

CLASSIFICATION OF SURFACES OF GENERAL TYPE WITH EULER NUMBER 3

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Abstract *The smallest topological Euler-Poincaré characteristic supported on a smooth surface of general type is 3. In this paper, we show that such a surface has to be a fake projective plane unless $h^{1,0}(M) = 1$. Together with the classification of fake projective planes given by Prasad and Yeung [PY], the recent work of Cartwright and Steger [CS], and a proof of the arithmeticity of the lattices involved in the present article, this gives a classification of such surfaces.*

This paper is a corrected version of the paper [1]. Changes comparing to [1] are summarized in the last section.

1. Introduction

1.1 The main purpose of this article is to prove the following result on classification of smooth surfaces of general type with the smallest possible topological Euler-Poincaré characteristic. The topological Euler-Poincaré characteristic, denoted by $e(M)$, is the same as the second Chern number $c_2(M)$ of the surface M . It is also simply called the Euler-Poincaré characteristic or the Euler number in this paper.

Theorem 1. (a). *Let M be a smooth surface of general type. Then the Euler-Poincaré characteristic $e(M)$ of M is at least 3.*

(b). *Suppose $e(M) = 3$. Then $M = B_{\mathbb{C}}^2/\Gamma$ is the quotient of a complex hyperbolic space by a torsion free lattice of $PU(2,1)$. Furthermore, unless $h^{1,0}(M) = 1$, M is a fake projective plane.*

(c). *Up to biholomorphism, there are only two examples of M with $e(M) = 3$ and $h^{1,0}(M) = 1$. The two examples are complex conjugate of one another.*

(d). *The moduli space of minimal surfaces of general type with $e(M) = 3$ is reduced and consists of 101 points. 100 of such points correspond to fake projective planes with $h^{1,0} = 0$. One of such points correspond to surfaces with $h^{1,0} = 1$.*

1.2 The main examples of smooth surfaces of general type with Euler-Poincaré characteristic 3 are provided by fake projective planes, which are smooth surfaces with the same Betti numbers as the projective plane but are not biholomorphic to the projective plane. An example of fake projective plane was first constructed by Mumford [Mu3], followed by constructions of Ishida-Kato [IK] and Keum [Ke]. Recently fake projective planes have been classified by Prasad and Yeung [PY] into twenty-eight classes, each of which was shown to consist of at least of two fake

Key Words: surfaces of general type, fake projective planes, harmonic maps and rigidity

1991 Mathematics Subject Classification. Primary 14J29, 32J25, 14J10

The author was partially supported by a grant from the National Security Agency and a grant from the National Science Foundation.

projective planes up to biholomorphism. Subsequently, Cartwright and Steger [CS] showed that there were precisely 50 non-isometric fake projective planes among the twenty eight classes. It is known that for each fake projective plane as a Riemannian manifold, it supports precisely two different conjugate complex structures (cf. [KK]).

It is natural to ask whether fake projective planes exhaust all possibilities of smooth surfaces of general type with Euler-Poincaré characteristic 3. In their work [CS] to enlist all the fake projective planes in the twenty-eight classes classified in [PY], Cartwright and Steger come up with an interesting surface with Euler-Poincaré characteristic 3 and the first Betti number 2. The results of this article show that fake projective planes as classified in [PY] and [CS], and the examples of Cartwright-Steger in [CS] mentioned above, exhaust all smooth surfaces of general type with Euler-Poincaré characteristic 3. The detailed computations are located in the weblink provided in [CS].

1.3 The following is an outline of proof of Theorem 1. Part (a) follows from classical results in geometry, as explained in §1. The main part of the article is in the proof of the statement of (b) and (c). (b) and (c) are proved using combination of classical methods from algebraic geometry as well as techniques of harmonic mappings into appropriate Bruhat-Tits buildings. Finally, the classification of Prasad-Yeung [PY] and Cartwright-Steger [CS] is applied to conclude the proof.

1.4 The author would like to thank Jungkai Chen, Matteo Penegini and Wing Keung To for reading an early draft of the paper and pointing out some errors, to thank Jiu-Kang Yu for helpful discussions, and to thank the referees for various suggestions. The author would also like express his appreciation to Domingo Toledo for pointing out a gap in Archimedean rigidity in the original argument and for helpful discussions.

2. Preliminaries

2.1 Let us denote by $c_i = c_i(M)$ the Chern numbers, $b_i = b_i(M)$ the Betti numbers of M and $h^{i,j} = h^{i,j}(M) = \dim_{\mathbb{C}} H^j(M, \Omega_M^i)$ the corresponding Hodge numbers. $c_2(M)$ is just the Euler-Poincaré characteristic of M . For our convenience, let us recall some standard identities.

$$\begin{aligned}
 (1) \quad c_2 &= 2b_0 - 2b_1 + b_2 \\
 (2) \quad \frac{1}{12}(c_1^2 + c_2) &= h^{0,0} - h^{1,0} + h^{2,0} \\
 (3) \quad b_1 &= 2h^{1,0}, \\
 (4) \quad b_2 &= 2h^{2,0} + h^{1,1}, \\
 (5) \quad h^{i,j} &= h^{j,i},
 \end{aligned}$$

where the second one is the Noether formula and the third one comes from Hodge decomposition.

For Theorem 1(a), let first M be a minimal surface of general type so that $c_1^2 > 0$. It is clear from Miyaoka-Yau inequality that $c_1^2 \leq 3c_2$. Noether's Formula implies that

$$0 < \frac{1}{12}(c_1^2(M) + c_2(M)) \leq \frac{1}{3}c_2(M).$$

It follows that $e(M) = c_2(M) \geq 3$.

Suppose now that M is an arbitrary surface of general type. Let M' be a minimal surface of general type obtained by contracting some -1 curves on M . Since contracting a -1 curve decreases the Euler Poincaré characteristic by 1, we know that $e(M) \geq e(M') \geq 3$. In particular, if $e(M) = 3$, the above discussions imply that $M = M'$. This concludes the proof of Theorem 1(a).

Since the far right hand side of the above sequence of inequalities is 1, it follows that the inequality sign \leq is actually an equality. We conclude that $c_1^2(M) = 9 = 3c_2(M)$. It is well known that a compact complex surface M with $c_1^2(M) > 0$ is projective algebraic (cf. [BHPV], page 161). The results of Aubin [A] and Yau [Ya] on the Calabi Conjecture in the case of negative scalar curvature implies the existence of Kähler-Einstein metric, see also [Mi]. This in turn implies that the metric is the standard hyperbolic metric using the fact that $c_1^2(M) = 3c_2(M)$, see for example the survey in [Y2], page 391. Hence M is a compact complex ball quotient.

We summarize the observation above as follows.

Proposition 1. *Let M be a smooth surface of general type. Then the Euler-Poincaré characteristic $e(M) \geq 3$. Moreover, the equality occurs if and only if $M = B_{\mathbb{C}}^2/\Gamma$ is the quotient of a complex hyperbolic space by a torsion free lattice of $PU(2, 1)$.*

2.2 The moduli space of such surfaces M with $e(M) = 3$ is well-known to come with a natural scheme structure. The infinitesimal deformation of any such M in a local Kuranishi family of deformation is given by an element in $H^1(M, \Theta)$, where Θ is the sheaf of holomorphic vector fields on M . Since any such M is a locally Hermitian symmetric space, according to the local rigidity of Calabi and Vesentini [CV], $H^1(M, \Theta) = 0$. It follows that the virtual dimension of any deformation space is zero, which implies that the actual deformation space is of dimension 0. As the dimension of the virtual deformation is the same as the dimension of the actual deformation, we conclude that the moduli space is reduced. As the dimension is zero, the moduli space consists of a finite number of points.

Hence to prove Theorem 1, our working assumption from this point on is that M is a compact complex two ball quotient. From Noether's formula (0.2),

$$h^{0,0}(M) - h^{1,0}(M) + h^{2,0}(M) = \frac{1}{12}(c_1^2(M) + c_2(M)) = 1.$$

We conclude that $h^{1,0}(M) = h^{2,0}(M)$. The purpose of §3-5 is to employ classical algebraic geometric method to prove that $h^{1,0}(M) \leq 2$. The case $h^{1,0}(M) = 0$ corresponds to fake projective plane and has been classified in [PY] and [CS]. For the cases $h^{1,0}(M) = 1$ and 2, we show that the arguments in [Ye2] and [Ye3] can be modified to prove the arithmeticity of the lattice involved in §6, from which the classification results in [PY] and [CS] can be applied again.

2.3

Lemma 1. *The canonical line bundle K_M of a smooth surface M of general type with $e(M) = 3$ satisfies $c_1^2(K_M) = 9 \int_M \Theta(H_{\widetilde{M}}) \wedge \Theta(H_{\widetilde{M}})$, where $\Theta(H_{\widetilde{M}})$ is the curvature form of an ample line bundle $H_{\widetilde{M}}$ on \widetilde{M} , and $\int_M \Theta(H_{\widetilde{M}}) \wedge \Theta(H_{\widetilde{M}}) \in \mathbb{Z}$.*

Proof Denote by $H_{\widetilde{M}}$ the $SU(2, 1)$ -equivariant line bundle discussed in §10.4 of [PY] as well as in [Ko]. Let $\pi : \widetilde{M} \rightarrow M = \widetilde{M}/\Pi$ be the uniformization map. On

\widetilde{M} , $H_{\widetilde{M}}$ is a third root of π^*K_M as an $SU(2, 1)$ line bundle, but may not descend to M as a holomorphic line bundle since Π in general only lives in $PU(2, 1)$. We only know that $H_{\widetilde{M}}$ descends as a multivalued line bundle with ambiguity lying in \mathbb{Z}_3 . However, the curvature form $\Theta(H_{\widetilde{M}})$ descends as a genuine $(1, 1)$ form on M . Let \mathcal{F} be a fundamental domain of M in \widetilde{M} , It follows that

$$\int_M \Theta(H_{\widetilde{M}}) \wedge \Theta(H_{\widetilde{M}}) = \int_{\mathcal{F}} \Theta(H_{\widetilde{M}}) \wedge \Theta(H_{\widetilde{M}}) = \frac{1}{9} c_1(K_M) \cdot c_1(K_M).$$

However, we observe from Chern number equality $c_1^2(M) = 3c_2(M)$ and the Noether Formula that $c_2(M) = \frac{1}{4}(c_1^2(M) + c_2(M)) = 3\chi(\mathcal{O}_M)$. Hence $c_1^2(M) = 9\chi(\mathcal{O}_M)$ and $\int_M \Theta(H_{\widetilde{M}}) \wedge \Theta(H_{\widetilde{M}}) \in \mathbb{Z}$. \square

3. Case of irregularity ≥ 3 .

3.1 We note that for any two linearly independent holomorphic one forms ω_1 and ω_2 on M , the wedge product $\omega_1 \wedge \omega_2$ cannot be identically zero on M . Otherwise Castelnuovo-de Franchi Theorem implies that there is a fibration $\pi : M \rightarrow S$ of M over a Riemann surface S of genus at least 2 (cf. [BPHV], page 157). Let $g(S)$ be the genus of S and $g(M_s)$ be the genus of a generic fiber M_s of π . Denote by $e(M)$ the Euler-Poincaré characteristic of a manifold M . It follows that

$$(6) \quad e(M) = e(S)e(M_s) + \sum_{s_o} n_{s_o},$$

where the sum is taken over the finite number of singular fibers M_{s_o} of π , of which each $n_{s_o} = e(M_{s_o}) - e(M_s)$ is a non-negative integer, and is positive unless M_{s_o} is a multiple fiber with $(M_{s_o})_{red}$ nonsingular elliptic (cf. [BHPV], page 118). Hence $e(M) \geq (2g(S) - 2)(2g(M_s) - 2) \geq 4$. This contradicts our assumption that $c_2(M) = 3$. Hence $\omega_1 \wedge \omega_2$ is a non-trivial holomorphic two form on M whenever ω_1 and ω_2 are linearly independent.

Since $\omega_1 \wedge \omega_2 \neq 0$, this implies that $h^{2,0} \geq 2h^{1,0} - 3$ by considering $\omega_1 \wedge \omega_i$ and $\omega_i \wedge \omega_2$, where $\{\omega_i\}_{1 \leq i \leq h^{1,0}(M)}$ is a basis of $H^0(M, \Omega)$. Since we know that $h^{1,0}(M) = h^{2,0}(M)$, we conclude that $h^{1,0}(M) = h^{2,0}(M) \leq 3$.

3.2 The case of $h^{1,0}(M) = h^{2,0}(M) = 3$ was ruled out from the classification of Hacon and Pardini ([HP], Theorem 2.2, see also [CCM]).

Denote by $A = A(M)$ the Albanese variety of M and $\alpha : M \rightarrow A$ the Albanese mapping. We are going to eliminate the case of $h^{1,0} = 1$ and 2 in §5.

4. Example of M with irregularity 1 and $e(M) = 3$

4.1 In trying to enumerate the set of all fake projective planes in the class \mathcal{C}_{11} according to [PY], Cartwright and Steger [CS] came across a torsion free lattice of Euler-Poincaré characteristic 3 and $h^{1,0}(M) = 1$. This is surprising since before their work, it was generally expected that smooth surfaces of general type with $c_2 = 3$ were fake projective planes. To describe the example, we need to explain the scheme of classification in [PY] briefly. We will refer the readers to [PY] for all the unexplained notations in the following discussions.

An arithmetic lattice Λ for $PU(2, 1)$ is described as follows (cf. [PY]). We refer the readers to [PY] for all the unexplained notations. Let k be a totally real number field. Let ℓ be a totally imaginary quadratic extension of k . Let \mathcal{D} be a division algebra with center ℓ of degree 3 equipped with an involution σ of second kind, such

that for the hermitian form h_0 on \mathcal{D} defined by $h_0(x, y) = \sigma(x)y$, the group $\mathrm{SU}(h_0)$ is isotropic at v_o , and is anisotropic at every other real place of k . For $x \in \mathcal{D}^\times$, let $\mathrm{Int}(x)$ denote the automorphism $z \mapsto xzx^{-1}$ of \mathcal{D} . Let $\mathcal{D}^\sigma = \{z \in \mathcal{D} \mid \sigma(z) = z\}$. Observe that for all $x \in \mathcal{D}^\sigma$, $\mathrm{Int}(x) \cdot \sigma$ is again an involution of \mathcal{D} of the second kind, and any involution of \mathcal{D} of the second kind is of this form. Now for $x \in \mathcal{D}^\sigma$, given an hermitian form h' on \mathcal{D} with respect to the involution $\mathrm{Int}(x) \cdot \sigma$, the form $h = x^{-1}h'$ is a hermitian form on \mathcal{D} with respect to σ , and $\mathrm{SU}(h') = \mathrm{SU}(h)$. Therefore it suffices to work just with the involution σ , and to consider all hermitian forms h on \mathcal{D} , with respect to σ , of determinant 1, such that the group $\mathrm{SU}(h)$ is isotropic at v_o , and is anisotropic at all other real places of k . Let h be such a hermitian form. Then $h(x, y) = \sigma(x)ay$, for some $a \in \mathcal{D}^\sigma$. The determinant of h is $\mathrm{Nrd}(a)$ modulo $N_{\ell/k}(\ell^\times)$. As the elements of $N_{\ell/k}(\ell^\times)$ are positive at all real places of k , we see that the signatures of h and h_0 are equal at every real place of k , which leads to the isometry between the hermitian forms h and h_0 . Hence, $\mathrm{SU}(h)$ is k -isomorphic to $\mathrm{SU}(h_0)$. Thus \mathcal{D} determines a unique k -form G of $\mathrm{SU}(2, 1)$, up to a k -isomorphism, namely $\mathrm{SU}(h_0)$, with the desired behavior at the real places of k . The group $G(k)$ of k -rational points of this G is

$$G(k) = \{z \in \mathcal{D}^\times \mid z\sigma(z) = 1 \text{ and } \mathrm{Nrd}(z) = 1\}.$$

Let $P = (P_v)_{v \in V_f}$ be a coherent collection of parahoric subgroups P_v for each place $v \in V_f$, the set of all finite places of k , chosen as in [PY] (see also the Addendum). In [PY], the set of all possible arithmetic lattices Γ with $e(B_{\mathbb{C}}^2/\Gamma) = 3$ was classified into a small number of classes. Each of these classes determines a unique principal arithmetic subgroup $\Lambda (= G(k) \cap \prod_{v \in V_f} P_v)$, whose normalizer in $\overline{G}(k_{v_o})$ is denoted by $\overline{\Gamma}$. Each Λ determines a class of fake projective planes with fundamental group given by a lattice Π of $PU(2, 1)$, where Π is an element in

$$A_\Lambda = \{\Pi < \overline{\Gamma} : [\overline{\Gamma} : \Pi] = \frac{3}{\chi(\overline{\Gamma})}, |\Pi/[\Pi, \Pi]| < \infty, \text{ and } \Pi \text{ is torsion-free}\}$$

It follows that from the work of [PY] that there are twenty-eight distinct set $\{k, \ell, G, (P_v)_{v \in V_f}\}$ which can support fake projective planes. Five more classes which may contain smooth surfaces of Euler number 3 are listed in [PY] but are not expected to support fake projective planes. The latter was confirmed by [CS] which also shows that there is precisely one class containing Γ with $e(B_{\mathbb{C}}^2/\Gamma) = 3$ and $h^{1,0}(B_{\mathbb{C}}^2/\Gamma) = 1$, and there is only one such Γ . The defining number fields (k, ℓ) for the example of $h^{1,0}(M) = 1$ found in [CS] are given by $k = \mathbb{Q}(\sqrt{3})$ and $\ell = \mathbb{Q}(\zeta_{12})$, the cyclotomic field associated to the 12th root of unity. The pair of number fields is denoted by \mathcal{C}_{11} in [PY]. The division algebra in the definition of the lattice is chosen to be $\mathcal{D} = \ell$. There is a maximal arithmetic lattice $\overline{\Gamma}$ defined over \mathcal{C}_{11} which may contain a torsion free lattice of Euler number 3 as explained in page §8.2 of [PY]. It follows from the volume formula of Prasad [P], (see the table on page 354 of [PY]), that orbifold characteristic $e(B_{\mathbb{C}}^2/\overline{\Gamma}) = 1/288$. Cartwright and Steger showed in [CS] that indeed a torsion-free subgroup Γ of index 864 existed in $\overline{\Gamma}$. Moreover, Γ is in fact a congruence subgroup of $\overline{\Gamma}$. This is obtained by writing down explicitly a set of generators for $\overline{\Gamma}$, from which a torsion-free subgroup of index 864 is found.

From explicit computation, Cartwright and Steger verify that the first Betti number of $B_{\mathbb{C}}^2/\Gamma$ is 2.

In the next section, we will sketch a proof that the example constructed by Cartwright and Steger is unique in the sense that the fundamental group of any such example has to be conjugate to the one constructed by Cartwright and Steger.

5. Arithmeticity and integrality

5.1 In this and the next section, we approach the remaining cases, $h^{1,0} = 1$ and 2, by a method very different from §4. It is a modification of the approach developed in [Kl], [Ye2], [PY] and [CS] for the classification of fake projective planes. Before we go to the actual proof, we would like to outline the principle involved and the idea of proof.

Here are the main steps, working under the assumption that the smooth surface M satisfies the conditions that $K_M \cdot K_M = 9$.

Step 1: To show that M is a complex two ball quotient $B_{\mathbb{C}}^2/\Gamma$, where Γ is a torsion-free lattice in $PU(2,1)$.

Step 2: To show that the lattice Γ is an arithmetic lattice in $PU(2,1)$.

Step 3: To classify all possible torsion-free arithmetic lattices Γ for which the corresponding M satisfies the topological condition above.

Step 1 was already achieved in §2. Step 2 is the key step in this section and is a modification of the argument used in [Ye2]. Step 3 has already been achieved by the results in [PY] and [CS], since the setting of [PY] actually aims at classification of all torsion-free arithmetic lattices in $PU(2,1)$ with Euler number 3. As explained in §5, it is shown in [PY] and [CS] that among all such arithmetic lattices, there is only one lattice with $h^{1,0} = 1$. All the others have $h^{1,0} = 0$ and are fake projective planes.

5.2 Step 2 consists of two steps, integrality and Archimedean rigidity. We would study integrality in this section, and consider Archimedean rigidity in the next section.

Proposition 2. *Let $M = B_{\mathbb{C}}^2/\Gamma$ be a smooth compact complex two ball quotient with $e(M) = 3$ and $h^1(M) \leq 2$. Then Γ is an integral lattice.*

Proof The structure of proof will be similar to the approach we took in [Ye2], which was originally designed for a lattice corresponding to a fake projective plane. In [Ye2], we need the assumptions that the Picard number is 1 and $h^{1,0} = 0$. The argument of the original article of [Ye2] and its corrections in the erratum was presented in a self-contained and coherent way in [Ye3], stated as Theorem 7 in [Ye3]. For Proposition 4, $M = B_{\mathbb{C}}^2/\Gamma$ is a torsion free compact complex ball quotient with $e(M) = 3$. Comparing to the conditions required for the results in [Ye2] or [Ye3], the only modification needed is that the Picard number may not be equal to 1.

Before we go to the actual places of [Ye3] where modification is needed, we would like to outline the main idea of proof.

A result of Weil tells us that any cocompact lattice Γ of $PU(2,1)$ is locally rigid, from which it follows that Γ can be defined over a number field, that is, there exists an injective homomorphism $\rho : \Gamma \rightarrow G(k)$, G an algebraic group defined over a number field k with a Archimedean place v_o such that $G(k_{v_o}) \cong PU(2,1)$. We say that Γ is integral if there exists a subgroup Γ' of finite index in Γ so that $\rho(\Gamma') \subset G(\mathcal{O}_k)$. The details of the above can be found in §1 of [Ye2].

5.3 As in the proof of [Ye2], [Ye3], there are two main steps for Step 2 above.

Step 2A, to prove that Γ is integral, and

Step 2B, to prove an analogue of archimedean superrigidity as in §4.7 of [Ye3], in the sense that there is ρ as above satisfying the condition that $G(k_v)$ is compact for all $v \neq v_o$.

Let us first give an overview of the proofs of Step 2A and Step 2B before we work on the details.

Step 2A is a modification of the proof of Integrality in §4.4 and §4.5 of [Ye3] (§2-4 of [Ye2]). For the sake of proof by contradiction, assume that Γ is not integral so that there exists for a finite place v a non-trivial unbounded representation $\rho_v : \Gamma \rightarrow G(k_v)$. Then there exists a faithful energy minimizing ρ_v -equivariant harmonic map $f : \widetilde{M} \rightarrow X$, the Bruhat-Tits building associated to $G(k_v)$, as explained in the first two paragraphs of §4.4 in [Ye3], using a result of Gromov and Schoen. The Bruhat-Tits building is either of rank 1 or 2, from which the pull back of the differentials of the affine coordinate functions on an apartment by the harmonic map leads to harmonic forms on a finite sheeted cover M_1 , namely a spectral covering, of M . The covering group is given by \overline{W}_1 , a subgroup of the Weyl group associated to the root system of $G(k_v)$. Bochner formula implies the existence of non-trivial holomorphic one forms ω on M_1 , from which we construct a non-trivial Albanese map associated to ω 's from A_1 to an abelian variety $Alb_{\overline{W}_1, \omega}$, to be explained in more details below. The key point of the argument is to use properties of the Albanese map, its relation to the original harmonic map into the buildings and the geometry of the Bruhat-Tits building to prove that M_1 is an unramified covering of M , from which we conclude a contradiction as explained in [Ye2] or page 406 of [Ye3]. The details are given in this sections.

For Step 2B, the idea is a substantial modification of §4.7 of [Ye3] (§5 of [Ye2]). Let $v_i, i = 2, \dots, n$ be the other Archimedean places of k so that we may consider $R_{k/\mathbb{Q}}(G)(\mathbb{R}) = G(k_{v_o}) \times G(k_{v_2}) \times \dots \times G(k_{v_n})$. From the type of Lie algebra and a result of Simpson on complex variation of Hodge structure, we conclude that $G(k_{v_i})$ for $i \geq 2$ is either $PU(2, 1)$ or $PU(3)$. The key point is to rule out the possibility that $G(k_{v_i}) = PU(2, 1)$ for some $i \geq 2$. Assume that such an i exists. It follows that there exists a Γ -equivariant harmonic map Φ from $\widetilde{M} = B_{\mathbb{C}}^2$ to $B_{\mathbb{C}}^2$, the latter corresponding to the symmetric space associated to $G(k_{v_i}), i \geq 2$. In case that the real rank of the mapping Φ is at least 3, Bochner type argument implies that Φ gives rise to a holomorphic mapping. To rule out the case that the real rank of Φ is 2, we have to modify some argument of Carlson and Toledo in [CT] and study some natural foliation associated to Φ to show that Φ descends to a fibration of M over a curve, from which a contradiction is derived readily. Once we have proved that Φ is holomorphic of complex rank 2, the idea then is to show that Φ has no ramification divisor and hence is biholomorphic, making use of $c_1(M)^2 = 9$. This would then lead to a contradiction as in §4.7 of [Ye3]. The details in this step is given in the next section.

To carry out the scheme of proof of [Ye3] in our situation, we need to get rid of the use of Picard number 1 in both steps 2A and 2B above. For Step 2A, this is achieved by Lemma 4 below and the two paragraphs near the end of **5.4**, which will be used in the proof of Step 2A in [Ye3] to replace the restriction on the Picard

number 1. For Step 2B, we get rid of Picard number 1 by a more skillful use of the fact that $K_M \cdot K_M = 9$.

5.4 In this subsection, we will provide a more detailed exposition of Step 2A parallel to the discussions in [Ye2] and [Ye3]. We will also provide the details of the modification required for our case when it is needed.

Lemma 2. *Let V be a proper algebraic subvariety in a compact complex ball quotient M . Let $i : V \rightarrow M$ be the embedding. Let $\pi_1(V)$ be the fundamental group of V . Then $i_*(\pi_1(V)) \subset \pi_1(M)$ is non-trivial.*

Proof Suppose on the contrary that $i_*(\pi_1(V)) \subset \pi_1(M)$ is trivial. Let $p : \widetilde{M} \rightarrow M$ be the universal covering map. Let V_o be a connected component of $p^{-1}(V)$. The restriction $p : V_o \rightarrow V$ is an unramified covering. Let ℓ be any closed loop on V based at $p \in V$. Since $i_*(\pi_1(V))$ is trivial, the lift $\tilde{\ell}$ of ℓ to V_o has to be a contractible loop on V_o . Hence $p : V_o \rightarrow V$ is a one-sheeted cover of V and therefore is a diffeomorphism since it is a covering map. Hence V_o is a compact subvariety of \widetilde{M} . However on $\widetilde{M} = B_{\mathbb{C}}^n$, there exists a strictly plurisubharmonic function given by $|z|^2 = \sum_{i=1}^n |z_i|^2$, the restriction of which to V_o is still plurisubharmonic. As V_o is compact, the plurisubharmonic function has to attain a maximum. This however contradicts the Submean Value Inequality for a plurisubharmonic function, thereby concludes the proof of the Lemma.

In the following we will go through the structure of proof in [Ye3], explain in details places that need to be modified under our new weakened assumption, while refer the readers to [Ye3] for details that were already written there.

As mentioned in the brief overview above, if the lattice Γ involved is not integral in k , there exists a finite place v and a Γ equivariant harmonic map from \widetilde{M} , the universal covering of M , to X , the Bruhat-Tits building associated to the induced representation of Γ in the corresponding group $G(k_v)$. Pulling back the coordinate differentials on X by the harmonic map, we get some multivalued harmonic one forms on \widetilde{M} , which after descending to M and considering the $(1, 0)$ part provide multivalued holomorphic one forms. The fact that the one forms are multivalued follows from the construction, since there is an action of the affine Weyl group on X . The multivalued one forms become a set of single valued holomorphic one forms denoted by $\{\omega_i\}$ after going to a finite spectral covering M_1 of M , with the covering group \overline{W}_1 being a subgroup of the Weyl group of the root system of X involved. The details are given in the first seventh paragraphs in §4.4 of [Ye3].

The dimension of X may be 1 or 2 depending on the rank of G over k_v . Suppose first that $\text{rank}_{k_v}(G) = 1$ so that X is a tree. This corresponds to the argument given in page 400 of [Ye3]. In this case, there is a mapping $\alpha : M_1 \rightarrow \text{Alb}_{\overline{W}_1, \{\omega_i\}}(M_1)$, which is the quotient of the Albanese variety by the \overline{W}_1 -invariant abelian subvariety annihilated by all the ω_i obtained earlier. We also know that $\text{Alb}_{\overline{W}_1, \{\omega_i\}}(M_1)$ has complex dimension 1 in this case. Let x_o be a fixed point and x be an arbitrary point on M_1 . As in the proof of Lemma 3 in §4.6 of [Ye3], a generic fiber of the mapping $h_{\mathbb{R}} : \widetilde{M}_1 \rightarrow \mathbb{R}$ given by $x \rightarrow \int_{x_o}^x \text{Re}(\omega)$ is a generic fiber of $\tilde{f}_1 : \widetilde{M}_1 \rightarrow \widetilde{M} \xrightarrow{\alpha} X$. Since $h_{\mathbb{R}}$ can be considered as the real part of the universal covering of α , it follows that a generic fiber V_x of $\tilde{\alpha}$, where x is generic point in $\text{Alb}_{\overline{W}_1, \{\omega_i\}}(M_1)$, is mapped to a point by \tilde{f}_1 . Hence for a generic x , $\rho_v(i_*(\pi_1(V_x)))$ is acting trivially at $f \circ \pi(V_x)$, where i is the inclusion mapping. Since f is ρ -equivariant, we know that for each

$y \in \widetilde{M}$, $(\rho_v(\gamma))(f(y)) = f(\gamma(y))$ for each $y \in \widetilde{M}$ and $\gamma \in \pi_1(M)$. It follows that $\rho_v(i_*(\pi_1(V_x)))$ acts trivially on X . Hence $\rho_v(i_*(\pi_1(V_x)))$ is trivial. As ρ_v is one to one, $i_*(\pi_1(V_x))$ is trivial for a generic $x \in \text{Alb}_{\widetilde{W}_1, \{\omega_i\}}(M_1)$. This however contradicts Lemma 4. The claim is proved. The above is the first modification needed in Step 2A.

The more difficult case is that $\text{rank}_{k_v}(G) = 2$ so that the dimension of X is 2. In this case, an apartment in X can be written as $\Sigma = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\} \cong \mathbb{R}^2$, and there are three holomorphic one forms $\omega_i, i = 1, 2, 3$ on M_1 coming from pulling back of coordinate differentials by the harmonic map f into X as mentioned earlier. The Weyl group of the root system is the symmetric group of three elements S_3 and the spectral covering group \widetilde{W}_1 is a subgroup of the Weyl group S_3 . In this case, the corresponding Albanese map α as defined earlier may have dimension one or two image in $\text{Alb}_{\widetilde{W}_1, \{\omega_i\}}(M_1)$. First we claim that $\alpha(M_1)$ cannot be a dimension one subvariety in $\text{Alb}_{\widetilde{W}_1, \{\omega_i\}}(M_1)$. Assume on the contrary that $\alpha(M_1)$ is of complex dimension one. In such case, for a generic point on $\alpha(M_1)$, the inverse image in M_1 is a curve. Consider the mapping $\widetilde{h}_R : \widetilde{M}_1 \rightarrow \mathbb{R}^2$ defined by

$$\begin{aligned} \widetilde{h}_R(z) &= \left(\int_{z_o}^z (f \circ \pi)^* dx_1, \int_{z_o}^z (f \circ \pi)^* dx_2, \int_{z_o}^z (f \circ \pi)^* dx_3 \right) \\ &\in \left\{ (w_1, w_2, w_3) \in \mathbb{R}^3 \mid \sum_{i=1}^3 w_i = 0 \right\} \cong \mathbb{R}^2, \end{aligned}$$

where f is the harmonic map into the building, and x_i 's are the affine functions defining an apartment of X , cf. [Ye3], §4.4. Clearly $\widetilde{h}_R(z)$ is just the projection of $\widetilde{\alpha}$ onto the real part of \mathbb{C}^2 . Again, fibers of \widetilde{h}_R correspond to fibers of \widetilde{f} . Similar to the argument in the last paragraph, $\rho_v(i_*(\pi_1(V_x)))$ is trivial. As ρ_v is one to one, $i_*(\pi_1(V_x))$ is trivial for a generic $x \in \text{Alb}_{\widetilde{W}_1, \omega}(M_1)$. This again contradicts Lemma 4. The claim is proved. The argument of this paragraph is a replacement of a similar argument on page 400 of [Ye3] and is the second modification required.

Hence we know that $\alpha(M_1)$ has complex dimension 2. The key point of the argument is to show that in such a case, the spectral mapping $\pi : M_1 \rightarrow M$ is unramified. This is achieved via proof by contradiction. Hence assume that π is ramified. Let D be an irreducible codimension one component of the ramification divisor on M_1 corresponding to $\omega_i - \sigma\omega_i = 0$ for some ω_i and $\sigma \in \widetilde{W}_1$ in the construction of the spectral covering. There is at least one such component whose image on M is a branching divisor, otherwise π would be unramified. Since α is the Albanese map defined by the one forms $\{\omega_i\}$, the image $\alpha(D)$ is an algebraic curve in $\text{Alb}_{\widetilde{W}_1, \{\omega_i\}}(M_1)$ and is an Abelian subvariety in $\text{Alb}_{\widetilde{W}_1, \{\omega_i\}}(M_1)$, following from the same argument as in §4.5 of [Ye3]. To derive a contradiction, we relate α to the harmonic map f as given in §4.5 of [Ye3] so that the geometry of X comes into play. Consider a generic fiber C_a of the projection $M_1 \xrightarrow{\alpha} \text{Alb}_{\widetilde{W}_1, \{\omega_i\}}((M_1) \rightarrow \text{Alb}_{\widetilde{W}_1, \{\omega_i\}}(M_1) / \alpha(D))$. The contradiction is achieved by proving the following two statements. On the one hand, $\overline{p_*(\pi(C_a))}$, the Zariski closure of $p_*(\pi(C_a))$, is a non-trivial normal subgroup of the group G . Since G as a real Lie group is isomorphic to $PU(2, 1)$, it is simple. We conclude that $\overline{p_*(\pi(C_a))} = G$. On the other hand, $f(\pi(C_a))$ is a tree lying in X , of which the stabilizer is given by a proper subgroup of G . The two statements

contradict each other. Hence we conclude that $\pi : M_1 \rightarrow M$ is unramified once the above two statements are proved.

The proof of the two statements is given in details in §4.5 (page 400-405) of [Ye3], where no further assumption is used, except that we need to explain line 13 on page 403 of [Ye3], which corresponds to Sublemma on page 284 of the Erratum of [Ye2]. In the case that the Picard number of M is one in the setting of [Ye2] and [Ye3], Sublemma follows from the following observation. Suppose that D is a divisor contracted by α to a point on $\text{Alb}_{\overline{W}_1, \{\omega_i\}}(M_1)$. It follows that D lies in the kernel of ω_i for $i = 1, 2, 3$ from definition of the Albanese map, and hence in the kernel of $\sigma^* \omega_i$ for each $\sigma \in \overline{W}$. Since \overline{W} induces an action on $\text{Alb}_{\overline{W}_1, \{\omega_i\}}(M_1)$ and α induces a map $\beta : M \rightarrow \text{Alb}_{\overline{W}_1, \{\omega_i\}}/\overline{W}$, this implies that $\pi(D)$ is contracted by $\beta : M \rightarrow \text{Alb}_{\overline{W}_1, \{\omega_i\}}/\overline{W}$, which contradicts the assumption that the Picard number of M is 1. In our current situation we do not know if the Picard number of M is 1. However the Sublemma is only used to make sure of the assertion on line 20 of page 284 in Erratum of [Ye2] that $\alpha^* E_Q$ does not contain a contracted curve, where $Q \in \text{Alb}_{\overline{W}_1, \{\omega_i\}}(M_1)/\alpha(D)$. In the setting, Q is parametrized by $\alpha(\mathcal{S})$, where \mathcal{S} is a component of the singularity set of f at which C_a meet on M_1 . Note that as mentioned in [Ye2], [Ye3], there is nothing to be proved if C_a does not meet any singularity set of f on M_1 , and when such a singularity set \mathcal{S} is present, it is defined as $\omega_j - \sigma_k \omega_j = 0$ for some $\sigma_k \in \overline{W}_1$, and $\text{Alb}_{\overline{W}_1, \{\omega_i\}}(M_1)$ is isogeneous to $\alpha(D) \times \alpha(\mathcal{S})$. With a slight abuse of notation, we may consider $Q \in \alpha(\mathcal{S})$. It cannot be true that $\alpha^* E_Q$ contains a contracted divisor for all $Q \in \alpha(\mathcal{S})$, for otherwise $\alpha(M_1)$ would be of complex dimension 1. Hence for a generic choice of C_a corresponding to a generic choice of Q , $\alpha^* E_Q$ does not contain a contracted curve. We choose such a Q and the associated C_a . The rest of the arguments in [Ye2] as well as §4.5 of [Ye3] can then be applied to conclude the proof of the statements. This is the third modification required.

Hence the spectral mapping $\pi : M_1 \rightarrow M$ is unramified. However, unless \overline{W}_1 is trivial, this will contradict topological consideration arising from the action of \overline{W}_1 on M_1 as well as $\alpha(M_1)$ induced from the action of the affine Weyl group on an apartment of X . We refer the readers to §4.6 of [Ye3] for details of the argument. Hence \overline{W}_1 is trivial and therefore $M = M_1$. In such case, $h^{1,0} \geq 2$ as $\alpha(M) = \alpha(M_1)$ is of complex dimension 2, our working assumption. Hence the case of $h^{1,0} = 1$ is eliminated. The only remaining case is $h^{1,0} = 2$. Here we are going to utilize more the structure of the Bruhat-Tits building involved (cf. [Br]). Since \overline{W}_1 is trivial, $\rho(\Gamma)$ can only act on each fixed apartment Σ by translation. We observe that the only subgroup G_1 of $G(\mathbb{Q}_p)$ which acts by translation on each apartment of the Bruhat-Tits building X is trivial. To prove the claim, we note that the stabilizer G_Σ of an apartment Σ in $G(k_v)$ is precisely the normalizer of the split torus T associated to Σ . Moreover, $g \in G_\Sigma$ acts by translation if and only if $g \in Z_G(T)$, the centralizer of T . Hence $G_1 \subset H := \cap_T Z_G(T)$, where T ranges over the set of all maximal split torus T . Now H is normalized by $G(K_p)$. Since G is simple, H is trivial. The observation is proved. It follows from the observation that $\rho_* \gamma(\tilde{f}(\Sigma_i))$ is trivial, contradict to our assumption from the very beginning of proof of Theorem. This is the fourth modification required.

The contradiction above completes the proof for Step 2A, and in particular, Proposition .

□

6. Archimedean rigidity

6.1 Recall that M is a smooth complex two ball quotient with $h^0(M, \Omega) \leq 2$.

Lemma 3. *Let $\Phi : \widetilde{M} \rightarrow \widetilde{M}^\sigma$ be the harmonic map induced by the conjugate representation ρ_σ . Then Φ is holomorphic of real rank 4.*

Proof We are going to break the proof into several stages.

6.1.1 Suppose that $\text{rank}_{\mathbb{R}}(\Phi) \geq 3$. We may apply the result of Siu [Siu] to conclude that f is holomorphic or conjugate-holomorphic. In the second case, we take the complex conjugate of the image so that f becomes holomorphic. It follows that Φ is a holomorphic mapping of complex rank 2.

Suppose that $\text{rank}_{\mathbb{R}}(\Phi) = 1$. Then $\Phi(\widetilde{M})$ is a totally geodesic curve ℓ in $N^\sigma \cong B_{\mathbb{C}}^2$ according to a result of Sampson. From the fact that Π is Zariski dense and the action at $\partial B_{\mathbb{C}}^2$ does not have fixed points, we conclude that it cannot happen that $\text{rank}_{\mathbb{R}}(\Phi) = 1$.

Hence it suffices to rule out the case that $\text{rank}_{\mathbb{R}}(\Phi) = 2$.

6.1.2 Our first step is to show that a generic fiber of Φ is a complex curve on \widetilde{M} . To see this, there exists a Π -equivariant holomorphic map $\Phi : \widetilde{M} \rightarrow \widetilde{M}$ obtained as follows. Lemma 4.5 of [Sim] implies that ρ_σ has to come from a complex variation of Hodge structure, which means that ρ_σ induces an equivariant holomorphic mapping $\Psi : \widetilde{M} \rightarrow \widetilde{S}$ such that $\Phi = p \circ \Psi$, where \widetilde{S} is a Griffith's Period Domain over $N^\sigma = B_{\mathbb{C}}^2$ and $p : \widetilde{S} \rightarrow N^\sigma$ is the projection map. The only choices of \widetilde{S} above $B_{\mathbb{C}}^2 \cong PU(1, 2)/P(U(1) \times U(2))$ are $PU(1, 2)/P(U(1) \times U(2))$ or $PU(1, 2)/P(U(1) \times U(1) \times U(1))$. In the second case, the image of $\Phi(\widetilde{M})$ lies in a horizontal slice of $U(1, 2)/U(1) \times U(1) \times U(1)$. Since the only horizontal slice in $PU(1, 2)/P(U(1) \times U(1) \times U(1))$ is $PU(1, 2)/P(U(1) \times U(2))$, we conclude that

$$\Psi(\widetilde{M}) \subset PU(1, 2)/P(U(1) \times U(2)) \subset PU(1, 2)/P(U(1) \times U(1) \times U(1)).$$

Hence in either case, the fiber of Ψ and hence of Φ is a complex curve on \widetilde{M} .

The fibration given by Φ or $\Psi : \mathcal{M} \rightarrow \mathcal{C} := \Psi(\widetilde{M})$ is equivariant with respect to the action of Γ . When projected to $M = \widetilde{M}/\Gamma$, the fibers given by Ψ gives rise to a holomorphic foliation on M . The idea of our proof is to show that such a foliation leads to a fibration under our assumption of complex two ball quotient with $c_2 = 3$ and $h^1(M) \leq 2$. In the following we would just regards Ψ as Φ .

6.1.3 First of all, we consider the special case that the foliation on M is a fibration over an algebraic curve of genus at least 2. Applying (6), we conclude that $e(M) \geq 4$, which contradicts our assumption that $c_2(M) = 3$.

6.1.4 Observe now that if a fiber η of ψ has multiplicity more than 1, the image of the fiber η must be a closed curve in M . In fact, if the image of η is not a closed curve on M , it would have some limit points in M along some local transverse cross section to the foliation. This implies that Ψ has multiplicity greater than 1 for a generic fiber of Ψ , a contradiction. Hence multiple fibers correspond to compact invariant curves of the vector field on M , and there are at most a finite number of those leaves, denoted by E_1, \dots, E_k .

The foliation is defined locally by $d\Phi := \Phi^*dw = 0$, where w is a local holomorphic coordinates at a point on \widetilde{M} . The expression $d\Phi$ is non-degenerate on $M - \cup_{i=1}^k E_i$ apart from a finite number of points. The foliation on $M - \cup_{i=1}^k E_i$ is non-degenerate generically and extends naturally across $\cup_{i=1}^k E_i$ to give a foliation \mathcal{F} with a finite number of singularities on M . This is essentially the same as the foliation obtained from $d\Phi$ but neglecting the multiplicities of the closed curves $\cup_{i=1}^k E_i$. In other words, we consider the saturation of a foliation if it is defined locally by holomorphic one forms or vector fields as described in page 11 of [Bru2]. Hence we denote by $\text{Sing}(\mathcal{F})$ the singularity set of \mathcal{F} , a discrete set of points.

6.1.5 We refer for example to [B] for standard notations about holomorphic foliations on a complex surface. There is a short exact sequence associated to the foliation \mathcal{F} , cf. pages 10-11 of [Bru1].

$$(7) \quad 0 \rightarrow T_{\mathcal{F}} \rightarrow \mathcal{T}_M \rightarrow \mathcal{I}_Z N_{\mathcal{F}} \rightarrow 0,$$

which is dual to

$$(8) \quad 0 \rightarrow N_{\mathcal{F}}^* \rightarrow \Omega_M \rightarrow \mathcal{I}_Z T_{\mathcal{F}}^* \rightarrow 0,$$

after tensoring with K_M , where $N_{\mathcal{F}}$ is the normal bundle to the foliation, and \mathcal{I}_Z is the ideal sheaf with support on the singularity of the foliation.

The associated long exact sequence reads,

$$\begin{array}{ccccccc} 0 & \rightarrow & H^0(M, N_{\mathcal{F}}^*) & \xrightarrow{\iota} & H^0(M, \Omega_M) & \xrightarrow{\alpha} & H^0(M, \mathcal{I}_Z T_{\mathcal{F}}^*) \\ & & \xrightarrow{\beta} & H^1(M, N_{\mathcal{F}}^*) & \xrightarrow{\gamma} & H^1(M, \Omega_M) & \xrightarrow{\delta} & H^1(M, \mathcal{I}_Z T_{\mathcal{F}}^*) \\ & & \xrightarrow{\epsilon} & H^2(M, N_{\mathcal{F}}^*) & \xrightarrow{\sigma} & H^2(M, \Omega_M) & \xrightarrow{\tau} & H^2(M, \mathcal{I}_Z T_{\mathcal{F}}^*) \rightarrow 0. \end{array}$$

From the first line of the long exact sequence above, as $h^0(M, \Omega) \leq 2$, we know that $h^0(M, N_{\mathcal{F}}^*)$ takes the value of either 0, 1 or 2. For $h^0(M, N_{\mathcal{F}}^*) = 2$, the Castelnuovo-de Franchi argument (cf. [BHPV]) with two independent elements in $\iota(H^0(M, N_{\mathcal{F}}^*) \subset H^0(M, \Omega_M))$ leads to the conclusion that foliation comes from a holomorphic fibration over a curve of genus at least 2 and hence leads to a contradiction as discussed in **6.1.4**.

6.1.6. Suppose $h^0(M, N_{\mathcal{F}}^*) = 1$. Let $\omega \in H^0(M, N_{\mathcal{F}}^*)$. Denote by the same symbol its pull-back to \widetilde{M} . From definition, ω annihilates the tangent vectors to fibers of Φ on \widetilde{M} . We claim that any leaf of \mathcal{F} must be compact. Suppose on the contrary that a leaf \mathcal{L} of \mathcal{F} is not compact on M . Denote by $\widetilde{\mathcal{L}}$ a lift of \mathcal{L} in \widetilde{M} . $\widetilde{\mathcal{L}}$ is the fiber of $\Phi : \widetilde{M} \rightarrow C$ studied earlier. Suppose $\Phi(\widetilde{\mathcal{L}}) = o \in C$. Let V be a small coordinate neighborhood of o in C and z be a coordinate function on V . Then $\Phi^{-1}(V)$ is a neighborhood of \mathcal{L} . As both ω and Φ^*dz annihilate tangent vectors to \mathcal{L} on \widetilde{M} , we know that in a neighborhood of \mathcal{L} , $\omega = f\Phi^*dz$ for some function f which is holomorphic along \mathcal{L} . Taking exterior derivative

$$0 = d\omega = df \wedge \Phi^*dz + f d\Phi^*dz = df \wedge \Phi^*dz.$$

It follows that $df(v) = 0$ for all tangent vectors $v \in T\mathcal{L}$. Hence f is constant along \mathcal{L} . In other words, we may regard $\omega = \Phi^*dz$ along \mathcal{L} . Note that the same argument shows that $\omega_{\mathcal{L}} = \Phi^*\eta|_{\mathcal{L}}$ for any local holomorphic one form η on V as long as $\Phi^*\eta|_{\mathcal{L}} = \Phi^*dz$ on \mathcal{L} . In particular, this holds for $\eta = h(z)dz$, where h is a holomorphic function on V with $h(o) = 1$. Let W be a small coordinate neighborhood of \mathcal{M} which has non-empty intersection with U . Since L is dense on M , we may assume that the image $\pi(U \cap W)$ contains infinite number of disconnected

pieces $\mathcal{L}_i, i \in \mathbb{N}$ of $\mathcal{L} \cap \pi(U \cap W)$, by taking a smaller W if necessary. From the above discussions, $\omega|_{\mathcal{L}_i} = \Phi^* dz|_{\mathcal{L}_i}$ for each i , after pulling back to $U \cap W \subset \widetilde{M}$ and identifying $\Phi^{-1}(\mathcal{L}_1)$ with \mathcal{L} on W . As remarked above, the same argument shows that $\omega|_{\mathcal{L}_i} = \Phi^*(h(z)dz)|_{\mathcal{L}_i}$. Since we may choose $h(z)$ so that $h(z) \neq 1$ on $\Phi^{-1}(\mathcal{L}_2)$, we immediately reach a contradiction. Hence all leaves of \mathcal{F} on M are compact. It follows that Φ induces a fibration $\beta : M \rightarrow R$ to an algebraic curve R of genus at least 1, as in the proof of the classical Castelnuovo-de Franchi Theorem as mentioned above, see also Proposition 6.2 of [Bru2]. The fibration lifts to a fibration $\widetilde{\beta} : \widetilde{M} \rightarrow R$. Since the fibration in this case is actually the original one induced from Φ , which is Γ -equivariant, it follows that there is an induced covering map $\gamma : C = \Phi(\widetilde{M}) \rightarrow R$. As Γ acts freely on \widetilde{M} , $\Phi_*\Gamma$ acts faithfully on C as well. We conclude that the Poincaré metric of N restricted to C descends to R . Hence R is hyperbolic and has to be of genus at least 2. This leads to a contradiction as in **6.1.3** again.

6.1.7. Hence we assume that $h^0(M, N_{\mathcal{F}}^*) = 0$. We know that $h^0(M, \Omega_M)$ can be 0, 1 or 2. In the case that $h^0(M, \Omega_M) = 0$, it corresponds to the fake projective plane situation with $h^{1,1} = 1$. Since the pull-back $\Phi^*\omega_{\widetilde{M}^\sigma}$ of the standard Kähler form $\omega_{\widetilde{M}^\sigma}$ on \widetilde{M}^σ is Γ equivariant and descends to M , it has to be a non-trivial multiple of ω_M , the standard Kähler form on M . This however contradicts the fact that $\Phi^*\omega_{\widetilde{M}^\sigma} \wedge \Phi^*\omega_{\widetilde{M}^\sigma} = 0$ as the image of Φ has complex dimension 1.

Suppose now that $h^0(M, \Omega_M) = 2$. From the long exact sequence as above, we know that $h^0(M, \mathcal{I}_Z T_{\mathcal{F}}^*) - \dim \text{Im}(\beta) = 2$. Hence we may find two linearly independent elements $\omega_1, \omega_2 \in H^0(M, \Omega_M) \cong \text{Im}(\alpha) \subset H^0(M, \mathcal{I}_Z T_{\mathcal{F}}^*)$. As $T_{\mathcal{F}}^*$ is one dimensional, we may write $\omega_1 = f\omega_2$. Clearly $\omega_1 \wedge \omega_2 = 0$. Castelnuovo-de Franchi Theorem implies that there is a fibration of M over a curve R of genus at least 2 with generic fiber still denoted by M_s . In this case, the Euler number formula (6) still holds with $\chi_{top}(R) \geq 2$ and $\chi_{top}(M_s) \geq 2$, which leads to contradiction as in **6.1.3**.

6.1.8. Hence we are left with $h^0(M, \Omega_M) = 1, h^0(M, N_{\mathcal{F}}^*) = 0$ and hence $h^0(M, \mathcal{I}_Z T_{\mathcal{F}}^*) - \dim(\text{Im}(\beta)) = 1$. In this case, there is a non-trivial Albanese map $\text{alb} : M \rightarrow E$, an elliptic curve generated by the holomorphic one form $\theta \in H^0(M, \Omega_M)$. As θ can be considered as an element in $H^0(M, \mathcal{I}_Z T_{\mathcal{F}}^*)$, the fibers of α are actually transversal to the leaves of \mathcal{F} generically. Applying (6) to our Albanese fibration, we know that the contribution of the singular set Z_{alb} of alb is given by $\sum_{i \in Z_{\text{alb}}} \delta_{\text{alb}, i} = 3$. Since θ can be considered to be living in $H^0(M, \mathcal{I}_Z T_{\mathcal{F}}^*)$, we know that the singularity set of the foliation $Z \subset Z_{\text{alb}}$.

We now consider the singularities of \mathcal{F} . Recall an invariant of foliation introduced by Baum-Bott in [BB], see also [Bru1] for details in the following setting. In a neighborhood of a singular point $p \in \text{Sing}(\mathcal{F})$, we may suppose that \mathcal{F} is generated by a vector field v given in local coordinates by $v(z, w) = F(z, w) \frac{\partial}{\partial z} + G(z, w) \frac{\partial}{\partial w}$. Then

$$\text{Det}(p, \mathcal{F}) = \text{Res}_0 \left\{ \frac{\det J(z, w)}{F(z, w)G(z, w)} dz \wedge dw \right\},$$

is a non-negative integer. Define

$$\text{Det}(\mathcal{F}) = \sum_{p \in \text{Sing}(\mathcal{F})} \text{Det}(p, \mathcal{F}),$$

From [BB], see also Proposition 1 of [Bru1], we know that

$$(9) \quad \text{Det}(\mathcal{F}) = c_2(M) - c_1(T_{\mathcal{F}}) \cdot c_1(M) + c_1^2(T_{\mathcal{F}})$$

It follows from the equation that

$$(10) \quad c_1(T_{\mathcal{F}}) \cdot c_1(N_{\mathcal{F}}) = c_2(M) - \text{Det}(\mathcal{F}).$$

Note that apart from the finite number of singular points, the normal bundle $N_{\mathcal{F}}$ can locally be represented by the lift of the tangent vectors to the base of the fibration $\psi : \widetilde{M} \rightarrow C$. From Riemann-Roch formula for $T_{\mathcal{F}}$ on M , it follows that $c_1(T_{\mathcal{F}}) \cdot c_1(N_{\mathcal{F}}) = c_1(T_{\mathcal{F}}) \cdot (c_1(T_{\mathcal{F}}) - c_1(T_M))$ is an even integer. Hence as $c_2(M) = 3$ and $\text{Det}(\mathcal{F}) \geq 0$, identity (9) implies that $c_1(T_{\mathcal{F}}) \cdot c_1(N_{\mathcal{F}}) = 2$ and $\text{Det}(\mathcal{F}) = 1$, the latter implies that there is only one singularity Q for \mathcal{F} , with Milnor number 1.

From the earlier discussions, we conclude that $Q \in Z_{\text{alb}}$. As $\sum_{i \in Z_{\text{alb}}} \delta_{\text{alb}, i} = 3$ and the singularity of Q can only contribute 1 to the sum, we conclude that there is at least one more point, say $Q' \in Z_{\text{alb}} \setminus Z_{\mathcal{F}}$. Since by construction, the foliation is smooth around Q , we may choose a good local coordinate system $(x, y) \in W$ centered at Q' so that leaves of the foliation are given by $y = c$, a small constant. In such a coordinate, we may write $\theta = f dx$ for some local holomorphic function f on W . As θ vanishes at Q' , it follows that f has a non-trivial zero divisor passing through Q' on W . Since θ is the pull-back of a one form on the elliptic curve E by the Albanese map $\text{alb} : M \rightarrow E$, this is possible only if alb contains a singular fiber. In other words, θ contains a multiple fiber.

Recall now the expression (6) with respect to the Albanese fibration. Consider first the contributions from multiple fibers. For a multiple fiber s_o , the expression $n_{s_o} > 0$ unless the reduced M_{s_o} is an elliptic curve, which is not possible as M is hyperbolic. In such a case, the fact that we have a small Euler number 3 implies that a multiple fiber M_{s_o} has just one reduced irreducible component D_{s_o} and we may write $M_{s_o} = \alpha_{s_o} D_{s_o}$ for some integer $\alpha_{s_o} \geq 2$, cf. Corollary 2 in 5.3 of [CKY]. Let g be the genus of a generic fiber M_s of alb . Then $n_{s_o} = 2(g(M_{s_o}) - g(M_s))$ and $2(g(s) - 1) = 2\alpha_{s_o}(g(D_{s_o}) - 1)$. It follows that $n_{s_o} = 2(\alpha_{s_o} - 1)(g(D_{s_o}) - 1)$. Note that $g(D_{s_o}) \geq 2$ and $\alpha_{s_o} \geq 2$, we conclude from (7) that there is precisely one multiple fiber M_{s_o} , and furthermore, $g(D_{s_o}) = 2$ and $\alpha_{s_o} = 2$. There is also the contribution of Q to the formula (7). The point Q cannot lie on M_{s_o} , for otherwise, M_{s_o} would not be hyperbolic.

It follows from our setting that θ/ξ is precisely the element in $H^0(M, \mathcal{I}_Z T_{\mathcal{F}}^*)$, where ξ is the canonical section of D_{s_o} . Hence $T_{\mathcal{F}}^* = \pi^* \Omega_E + D_{s_o}$. Hence

$$(11) \quad \begin{aligned} 0 &= D_{s_o} \cdot D_{s_o} \\ &= c_1(T_{\mathcal{F}}^*) \cdot c_1(T_{\mathcal{F}}^*) - 2c_1(T_{\mathcal{F}}^*) \cdot \pi^* c_1(\Omega_E) + \pi^* c_1(\Omega_E) \cdot \pi^* c_1(\Omega_E) \\ &= c_1(T_{\mathcal{F}}^*) \cdot c_1(T_{\mathcal{F}}^*), \end{aligned}$$

where we used the fact that $c_1(\Omega_E) = 0$.

Now we observe that the foliation is defined on \mathcal{M} by the fiber of Φ . Hence the Chern form $C_1(N_{\mathcal{F}}^*)$ of $N_{\mathcal{F}}^*$ on \mathcal{T} is the pull-back of a $(1, 1)$ form on C , which of

dimension 1. Hence the pointwise product $C_1(N_{\mathcal{F}}^*) \wedge C_1(N_{\mathcal{F}}^*)$ vanishes on M , which implies that $c_1(N_{\mathcal{F}}) \cdot c_1(N_{\mathcal{F}}) = c_1(N_{\mathcal{F}}^*) \cdot c_1(N_{\mathcal{F}}^*) = 0$. This implies that

$$\begin{aligned} 0 &= (c_1(T_M) - c_1(T_{\mathcal{F}})) \cdot (c_1(T_M) - c_1(T_{\mathcal{F}})) \\ &= c_1(T_M) \cdot c_1(T_M) - 2c_1(T_M) \cdot c_1(T_{\mathcal{F}}) \\ &= c_1(T_M) \cdot c_1(T_M) - 2c_1(N_{\mathcal{F}}) \cdot c_1(T_{\mathcal{F}}), \end{aligned}$$

where we have used identity in (11). This however contradicts the fact that $c_1(T_M) \cdot c_1(T_M) = 9$, an odd number. The contradiction rules out the case of $h^0(M, \Omega_M) = 1$, $h^0(M, N_{\mathcal{F}}^*) = 0$.

This concludes the proof of Lemma 3. \square

6.2

Lemma 4. *Assume that $c_2(M) = 3$ and $\Phi : \widetilde{M} \rightarrow \widetilde{M}^\sigma$ is a holomorphic map of real rank 4. Then Φ is a biholomorphism.*

Proof Denote by $C_1(\widetilde{M}^\sigma)$ and $C_2(\widetilde{M}^\sigma)$ the Chern forms associated to the Poincaré metric on \widetilde{M}^σ . First of all, we observe that $\int_M \Phi^* C_1(\widetilde{M}^\sigma) \wedge \Phi^* C_1(\widetilde{M}^\sigma)$ and $\int_M \Phi^* C_2(\widetilde{M}^\sigma)$ are positive integers on M . Note that the integrands are equivariant under Γ and hence descends to M . Clearly the first integral $\int_M \Phi^* C_1(\widetilde{M}^\sigma) \wedge \Phi^* C_1(\widetilde{M}^\sigma) \in \mathbb{Z}$, as Chern number of the pull-back line bundle. Now it is well-known that on any complex surface S with $p : P(S) \rightarrow S$ the projection map of the projectivized tangent bundle, there is pointwise identification as a differential form

$$p_* C_1(L)^3 = -(C_1^2 - C_2),$$

where L is the tautological line bundle on PS , cf. [Ye1], page 493-494. In particular, pulling back to our manifold M ,

$$\int_{P(M)} \Phi^* C_1(L)^3 = - \int_M \Phi^* (C_1^2 - C_2),$$

Now the mapping Φ induces a mapping between the corresponding projectivized tangent bundles, which implies that $\int_{PM} \Phi^* C_1(L)^3 \in \mathbb{Z}$. Using the identity above and the fact that $\int_M \Phi^* C_1^2(\widetilde{M}^\sigma) \in \mathbb{Z}$, we conclude that $\int_M \Phi^* C_2(\widetilde{M}^\sigma) \in \mathbb{Z}$ as well. Note that the Chern forms of the bundles are positive on M^σ in terms of the Poincaré metric there. Hence the integrals are positive as well. The observation is proved.

For simplicity of notation, denote $\Phi^* c_2 = \int_M \Phi^* C_2(\widetilde{M}^\sigma)$ and $\Phi^* c_1^2 = \int_M \Phi^* C_1(\widetilde{M}^\sigma) \wedge \Phi^* C_1(\widetilde{M}^\sigma)$. It follows that $\Phi^* c_2 \in \mathbb{Z}$. As the Chern forms of the Poincaré metric on M^σ satisfies $C_1^2 = 3C_2$ pointwise on the form level, we conclude that $\Phi^* c_1^2 = 3\Phi^* c_2$ and hence is a positive multiple of 3.

Assume that the mapping is not etale and hence R exists. Let $R = \sum_{i=1}^k b_i R_i$ be the ramification divisor of Φ , where R_i are the irreducible components and $b_i + 1$ is the local branching order along R_i . From Riemann-Hurwitz Formula,

$$(12) \quad c_1^2(M) = \Phi^* c_1^2(\widetilde{M}^\sigma) - 2\Phi^* c_1(\widetilde{M}^\sigma) \cdot R + R \cdot R,$$

$$(13) \quad = \Phi^* c_1^2(\widetilde{M}^\sigma) + \Phi^* K_{\widetilde{M}^\sigma} \cdot R + K_{\widetilde{M}} \cdot R$$

$$(14) \quad = \Phi^* c_1^2(\Phi(\Sigma)) + \Phi^* K_{\widetilde{M}^\sigma} \cdot \sum_{i=1}^k b_i R_i + K_{\widetilde{M}} \cdot \sum_{i=1}^k b_i R_i.$$

In the above, we have used $K_{\widetilde{M}} = \Phi^* K_{\widetilde{M}^\sigma} + R$, where $K_{\widetilde{M}} = K_M$.

From equation (13), we conclude that $\Phi^* K_{\widetilde{M}^\sigma} \cdot R + K_{\widetilde{M}} \cdot R = 6$, where the first term is non-negative and the second term is positive. Note also that $\Phi^* K_{\widetilde{M}^\sigma} \cdot R = (-R + K_M) \cdot R$ is even after applying Riemann-Roch applied to R on M . Hence one of the following cases hold,

(a) $\Phi^* K_{\widetilde{M}^\sigma} \cdot R = 0, K_M \cdot R = 6$

(b) $\Phi^* K_{\widetilde{M}^\sigma} \cdot R = 2, K_M \cdot R = 4$, or

(c) $\Phi^* K_{\widetilde{M}^\sigma} \cdot R = 4, K_M \cdot R = 2$. For Case (a), R has to be contracted by Φ since $K_{\widetilde{M}^\sigma}$ is positive. This is possible only if $R \cdot R$ is negative from contraction criterion on surface, which would violate equation (12). For Case (c), $R \cdot R = K_M \cdot R - \Phi^* K_{\widetilde{M}^\sigma} \cdot R = -2$. From Adjunction formula, it follows that the genus of R is 1, which violates the fact that M is hyperbolic. Hence only (b) is possible, with $R \cdot R = 2$ as a consequence.

As $-\Phi^* c_1(\widetilde{M}^\sigma) \cdot R = 2$, it is easy to see that there can be at most two irreducible components in R . First assume that there are two irreducible components R_1 and R_2 in R . The above constraint leads to $-\Phi^* c_1(\widetilde{M}^\sigma) \cdot R_i = 1$ and $b_i 1$ for $i = 1, 2$. For each i , $R_i \cdot R_i > 0$, for otherwise adjunction formula implies that R_i has genus less than 2, contradicting the fact that M is hyperbolic. It follows that $R_i \cdot R_i = 1$ for $i = 1, 2$ and $R_1 \cdot R_2 = 0$. But this implies again from Riemann-Roch for R_1 that $-\Phi^* c_1(\widetilde{M}^\sigma) \cdot R_1 = (-c_1(M) - R_2) \cdot R_1 = -c_1(M) \cdot R_1$ is even and positive. Similarly $-c_1(M) \cdot R_2$ is even positive. This contradicts the earlier conclusion that $2 = -\Phi^* c_1(\widetilde{M}^\sigma) \cdot R = -\Phi^* c_1(\widetilde{M}^\sigma) \cdot (R_1 + R_2)$.

Hence we conclude that $R = R_1$ with only one irreducible component. From $b_1^2 R_1 \cdot R_1 = R \cdot R = 2$. We conclude that $b_1 = 1$. This implies that the local branching order of Φ around R is 2. From Adjunction Formula, we know that the arithmetic genus $\chi(\mathcal{O}_R) = 3$.

Let Σ be a fundamental domain of M in \widetilde{M} . It is known that Σ can be taken as a polyhedron and Poincaré Polyhedron Theorem holds, with a finite number of faces in the boundary $\partial\Sigma$. The faces of $\partial\Sigma$ are identified by actions of elements $\gamma_1, \dots, \gamma_k$ in Γ to give rise to a compact manifold isometric to M . Similarly, the boundary components of $\Phi(\Sigma)$ are identified by induced actions of $\gamma_1, \dots, \gamma_k$. By identifying the corresponding boundary components, we get a compact topological space which we denote by Σ^σ . From the earlier discussions, Σ is a two-fold branched cover of Σ^σ . Since any two-fold covering is a Galois covering, Σ^σ is obtained from Σ by a \mathbb{Z}_2 quotient.

As $R = R_1$ is fixed by an automorphism of M coming from the generator of the \mathbb{Z}_2 action, we conclude that R is totally geodesic. However, for a totally geodesic curve R_1 on a complex two ball quotient with possibly self-intersection, we have the formula $K_M \cdot R_1 = 3(g(\hat{R}_1) - 1)$, where \hat{R}_1 is the normalization of R_1 , cf. Lemma 6 of [CKY]. This violates our conclusion in (b) that $K_M \cdot R = 4$. The contradiction concludes our proof for Lemma 4.

In conclusion, Φ is a biholomorphism. \square

We remark that Lemma 4 was known to Domingo Toledo [T] with a different proof.

6.3 From the above lemma, we conclude that Φ is a biholomorphism. This implies that Φ gives rise to an isometry with respect to the Killing metrics on the domain and the image. Hence it corresponds to a bihomomorphism from G to G^σ . This

however leads to a contradiction, since the Galois conjugate σ is not even continuous with respect to the standard topology on \mathbb{C} . The contradiction implies that G^σ is compact and hence Γ is arithmetic. Alternately, the argument above shows that the homomorphism induced by σ is standard in the sense of Margulis [Ma], page 367, which implies that the lattice is arithmetic. In summary, we conclude the following proposition. Recall that the condition $h^1(M) \leq 2$ is automatically satisfied according to the previous sections.

Proposition 3. *Let $M = B_{\mathbb{C}}^2/\Gamma$ be a smooth compact complex two ball quotient with $e(M) = 3$. Then Γ is an arithmetic lattice.*

6.4 We can now state the uniqueness of the examples with $h^{1,0} = 1$.

Proposition 4. *Let $M = B_{\mathbb{C}}^2/\Gamma$ be an arithmetic complex two ball quotient with $e(M) = 3$ and $h^{1,0}(M) = 1$. Then M is holomorphic to the surface constructed by Cartwright and Steger mentioned in §4.*

Proof The classification of Prasad and Yeung [PY] covers all arithmetic complex two ball quotients of $e(M) = 3$. In particular, the set of all arithmetic lattices of $e(M) = 3$ consists of 28 classes with $[\mathcal{D}, \ell] > 1$, each of which contains fake projective planes, and five more classes, $\mathcal{C}_1, \mathcal{C}_8, \mathcal{C}_{11}, \mathcal{C}_{18}$, or \mathcal{C}_{21} with $\mathcal{D} = \ell$ in the notation of [PY] that may contain examples. Finally in [CS], Cartwright and Steger show that there are precisely 100 fake projective planes within the first twenty-eight classes, and there exists precisely one arithmetic lattice Γ with $e(M) = 3$ in the remaining five classes above, lying within the class with number field given by \mathcal{C}_{11} and has $h^{1,0}(B_{\mathbb{C}}^2/\Gamma) = 1$. We conclude that the fundamental group of a torsion free ball quotient M with $e(M) = 3$ and $h^{1,0}(M) = 1$ has to be isomorphic to the example of Cartwright and Steger mentioned in §4.

Let us first restrict the possibility of the fixed point set for a hypothetical complex conjugate diffeomorphism h on the Cartwright-Steger surface. In such case, it is known from the work of Cartwright Steger that the automorphism group of the surface has order 3. Suppose on the contrary that there is a complex conjugate diffeomorphism h on the surface. It follows that either h or h^3 is a conjugate involution, which has a totally real manifold as the fixed point set F . From (4), $h^{1,1} = 3$. Hence it follows from Lefschetz Fixed Point Formula that F contains a component which is one of the followings, sphere S^2 , real projective plane $P_{\mathbb{R}}^2$, torus T^2 or $P_{\mathbb{R}}\#T^2$. Neither of the first two cases is possible from the proof in Lemma 4. In fact, the lift \tilde{F} of either set to the universal covering $\tilde{M} \cong B_{\mathbb{C}}^2 \subset \mathbb{C}^2$ has to be compact. The square of the Euclidean distance function with respect to the origin on \tilde{M} , $d_E(0, z)^2 = |z_1|^2 + |z_2|^2$, is convex with respect to the Killing metric and hence the restriction of r^2 to \tilde{F} is constant by the Maximum Principle. However as $B_{\mathbb{C}}^2$ is homogeneous, we may identify the origin 0 of $B_{\mathbb{C}}^2$ as an arbitrary point on \tilde{M} and repeat the same argument to conclude that \tilde{F} is a point, a contradiction. For the case of T^2 , by considering a harmonic map from T^2 to M induced by the immersion, the same argument together with Pressman's Theorem (cf. [CE]) leads to a contradiction as well. Hence the possibility is $P_{\mathbb{R}}\#T^2$.

In fact, from [BY], since M can be defined over \mathbb{Q} and the complex conjugate of M is biholomorphic to M , see Remark 5.1 of [BY]. Hence it follows that there is

exactly one Cartwright-Steger surface up to biholomorphism. This concludes the proof of the proposition.

6.5

Proof of Theorem 1 This follows by combining the results of Proposition 1, Proposition 3 and Proposition 4.

7. Modifications comparing to [1]

7.1 In the following, we list some main changes in this paper comparing to [1].

(a) The original scheme of proof for irregularity 2 in §4 of [1] was incorrect; in particular, Lemma 2 is not correct. In this corrected version, §4 of [1] was removed, and arguments in §6 of [1] is modified to cover the case of irregularity 2. §6 of [1] is replaced by §5 and §6 in this version. The main modification in the proof of Integrality of Γ is given at the end of §5. The modification in the proof of super rigidity is given in §7 with analysis of some associated foliations.

(b) Lemma 1 of [1] was incorrectly stated and should be discarded. It is not needed in the revised argument above.

(c) Reference [Mi] is added in the section corresponding to **2.1** of [1] which proves ampleness of K_M in case that $c_1^2(M) = 3c_2(M)$.

(d) Details are added to §6 to explain that the complex conjugate of M cannot be isomorphic to M . It was taken from [Ye4].

7.2 We also remark that Proposition 2 is now a special case of a general result of [EG].

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