathcal{C}_{18},\{v_3\})$ | $(\mathcal{C}_{18},p=3,\emptyset)$ | $(\mathcal{C}_{18},p=3,\emptyset,d_3,D_3)$ | 3 | $(\mathcal{C}_{18},p=3,\emptyset,d_3)$ | $Q_8$ | $15P^2_{\mathbb{C}}\#55\overline{P}^2_{\mathbb{C}}$ |
| | | | | $(\mathcal{C}_{18},p=3,\emptyset,D_3)$ | $\mathbb{Z}_{13}$ | $25P^2_{\mathbb{C}}\#90\overline{P}^2_{\mathbb{C}}$ |
| | | | | $(\mathcal{C}_{18},p=3,\emptyset,(dD)_3)$ | $\{1\}$ | $P^2_{\mathbb{C}}\#6\overline{P}^2_{\mathbb{C}}$ |
| | | | | $(\mathcal{C}_{18},p=3,\emptyset,(d^2D)_3)$ | $\{1\}$ | $P^2_{\mathbb{C}}\#6\overline{P}^2_{\mathbb{C}}$ |
| | | | 9 | $(\mathcal{C}_{18},p=3,\emptyset)$ | $\{1\}$ | $P^2_{\mathbb{C}}\#6\overline{P}^2_{\mathbb{C}}$ |
| | $(\mathcal{C}_{18},p=3,\{2\})$ | $(\mathcal{C}_{18},p=3,\{2\},D_3)$ | 3 | $(\mathcal{C}_{18},p=3,\{2\})$ | $\mathbb{Z}_2\times\mathbb{Z}_3$ | $11P^2_{\mathbb{C}}\#41\overline{P}^2_{\mathbb{C}}$ |
| | | $(\mathcal{C}_{18},p=3,\{2\},(dD)_3)$ | 3 | $(\mathcal{C}_{18},p=3,\{2\})$ | $\mathbb{Z}_2\times\mathbb{Z}_3$ | $11P^2_{\mathbb{C}}\#41\overline{P}^2_{\mathbb{C}}$ |
| | | $(\mathcal{C}_{18},p=3,\{2\},(d^2D)_3)$ | 3 | $(\mathcal{C}_{18},p=3,\{2\})$ | $\mathbb{Z}_2\times\mathbb{Z}_3$ | $11P^2_{\mathbb{C}}\#41\overline{P}^2_{\mathbb{C}}$ |
| | $(\mathcal{C}_{18},p=3,\{2I\})$ | $(\mathcal{C}_{18},p=3,\{2I\})$ | $N$ | $N$ | $N$ | $N$ |
| $(\mathcal{C}_{20},\{v_2\})$ | $(\mathcal{C}_{20},\{v_2\},\emptyset)$ | $(\mathcal{C}_{20},\{v_2\},\emptyset,D_3,2_7)$ | 3 | $(\mathcal{C}_{20},\{v_2\},\emptyset,2_7)$ | $\mathbb{Z}_2\times\mathbb{Z}_2$ | $7P^2_{\mathbb{C}}\#27\overline{P}^2_{\mathbb{C}}$ |
| | | | 7 | $(\mathcal{C}_{20},\{v_2\},\emptyset,D_3)$ | $\{1\}$ | $P^2_{\mathbb{C}}\#9\overline{P}^2_{\mathbb{C}}$ |
| | | | 21 | $(\mathcal{C}_{20},\{v_2\},\emptyset)$ | $\{1\}$ | $P^2_{\mathbb{C}}\#9\overline{P}^2_{\mathbb{C}}$ |
| | $(\mathcal{C}_{20},\{v_2\},\{3+\})$ | $(\mathcal{C}_{20},\{v_2\},\{3+\},D_3)$ | 3 | $(\mathcal{C}_{20},\{v_2\},\{3+\})$ | $\mathbb{Z}_4$ | $7P^2_{\mathbb{C}}\#27\overline{P}^2_{\mathbb{C}}$ |
| | | $(\mathcal{C}_{20},\{v_2\},\{3+\},\{3+\}_3)$ | 3 | $(\mathcal{C}_{20},\{v_2\},\{3+\})$ | $\mathbb{Z}_4$ | $7P^2_{\mathbb{C}}\#27\overline{P}^2_{\mathbb{C}}$ |
| | $(\mathcal{C}_{20},\{v_2\},\{3-\})$ | $(\mathcal{C}_{20},\{v_2\},\{3-\},D_3)$ | 3 | $(\mathcal{C}_{20},\{v_2\},\{3-\})$ | $\mathbb{Z}_4$ | $7P^2_{\mathbb{C}}\#27\overline{P}^2_{\mathbb{C}}$ |
| | | $(\mathcal{C}_{20},\{v_2\},\{3-\},\{3-\}_3)$ | 3 | $(\mathcal{C}_{20},\{v_2\},\{3-\})$ | $\mathbb{Z}_4$ | $7P^2_{\mathbb{C}}\#27\overline{P}^2_{\mathbb{C}}$ |

In the above, the pairs of number fields $\mathcal{C}_2, \mathcal{C}_{10}, \mathcal{C}_{18}$ and $\mathcal{C}_{20}$ follows the original notation in [PY]. We recall the definition here. We use $\zeta_n$ to denote a root of unity of order $n$.

| $(k,\ell)$ | $k$ | $\ell$ |
|---|---|---|
| $\mathcal{C}_2$ | $\mathbb{Q}(\sqrt{5})$ | $\mathbb{Q}(\sqrt{5},\zeta_3)$ |
| $\mathcal{C}_{10}$ | $\mathbb{Q}(\sqrt{2})$ | $\mathbb{Q}(\sqrt{2},\mathbb{Q}(\sqrt{-7+4\sqrt{2}}))$ |
| $\mathcal{C}_{18}$ | $\mathbb{Q}(\sqrt{6})$ | $\mathbb{Q}(\sqrt{18},\zeta_3)$ |
| $\mathcal{C}_{20}$ | $\mathbb{Q}(\sqrt{7})$ | $\mathbb{Q}(\sqrt{7},\zeta_4)$ |

In the second table a prime $v_p$ in $k$ of the above four pairs of number fields refers to a prime ideal sitting above a rational prime $p \in \mathbb{Q}$.

**(3.5)** We also recall the following information about the singularities of a finite group action on a fake projective plane.

For each $\overline{\Gamma}$ mentioned above, we consider the set of all subgroups $\Lambda$ of $\overline{\Gamma}$ so that $\Pi$ is a proper subgroup of $\Lambda$. Since $B^2_{\mathbb{C}}/\Pi$ has the smallest Euler-Poincaré characteristics, which is 3, among smooth surfaces of general type (cf. [Y] §2.4), it follows that $\Lambda$ cannot be torsion-free. Since $\overline{\Gamma}$ is an arithmetic lattice of type $[D : \ell] > 1$, it is well-known that there cannot be a totally geodesic complex hyperbolic space-form of complex dimension 1 in $B^2_{\mathbb{C}}/\overline{\Gamma}$. Hence $\Lambda$ acts with isolated fixed points on $B^2_{\mathbb{C}}$ and $B^2_{\mathbb{C}}/\Lambda$ has isolated points as singularities. The set of all possible $\Lambda$ has been written down by Cartwright-Steger. The singularities of $B^2_{\mathbb{C}}/\Lambda$ were listed by Keum in [K], and by Cartwright and Steger, see the weblink in [CS1].

**(3.6) Theorem**(a)([K], [CS1]) *Let $H$ be a finite group acting on a fake projective plane $M$. Let $N = M/H$.*
*(i). If $|H| = 3$, $N$ has 3 singularities of type $\frac{1}{3}(1,2)$.*
*(ii). If $|H| = 7$, $N$ has 3 singularities of type $\frac{1}{7}(1,3)$.*
*(iii). If $|H| = 9$, $N$ has 4 singularities of type $\frac{1}{3}(1,2)$.*
*(iv). If $|H| = 21$, $N$ has 3 singularities of type $\frac{1}{3}(1,2)$ and one singularity of type $\frac{1}{7}(1,3)$.*
*(b). ([K]) The Kodaira dimension of the minimal resolution of $N$ is either 1 or 2.*

We also note that the singularities of the quotient of the original Mumford's fake projective plane in [Mu] was first found by Ishida [I], who also studied the minimal resolution and the fundamental group in this case. Mumford's fake projective plane is the one denoted by $(a = 7, p = 2, \{7\}, 7_{21})$ in the table of the last subsection.

**(3.7)** Since $N$ is a normal surface with isolated singularities, we recall the facts related to calculations of Chern numbers as studied by Laufer [L] and Brenton [B]. According to Theorem 1 of [B], there is a rational cohomology class $c_1 \in H^2(N, \mathbb{Q})$ so that for every holomorphic line bundle $L$ on $N$, the Riemann-Roch equation

$$\chi(L) = \frac{1}{2}(c_1^2(L) + c_1(L) \cdot c_1) + \chi(\mathcal{O}).$$

In our case the rank of the group of all cycles of complex dimension in $N$ is 1, since the same is true on $M$. Suppose $N$ is the quotient of $M$ by a finite group $G$. Then $c_1^2 = \frac{1}{|G|}c_1^2(M)$. Let $\pi : Y \to N$ be the minimal resolution of singularities of $N$. Then the first chern class of $Y$ is given by

$$c_1(Y) = \pi^*(c_1) + \sum_{i=1}^{s} t_i [C_i], \tag{1}$$

where $C_i, i = 1, \ldots, s$, are all the irreducible components of the exceptional divisors on $Y$. In our case, each $C_i$ is a rational curve and $t_i$ can be found by applying Adjunction Formula. In this way, $c_1^2(Y)$ can be easily calculated. The Euler-Poincaré characteristic of $N$ can be found from the corresponding one on $M$ by definition, from which the Euler-Poincaré characteristic of $Y$ can be computed since the resolution of singularities are explicit. In this way, $c_2(Y)$ can be computed rather easily.

**(3.8)** In Theorem 1, all the exotic $pP^2_{\mathbb{C}}\#q\overline{P}^2_{\mathbb{C}}$ have values $p$ in odd number. This follows from the following simple observation.

**Lemma 1.** *Suppose $M$ is an algebraic surface with the same Euler number and index as $pP^2_{\mathbb{C}}\#q\overline{P}^2_{\mathbb{C}}$ for some non-negative integers $p$ and $q$ with trivial first Betti number. Then $p$ is odd.*

**Proof** Since both the first and the third Betti numbers are trivial from Duality, the value of $(p, q)$ are related to the Euler number $e$ and index $\sigma$ by the following system of equations.

$$p + q = e(M) - 2 \tag{2}$$

$$p - q = \sigma(M) \tag{3}$$

Since $e(M) = c_2(M)$ and $\sigma(M) = \frac{1}{3}(c_2(M) - 2c_1^2(M))$, we solve easily from the above that

$$p = (\frac{1}{6}(c_1^2 + c_2) - 1).$$

Since $c_1^2 + c_2 \equiv 0 \pmod{12}$ from Noether's Formula (cf. [BHPV]), it follows that $p$ is odd.

**(3.9)** We also need the following fact in our proof. Suppose that $N$ is a singular complex surface with isolated singularities. Suppose $M$ is a resolution of $N$ in which each isolated singularity is resolved to a chain of rational curves. Then $\pi_1(M) = \pi_2(N)$.

The observation follows from the fact that the subvariety of $M$ to be collapsed by the projection has trivial fundamental group. Hence the projection from $M$ to $N$ does not kill any element in $\pi_1(M)$.

## §4. Proof of the Main Theorem.

**(4.1)** We are going to list case by case the construction. Even though the general idea is the same, each case needs a somewhat different treatment. The candidate is obtained by considering the universal covering of the minimal resolution of a finite quotient of a fake projective plane according to the list of [PY] and [CS]. In each case, there are two basic steps of proof. The first is to show that the candidate is homeomorphic to some appropriate $pP^2_{\mathbb{C}}\#q\overline{P}^2_{\mathbb{C}}$, using Theorem (2.2), Theorem (2.3) and some explicit geometric facts of the manifold. The second step is to show that the candidate cannot be diffeomorphic to the same $pP^2_{\mathbb{C}}\#q\overline{P}^2_{\mathbb{C}}$, using Theorem (2.5) and (3.6)(a) in case that $p = 1$, and Theorem (2.4) in case that $p > 1$. We denote by $e(Y)$ and $\sigma(Y)$ the Euler-Poincaré characteristic and the index respectively.

**(4.2)** $(p, q) = (1, 6)$. To illustrate the idea taken, we are going to explain this case in details essentially from scratch. Consider a fake projective plane $M$ given by $(C_2, p = 2, \{3\}, d_3, D_3)$ in the class of $(C_2, p = 2, \{3\})$ listed in the table in **(3.3)**. There is finite group $H$ of order 3 acting on $M$, giving rise to a quotient manifold $N$ denoted by $(C_2, p = 2, \{3\}, d_3)$ as shown in the table. $N$ has three isolated singularities of type $\frac{1}{3}(1, 2)$ as mentioned in Theorem (3.6). Since the singularities are rational and the minimal resolution consists of only rational curves of self-intersection $(-2)$, the canonical line bundle $K_N$ is $\mathbb{Q}$-Cartier and $K_N \cdot K_N = 3$.

Since the Euler-Poincaré characteristic of $M$ is 3 and there are three fixed points in the covering group of $M \to N$, the Euler-Poincaré characteristic of $N$ is given by $e(N) = 3$. Resolution of each singularity $\frac{1}{3}(1,2)$ of $N$ gives rise to a chain of two rational curves of self-intersection $-2$ (cf. [BHPV]). Again denote by $Y$ the minimal resolution of singularity of $N$. Since $N$ has Picard number 1, it follows that $\sigma(Y) = 1-6 = -5$. Since each of the three singularities of $N$ are replaced by chains of two rational curves, we conclude that $e(Y) = 3+6 = 9$. As the Picard number of $M$ is 1, so is $N$. It follows that $K_N$ is $\mathbb{Q}$-Cartier and $K_N^2 = K_M^2/3 = 3 > 0$ and $K_N$ is effective.

The Riemann-Roch formula for surface implies that

$$h^0(Y, lK_Y) = h^1(Y, lK_Y) - h^2(Y, lK_Y) + \frac{l(l-1)}{2}(c_1^2(Y)) + \frac{1}{12}(c_1^2(Y) + c_2(Y)) \geqslant 1.$$

From Serre Duality, $H^2(Y, lK_Y) = H^0(Y, (1-l)K_Y)$. Suppose there exists a non-trivial $\varphi \in H^0(Y, (1-l)K_Y)$ for $l > 1$. Then the intersection of the cohomology class represented by $\varphi$ satisfies $[\varphi] \cdot \pi^* K_N = (1-l)K_N \cdot K_N$. The left hand side is non-negative as $K_N$ is nef, but the right hand side is negative as $K_N \cdot K_N > 0$. The contradiction implies that $h^2(Y, lK_Y) = 0$. Hence we conclude that $K_Y$ is big and $Y$ is a projective algebraic manifold of general type.

It is also shown by [CS1] that $\pi_1(N) = 1$ according to the Table (II) in **(3.4)**. We conclude that $\pi_1(Y) = 1$ as well. Since the singularities of $N$ resolve to $(-2)$ curves, we conclude that $K_Y = \pi^* K_N$ and hence $K_Y \cdot K_Y = K_N \cdot K_N = 3$ is odd. It follows that the intersection matrix of $Y$ is of odd type. It follows from Theorem (2.2), the result of Freedman, that $Y$ is homeomorphic to $pP_{\mathbb{C}}^2 \# q\overline{P}_{\mathbb{C}}^2$ for some pair of $(p,q)$. We may now solve $(p,q)$ directly equations (2) and (3). Here the first equation follows from the fact that the first and third Betti numbers of $N$ are trivial since $\pi_1(N) = \{1\}$.

Since $(e(Y), \sigma(Y)) = (9, -5)$, we conclude that $(p,q) = (1,6)$.

Since $Y$ is of general type and $P_{\mathbb{C}}^2 \# 6\overline{P}_{\mathbb{C}}^2$ as blow-up of $P_{\mathbb{C}}^2$ at six points is a rational surface, we conclude from Theorem (2.5) that $Y$ cannot be diffeomorphic to $P_{\mathbb{C}}^2 \# 6\overline{P}_{\mathbb{C}}^2$. Hence $Y$ gives rise to an exotic $P_{\mathbb{C}}^2 \# 6\overline{P}_{\mathbb{C}}^2$.

Exotic $P_{\mathbb{C}}^2 \# 6\overline{P}_{\mathbb{C}}^2$ are also obtained from $(C_{10}, p = 2, \{17-\})$ and $(C_{18}, p = 3, \emptyset, (dD)_3)$ in Table 2 in **(3.4)** by the above argument.

**(4.3)** $(p,q) = (1,8)$.

Consider $M$ in the class of $(\mathcal{C}_2, p = 2, \{3\}, d_3, D_3)$ in Table 2 in **(3.4)**, and an order 9 quotient $N$ of $M$ which gives rise to $\mathcal{C}_2, p = 2, \{3\})$. $N$ has four isolated singularities of type $\frac{1}{3}(1,2)$ from Theorem (3.6). Again, since the minimal resolution consists of curves with self-intersection $(-2)$, the canonical line bundle $K_N$ is $\mathbb{Q}$-Cartier and $K_N \cdot K_N = K_M \cdot K_M/9 = 1$. Since there are four singularities in $N$ and the order of the covering group of $M \to N$ is 3, we see that there are 12 fixed points of the covering group. As the Euler-Poincaré characteristic of $M$ is 3, the Euler-Poincaré characteristic of $N$ is given by $e(N) = 4 + (3-12)/9 = 3$. Again, the resolution of each singularity $\frac{1}{3}(1,2)$ of $N$ gives rise to a chain of two rational curves of self-intersection $-2$ (cf. [BHPV]). Hence the minimal resolution of $N$ is a projective algebraic surface $Y$. Since $N$ has Picard number 1, it follows that $\sigma(Y) = 1-8 = -7$. Since the four singularities of $N$ are replaced by chains of two rational curves, we conclude that $e(Y) = 3+8 = 11$.

In this case, it is shown by Cartwright and Steger that $N$ is simply-connected, according to tables (I) and (II).. Since $(e(Y), \sigma(Y)) = (11, -7)$, we conclude from equation (2) and (3) that $(p, q) = (1, 8)$. Applying (3.6)(b) or the argument in (4.2), we obtain an algebraic surface of general type which is homeomorphic but not biholomorphic to $P^2_{\mathbb{C}} \# 8\overline{P}^2_{\mathbb{C}}$.

**(4.4)** $(p, q) = (1, 9)$.

Consider $M$ given by $(a = 7, p = 2, \{7\}, D_3, 2_7)$ in the class of $(k, \ell) = (\mathbb{Q}, \mathbb{Q}(\sqrt{-a})$ according to Table 1 in **(3.4)**. There is a finite group action $H$ of order 7, giving rise to a quotient $N$ of $M$ denoted by $(a = 7, p = 2, \{7\}, D_3)$ in the Table 1. According to Theorem (3.6), $N$ has three isolated singularities $p_1, p_2$ and $p_3$ of type $\frac{1}{7}(1, 3)$. Resolution of the singularity at $p_i$ gives rise to a chain of three rational curves $E_{i1}, E_{i,2}, E_{i,3}$ of self-intersections $-3, -2, -2$ respectively. We can now compute $K_Y^2$ according ot the discussions in **(3.7)**.

$$K_Y = \pi^* c_1 - \sum_{i=1}^{3} (\frac{3}{7} E_{i1} + \frac{2}{7} E_{i2} + \frac{1}{7} E_{i3}).$$

The expression $c_1$ used in equation (1) can be computed by $c_1^2 = \frac{1}{7} K_M^2 = \frac{9}{7}$. Hence from the above expression of $K_Y$, we conclude that $K_Y^2 = 0$. It is also clear that $e(N) = 3$ and hence $e(Y) = 3 + 3 \cdot 3 = 12$. Moreover, $\sigma(Y) = -8$.

In this case, it is also shown in [CS1] that $N$ is simply-connected. From Theorem (2.3) and the fact that $\sigma(Y)$ is not divisible by 16 we conclude that the intersection form $Q_Y$ is of odd type. Hence from $(e(Y), \sigma(Y)) = (12, -8)$, we conclude from equations (2) and (3) that $(p, q) = (1, 9)$. Hence from Theorem (2.2), $Y$ is homeomorphic to $P^2_{\mathbb{C}} \# 9\overline{P}^2_{\mathbb{C}}$, which is rational and has Kodaira dimension $-\infty$.

Again, (3.6) and (2.5) implies that $M$ is not diffeomorphic to $P^2_{\mathbb{C}} \# 9\overline{P}^2_{\mathbb{C}}$. This follows from the fact that $c_1^2(Y) = 3\sigma(Y) + 2e(Y) = 0$

Exotic $P^2_{\mathbb{C}} \# 9\overline{P}^2_{\mathbb{C}}$ are also obtained from quotient of order 21 of the above $M$, the case of $(a = 7, p = 2, \{7\}, D_3, 2_7)$. Similarly, if we consider a fake projective plane $M$ given by $(\mathcal{C}_{20}, \{v_2\}, \emptyset, D_3, 2_7)$ in Table 2 in **(3.4)**, there are still order 7 and order 21 finite group actions on $M$. $M/H$ in these two cases are denoted by $(C_{20}, p = 2, \emptyset, D_3)$ and $(C_{20}, p = 2, \emptyset)$ in the Table 2. For these two case, resolution of singularities as above yields examples of exotic $P^2_{\mathbb{C}} \# 9\overline{P}^2_{\mathbb{C}}$.

**(4.5)** $(p, q) = (3, 13)$.

Consider $M$ the fake projective plane denoted by $(a = 7, p = 2, \{7\}, \emptyset, D_3, 2_7)$ in Table 1 in **(3.4)**. There exists a $\mathbb{Z}_3$ action on $M$, giving rise to an order 3 quotient $N$ of $M$ denoted by $(a = 7, p = 2, \{7\}, \emptyset, 2_7)$ in Table 1.

As discussed in **(4.2)**, $N$ has three isolated singularities of type $\frac{1}{3}(1, 2)$. The same argument as in **(4.2)** leads to a desingularization $Y$ of $N$ with $(e(Y), \sigma(Y)) = (9, -5)$,

In this case, it is also shown in [CS1] that $\pi_1(Y) = \mathbb{Z}_2$. Let $\widetilde{Y}$ be the universal covering of $Y$. Since both $e$ and $\sigma$ are multiplicative with respect to unramified coverings, we conclude that $(e(\widetilde{Y}), \sigma(\widetilde{Y})) = (18, -10)$, From Theorem (2.3) and the fact that $\sigma(\widetilde{Y})$ is not divisible by 16 we conclude that the intersection form $Q_Y$ is of odd type. We conclude from equations (2) and (3) that $(p, q) = (3, 13)$. Hence from Theorem (2.2) and the fact that $\widetilde{Y}$ is simply connected, $\widetilde{Y}$ is homeomorphic to $3P^2_{\mathbb{C}} \# 13\overline{P}^2_{\mathbb{C}}$.

Clearly $\widetilde{Y}$ is algebraic since $Y$ is algebraic. In fact, the argument of **(4.2)** shows that $\widetilde{Y}$ is of general type as well. On the other hand, from the result of Donaldson in Theorem (2.4), we know that $3P^2_{\mathbb{C}}\#13\overline{P}^2_{\mathbb{C}}$ as a differentiable manifold does not support any complex structure. Hence $\widetilde{Y}$ gives rise to an exotic $3P^2_{\mathbb{C}}\#13\overline{P}^2_{\mathbb{C}}$ and $\widetilde{Y}$ is a surface of general type.

Exotic $3P^2_{\mathbb{C}}\#13\overline{P}^2_{\mathbb{C}}$ are also obtained from quotient of order 3 of fake projective planes from the case of $(a=7, p=2, \{7\}, D_3, 7_7)$, $(a=15, p=2, \{5\}, D_3)$ and $(\mathcal{C}_2, p=2, \emptyset, d_3, D_3)$ according to Table 1 and Table 2 in **(3.4)**. The corresponding quotients are denoted by $(a=7, p=2, \{7\}, 7_7)$ and $a=15, p=2, \{5\})$ for the first two fake projective planes above respectively, and $(C_{10}, p=2, \emptyset, (dD)_3)$ or $(C_{10}, p=2, \emptyset, (d^2D)_3)$ for the third one. Each of these $N$ has fundamental group given by $\mathbb{Z}_2$. Hence the same argument as above gives rise to surfaces of general type which are exotic $3P^2_{\mathbb{C}}\#13\overline{P}^2_{\mathbb{C}}$.

**(4.6)** $(p,q)=(3,17)$.

Consider $M$ the fake projective plane given by $(\mathcal{C}_2, p=2, \emptyset, d_3, D_3)$ in Table 2 in **(3.4)**. There exists a $\mathbb{Z}_3^2$ action on $M$, giving rise to an order 3 quotient $N$ of $M$ denoted by $(\mathcal{C}_2, p=2, \emptyset)$. $\pi_1(N)=\mathbb{Z}_2$ according to tables (I) and (II).

From Theorem (3.6), $N$ has 4 singularities of type $\frac{1}{3}(1,2)$. As discussed in **(4.2)**, each singularity can be resolved to a chain of two $(-2)$-curves. Discussion as in **(4.3)** implies that the minimal resolution $Y$ of singularities of $N$ have numerical invariants given by $(e(Y), \sigma(Y))=(11,-7)$, Moreover, $Y$ is a surface of general type.

By considering the universal covering $\widetilde{Y}$ of $Y$, which is a two-fold covering, as in **(4.4)**, we conclude that $(e(\widetilde{Y}), \sigma(\widetilde{Y}))=(22,-14)$, As $\sigma$ is not divisible by 16, Theorem (2.3) again implies that the cup form $Q_{\widetilde{Y}}$ is an odd form. It follows form equations (2) and (3) that $\widetilde{Y}$ is homeomorphic to $3P^2_{\mathbb{C}}\#17\overline{P}^2_{\mathbb{C}}$.

Same argument as in **(4.5)** making use of (2.4) implies that $M$ cannot be diffeomorphic to $3P^2_{\mathbb{C}}\#17\overline{P}^2_{\mathbb{C}}$. We get an exotic $3P^2_{\mathbb{C}}\#17\overline{P}^2_{\mathbb{C}}$.

Exotic $3P^2_{\mathbb{C}}\#17\overline{P}^2_{\mathbb{C}}$ are also obtained from quotient $N$ of order 9 of the fake projective planes denoted by $(\mathcal{C}_2, p=2, \{3\}, d_3, D_3)$ in Table 2 in **(3.4)**. The corresponding quotient $N$ is denoted by $(C_2, p=2, \{3\})$. $N$ has fundamental group given by $\mathbb{Z}_2$. Hence the same argument as above gives rise to surfaces of general type which are exotic $3P^2_{\mathbb{C}}\#17\overline{P}^2_{\mathbb{C}}$.

**(4.7)** $(p,q)=(3,19)$.

Consider $M$ to be the fake projective plane denoted by $(a=7, p=2, \emptyset, D_3, 2_7)$ according to Table 1 in **(3.4)**. There exists a group $H$ action of order 7 on $M$, giving rise to an order 7 quotient $N$ of $M$ denoted by $(a=7, p=2, \emptyset, D_3)$. $\pi_1(N)=\mathbb{Z}_2$ according to tables (I) and (II).

Since $|H|=7$, the discussion in the first paragraph of **(4.4)** implies that $N$ has three isolated singularities of type $\frac{1}{7}(1,3)$, each of those gives rise to a chain of three rational curves of self-intersection $-3,-2,-2$ respectively, and that the minimal resolution $Y$ of $N$ has numerical invariants given by $(e(Y), \sigma(Y)=(12,-8)$. It follows that the universal covering $\widetilde{Y}$ of $Y$ has numerical invariants given by $(e(\widetilde{Y}), \sigma(\widetilde{Y}))=(24,-16)$. Since $\widetilde{Y}$ contains an exceptional curve of self-intersection $-3$, the same is true on $\widetilde{Y}$, which can be obtained as the resolution of a 3-fold unramified covering of $N$. Hence the cup form on $\widetilde{Y}$ is odd. We may now apply

Theorem (2.2) and equations (2) and (3) to conclude that $\widetilde{Y}$ is homeomorphic to $3P^2_{\mathbb{C}}\#19\overline{P}^2_{\mathbb{C}}$.

Theorem (2.4) implies that $\widetilde{Y}$ is not diffeomorphic to $3P^2_{\mathbb{C}}\#19\overline{P}^2_{\mathbb{C}}$. Hence we get an exotic $3P^2_{\mathbb{C}}\#19\overline{P}^2_{\mathbb{C}}$.

Similar exotic $3P^2_{\mathbb{C}}\#19\overline{P}^2_{\mathbb{C}}$ are obtained from $(a = 7, p = 2, \emptyset, D_3)$ by similar arguments.

**(4.8)** $(p, q) = (5, 20)$.

Let $M$ be the fake projective plane denoted by $(a = 15, p = 2, \{3, 5\}, D_3)$ in Table 1 in **(3.4)**. There exists a group $H$ action of order 3 on $M$, giving rise to an order 3 quotient $N$ of $M$ denoted by $(a = 15, p = 2, \{3, 5\})$. $\pi_1(N) = \mathbb{Z}_3$ according to tables (I) and (II).

Since the order of the group $H$ is 3, the argument of **(4.2)** leads to $(e(Y), \sigma(Y) = (9, -5)$ and $Y$ is of general type. Hence the numerical invariants of the universal covering $\widetilde{Y}$ of $Y$ is given by $(e(\widetilde{Y}), \sigma(\widetilde{Y})) = (27, -15)$. As $\sigma(\widetilde{Y})$ is not divisible by 16, the intersection form of $\widetilde{Y}$ is odd according to Theorem (2.3). Hence we may apply equations (2) and (3) to conclude that $\widetilde{Y}$ is homeomorphic to $5P^2_{\mathbb{C}}\#20\overline{P}^2_{\mathbb{C}}$.

The result of Donaldson in Theorem (2.4) again implies that $\widetilde{Y}$ is not diffeomorphic to $5P^2_{\mathbb{C}}\#20\overline{P}^2_{\mathbb{C}}$. Hence $\widetilde{Y}$ is an exotic $5P^2_{\mathbb{C}}\#20\overline{P}^2_{\mathbb{C}}$.

Observe that according to Table 1 in **(3.4)**, $M$ may be chosen to be $(a = 15, p = 2, \{3, 5\}, 3_3)$ or $(a = 15, p = 2, \{3, 5\}, (D3)_3)$, but order 3 quotients give rise to the same $N$ given by $(a = 15, p = 2, \{3, 5\})$ studied above. Hence no new exotic $5P^2_{\mathbb{C}}\#20\overline{P}^2_{\mathbb{C}}$ is found apart from the one above.

**(4.9)** $(p, q) = (7, 27)$.

Consider $M$ to be the fake projective plane denoted by $(a = 7, p = 2, \emptyset, D_3)$ as in Table 1 in **(3.4)**. There exists a group $H$ of order 3 acting on $M$, giving rise to an order 3 quotient $N$ of $M$ denoted by $(a = 2, p = 3, \emptyset)$. $\pi_1(N) = \mathbb{Z}_2 \times \mathbb{Z}_2$ according to tables (I) and (II).

Since $|H| = 4$, the discussion in the first paragraph of **(4.1)** implies that $N$ has three isolated singularities of type $\frac{1}{3}(1, 2)$, each of those gives rise to a minimal resolution $Y$ of $N$ with numerical invariants given by $(e(Y), \sigma(Y) = (9, -5)$. It follows that the universal covering $\widetilde{Y}$ of $Y$ has numerical invariants given by $(e(\widetilde{Y}), \sigma(\widetilde{Y})) = (36, -20)$. Since $\sigma(\widetilde{Y})$ is not divisible by 16, Theorem (2.3) implies that the cup form of $\widetilde{Y}$ is an odd form. We may now apply Theorem (2.2) and equations (2) and (3) to conclude that $\widetilde{Y}$ is homeomorphic to $7P^2_{\mathbb{C}}\#27\overline{P}^2_{\mathbb{C}}$.

Again Theorem (2.4) applies to conclude that $\widetilde{Y}$ is not diffeomorphic to $7P^2_{\mathbb{C}}\#27\overline{P}^2_{\mathbb{C}}$. Hence we get an exotic $7P^2_{\mathbb{C}}\#27\overline{P}^2_{\mathbb{C}}$.

Similar examples can be obtained with $N$ given by $(a = 2, p = 3, \{2\}), (a = 7, p = 2, \{7\}, 7'_7), (a = 2, p = 3, \{3, 7\})$, $(a = 15, p = 2, \emptyset)$ and $(\mathbb{C}_{20}, p = 2, \{3-\})$ according to the two tables in **(3.5)**. Note again only $|H|$ matters.

**(4.10)** $(p, q) = (11, 41)$.

Consider $M$ to be the fake projective plane denoted by $(a = 15, p = 2, \{3\}, D_3)$ in Table 1 in **(3.4)**. There exists a group $H$ action of order 3 on $M$, giving rise to an order 3 quotient $N$ of $M$ denoted by $(a = 15, p = 2, \{3\})$. $\pi_1(N) = \mathbb{Z}_2 \times \mathbb{Z}_3$ according to tables (I) and (II).

The argument of the first paragraph of **(4.2)** implies that $N$ has a minimal resolutiion $Y$ with numerical invariants given by $(e(Y), \sigma(Y) = (9, -5)$. The universal covering $\widetilde{Y}$ of $Y$ has numerical invariants given by $(e(\widetilde{Y}), \sigma(\widetilde{Y})) = (54, -30)$. As $\sigma(\widetilde{Y})$ is not a multiple of 16, Theorem (2.3) implies that the cup form of $\widetilde{Y}$ is an odd form. We may now apply Theorem (2.2), equations (2) and (3) to conclude that $\widetilde{Y}$ is homeomorphic to $11P^2_{\mathbb{C}}\#41\overline{P}^2_{\mathbb{C}}$.

On the other hand, Theorem (2.3) applies to conclude that $\widetilde{Y}$ is not diffeomorphic to $11P^2_{\mathbb{C}}\#41\overline{P}^2_{\mathbb{C}}$. Hence we get an exotic $11P^2_{\mathbb{C}}\#41\overline{P}^2_{\mathbb{C}}$.

Exotic $11P^2_{\mathbb{C}}\#41\overline{P}^2_{\mathbb{C}}$ are also obtained with $N$ given by $(\mathcal{C}_2, p = 2, \emptyset, d_3)$ and $(\mathcal{C}_{18}, p = 3, \{2\})$ as listed in Table 2 in **(3.4)**.

**(4.11)** $(p, q) = (13, 48)$.

Consider $M$ to be the fake projective plane denoted by $(\mathcal{C}_2, p = 2, \{3\}, d_3, D_3)$ as in Table 2 in **(3.4)**. There exists a group $H$ of order 3 acting on $M$, giving rise to an order 3 quotient $N$ of $M$ denoted by $(\mathcal{C}_2, p = 2, \{3\}, D_3)$. $\pi_1(N) = \mathbb{Z}_7$ according to tables (I) and (II).

The argument of the first paragraph of **(4.1)** implies that $N$ has a minimal resolutiion $Y$ with numerical invariants given by $(e(Y), \sigma(Y)) = (9, -5)$. The universal covering $\widetilde{Y}$ of $Y$ has numerical invariants given by $(e(\widetilde{Y}), \sigma(\widetilde{Y})) = (63, -35)$. As $\sigma(\widetilde{Y})$ is not a multiple of 16, Theorem (2.3) implies that the cup form of $\widetilde{Y}$ is an odd form. Hence Theorem (2.2), equations (2) and (3) implies that $\widetilde{Y}$ is homeomorphic to $13P^2_{\mathbb{C}}\#48\overline{P}^2_{\mathbb{C}}$. Again, Theorem (2.4) shows that $\widetilde{Y}$ is not diffeomorphic to $13P^2_{\mathbb{C}}\#48\overline{P}^2_{\mathbb{C}}$. Hence we get an exotic $13P^2_{\mathbb{C}}\#48\overline{P}^2_{\mathbb{C}}$.

**(4.12)** $(p, q) = (15, 55)$.

Consider $M$ to be the fake projective plane denoted by $(a = 1, p = 5, \emptyset, D_3)$ in the notation of Table 1 in **(3.4)** There exists a group $H$ of order 3 acting on $M$, giving rise to an order 3 quotient $N$ of $M$ denoted by $(a = 15, p = 2, \emptyset)$. $\pi_1(N) = \mathbb{Z}_2 \times \mathbb{Z}_4$, as shown in the Tables (I) and (II).

The argument of the first paragraph of **(4.2)** implies that $N$ has a minimal resolutiion $Y$ with numerical invariants given by $(e(Y), \sigma(Y)) = (9, -5)$. The universal covering $\widetilde{Y}$ of $Y$ has numerical invariants given by $(e(\widetilde{Y}), \sigma(\widetilde{Y})) = (72, -40)$. As $\sigma(\widetilde{Y})$ is not a multiple of 16, Theorem (2.3) implies that the cup form of $\widetilde{Y}$ is an odd form. We may now apply Theorem (2.2), equations (2) and (3) to conclude that $\widetilde{Y}$ is homeomorphic to $15P^2_{\mathbb{C}}\#55\overline{P}^2_{\mathbb{C}}$. Theorem (2.4) can now be applied to conclude that $\widetilde{Y}$ is not diffeomorphic to $15P^2_{\mathbb{C}}\#55\overline{P}^2_{\mathbb{C}}$. Hence we get an exotic $15P^2_{\mathbb{C}}\#55\overline{P}^2_{\mathbb{C}}$.

Exotic $15P^2_{\mathbb{C}}\#55\overline{P}^2_{\mathbb{C}}$ are also obtained with $N$ given by $(a = 7, p = 2, \{3\})$ in Table 1 and $(\mathcal{C}_{18}, p = 3, \{2\})$ in Table 2. The same argument also applies to $(\mathcal{C}_{18}, p = 2, \{3\}, D_3)$.

**(4.13)** $(p, q) = (25, 90)$.

Consider $M$ to be the fake projective plane denoted by $(\mathcal{C}_{18}, p = 3, \emptyset, \{3\}, d_3, D_3)$ in Table 2 in **(3.4)**. There exists a group $H$ of order 3 acting on $M$, giving rise to an order 3 quotient $N$ of $M$ denoted by $(\mathcal{C}_{18}, p = 2, \emptyset, D_3)$. $\pi_1(N) = \mathbb{Z}_{13}$ according to Tables (I) and (II).

The argument of the first paragraph of **(4.1)** implies that $N$ has a minimal resolutiion $Y$ with numerical invariants given by $(e(Y), \sigma(Y)) = (9, -5)$. The universal covering $\widetilde{Y}$ of $Y$ has numerical invariants given by $(e(\widetilde{Y}), \sigma(\widetilde{Y})) = (117, -65)$. As

$\sigma(\widetilde{Y})$ is not a multiple of 16, Theorem (2.3) implies that the cup form of $\widetilde{Y}$ is an odd form. We may now apply Theorem (2.2), equations (2) and (3) to conclude that $\widetilde{Y}$ is homeomorphic to $25P^2_{\mathbb{C}}\#90\overline{P}^2_{\mathbb{C}}$. Again, Theorem (2.4) can be applied to conclude that $\widetilde{Y}$ is not diffeomorphic to $25P^2_{\mathbb{C}}\#90\overline{P}^2_{\mathbb{C}}$. Hence we get an exotic $25P^2_{\mathbb{C}}\#90\overline{P}^2_{\mathbb{C}}$.

**(4.14)** $(p, q) = (27, 97)$. Consider $M$ to be the fake projective plane denoted by $(\mathcal{C}_2, p = 2, \emptyset)$ in Table 2 in **(3.4)**. There exists a group $H$ of order 3 acting on $M$, giving rise to an order 3 quotient $N$ of $M$ denoted by $(\mathcal{C}_2, p = 2, \emptyset, D_3)$. $\pi_1(N) = \mathbb{Z}_2 \times \mathbb{Z}_7$ according to tables (I) and (II).

The argument of the first paragraph of **(4.1)** implies that $N$ has a minimal resolutiion $Y$ with numerical invariants given by $(e(Y), \sigma(Y)) = (9, -5)$. The universal covering $\widetilde{Y}$ of $Y$ has numerical invariants given by $(e(\widetilde{Y}), \sigma(\widetilde{Y})) = (126, -70)$. As $\sigma(\widetilde{Y})$ is not a multiple of 16, Theorem (2.3) implies that the cup form of $\widetilde{Y}$ is an odd form. We may now apply Theorem (2.2), equations (2) and (3) to conclude that $\widetilde{Y}$ is homeomorphic to $27P^2_{\mathbb{C}}\#97\overline{P}^2_{\mathbb{C}}$. Again, Theorem (2.4) can be applied to conclude that $\widetilde{Y}$ is not diffeomorphic to $27P^2_{\mathbb{C}}\#97\overline{P}^2_{\mathbb{C}}$. Hence we get an exotic $27P^2_{\mathbb{C}}\#97\overline{P}^2_{\mathbb{C}}$.

This concludes the proof of Theorem 1.

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