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EFFECTIVE ŁOJASIEWICZ INEQUALITY FOR ARITHMETICALLY DEFINED VARIETIES AND A GEOMETRIC APPLICATION TO BIHOMOGENEOUS POLYNOMIALS

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We establish two versions of effective Łojasiewicz inequality for arithmetically defined affine varieties. As an application, we consider bihomogeneous polynomials on the complex Euclidean space which are positive along the affine cone of an arithmetically defined projective variety, and we obtain effective estimates on certain modifications needed to turn them into sums of squares of pointwise norms of homogeneous polynomials. The latter can be interpreted as an effective result on isometric embeddings for the associated indefinite Hermitian holomorphic line bundles.

Keywords: Łojasiewicz inequality; arithmetically defined varieties; bihomogeneous polynomials.

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1. Introduction

In the study of complex algebraic varieties in $\mathbb{C}^n$, the Łojasiewicz inequality plays an important role, and it has been widely studied (see e.g. [2], [3], [6] and [9]; see also [10], [13], [14] for corresponding results in the real case). Roughly speaking, the Łojasiewicz inequality estimates the Euclidean distance of a point of $\mathbb{C}^n$ to a given variety M in terms of the value at this point of a set of defining polynomials $\{f_1, \cdots, f_r\}$ of M. More precisely, one has an inequality of the form

$$\Big(\frac{\mathrm{dist}\,(z, M)}{1 + \|z\|}\Big)^{\alpha} \leq C \cdot \max_{1 \leq i \leq r} |f_i(z)|. \tag{1.1}$$

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for some positive constants α and C independent of $z \in \mathbb{C}^n$ (cf. e.g. [9], p. 814, for equivalent variants of the inequality). Effective estimates on the exponent α were given by Brownawell [2], Ji-Kollár-Shiffman [9] and Cygan-Krasiński-Tworzewski [6], while there appears to be no known effective results in literature on estimating the constant C. Kollár has raised the problem of finding effective estimates on the constant C, at least for arithmetically defined varieties.

Our main results in this paper give two versions of the Łojasiewicz inequality for affine varieties defined by polynomials over a number field with effective estimates on both α and C (cf. Theorem 2.1 and Theorem 2.3). In particular, the constant C is expressed in terms of the cardinality, maximum degree, and height of a given set of defining polynomials of the variety. In Theorem 2.1, the right hand side is replaced by a number of defining polynomials of M depending intrinsically on the geometry of M in $\mathbb{C}^n$. Theorem 2.1 has a direct application to an isometric embedding problem for projective varieties and an analogue of Hilbert Seventeenth Problem, see below. From Theorem 2.1, a version of the Łojasiewicz inequality of the same form as (1.1) with effective constant C is obtained in Theorem 2.3. The constant C in Theorem 2.3 is not as good as the one in Theorem 2.1, but Theorem 2.3 has the advantage of being more applicable in terms of an explicitly given set of polynomials.

Our main result is motivated in part by a recent work of the authors [17] on real-valued bihomogeneous polynomials f which are positive on $\mathbb{C}^n$ or an affine hypersurface V in $\mathbb{C}^n$ (outside the origin). Results in this direction can be considered as analogue for varieties in $\mathbb{C}^n$ instead of Euclidean spaces of the Hilbert Seventeenth Problem, which seeks to represent a positive definite form in the Euclidean space as sums of squares of rational functions. In [17], the authors gave effective estimates on the exponent ℓ so that the values of $\|z\|^{2\ell} f$ on V will coincide with those of a sum of squares of pointwise norms of homogenous polynomials (see also [7] and [16] for some related results). As an application of our Łojasiewicz inequality in Theorem 2.1, we give a generalization of the above result to affine varieties defined by homogeneous polynomials over a number field (cf. Theorem 2.2 for the precise statement). The latter can be interpreted geometrically as an effective result on isometric embeddings for the associated indefinite Hermitian holomorphic line bundles (cf. Remark 2.2).

The starting point of the proof of our effective Łojasiewicz inequality in Theorem 2.1 is a projection argument of Ji-Kollár-Shiffman [9]. But unlike [9], we will also need to give effective estimates on the "sizes" of the coefficients of the polynomials involved at various stages. This is made possible by using some results on geometric properties of heights of polynomials and varieties as developed in the works of Bost-Gillet-Soulé [1], Philippon [15] and Krick-Pardo-Sombra [11].

At present we do not know whether our approach can be modified to get an effective Łojasiewicz inequality for arbitrary affine varieties of codimension ≥ 2 in $\mathbb{C}^n$ (i.e., without the assumption on arithmeticity). The main difficulty appears to be a lack of natural effective measure of the complexity of the coefficients of the polynomials involved (as provided by the heights of polynomials in the arithmetic case).

The applications of expressing twisted forms in terms of squares in Theorem 2.2 is a consequence of Theorem 2.1 and geometric arguments generalizing some results of [17]. The effective version of Łojasiewicz inequality in the form of Theorem 2.3 follows from Theorem 2.1, an effective Nullstellensatz statement in [11], and a trick of Rabinowitsch as explained in [4].

This paper is organized as follows. In Section 2, we introduce some definitions and state our main results. In Section 3, we give some background results on heights of polynomials. In Section 4, we establish our main effective Łojasiewicz inequality for affine varieties defined by polynomials over a number field as stated in Theorem 2.1. The deduction of Theorem 2.2 is given in Section 5, and the proof of Theorem 2.3 is given in Section 6 with some of the technical details given in the Appendix in Section 7.

2. Notation and Statement of Results

2.1. *First form of effective Łojasiewicz inequality*

In this paper, we consider affine varieties in $\mathbb{C}^n$ defined by polynomials over a number field. For simplicity, we just call them arithmetically defined affine varieties.

Let $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$. For a finite number of polynomials $f_1, \cdots, f_r \in \overline{\mathbb{Q}}[z_1, z_2, \cdots, z_n]$, we recall that their affine height is defined as the absolute logarithmic height of the tuple formed by 1 and all the coefficients of the f_i's. (see Section 3 for the precise definition; see also [8], p. 224-225, or [11], Section 1.1). For positive integers n, r, δ and real number $\lambda > 0$, we denote by $\mathcal{M}_{\overline{\mathbb{Q}}}(n, r, \delta, \lambda)$ the class of affine varieties in $\mathbb{C}^n$ of pure dimension, which are given as the common zero set of not more than r polynomials in $\overline{\mathbb{Q}}[z_1, z_2, \cdots, z_n]$ with degrees at most δ and whose affine height is at most λ, i.e.,

$$\begin{aligned}\mathcal{M}_{\overline{\mathbb{Q}}}(n, r, \delta, \lambda) = \big\{M \subset \mathbb{C}^n \,\big|\, M = Z(f_1, \cdots, f_r) \quad \exists f_i \in \overline{\mathbb{Q}}[z_1, z_2, \cdots, z_n] \\ \text{with } \deg(f_i) \le \delta,\ i = 1, \cdots, r,\ h^A(f_1, \cdots, f_r) \le \lambda, \\ \text{and } M \text{ is of pure dimension}\big\}. \end{aligned} \tag{2.1}$$

Here $Z(f_1, \cdots, f_r)$ denotes the common zero set of $f_1, \cdots, f_r$ with possibly some f_i being the zero polynomial, and $\deg(f_i)$ denotes the degree of f_i. We remark that M may be non-smooth or reducible. For $z \in \mathbb{C}^n$ and $M \subset \mathbb{C}^n$, we denote the Euclidean distance of z from M by

$$\operatorname{dist}(z, M) := \inf_{y \in M} \|z - y\|. \tag{2.2}$$

Here $\| \ \|$ denotes the Euclidean norm on $\mathbb{C}^n$. Our main result is the following

Theorem 2.1. *For positive integers n, r, δ and real number $\lambda > 0$, let $M \in \mathcal{M}_{\overline{\mathbb{Q}}}(n, r, \delta, \lambda)$ be an arithmetically defined affine variety.*
(i) Let $C = C(n, r, \delta, \lambda)$ be given by

$$C := \delta^{2r}\big(2n\lambda + n^2 3^{n+1} \log n + rn^2 3^{n+2} \log \delta\big). \tag{2.3}$$

Then there exists a finite number of nonzero polynomials $\{q_j\}_{1\le j\le j_o}$ *in* $\overline{\mathbb{Q}}[z_1,\cdots,z_n]$ *with* $j_o \le (\delta^r+1)^{n-1}$ *such that each* q_j *is of degree at most* δ^r *and vanishes on* M*,*

$$h^A(q_j) \le C, \quad j=1,\cdots,j_o, \quad \textit{and} \tag{2.4}$$

$$\textit{dist}\,(z,M)^{\delta^r} \le \max_{1\le j\le j_o} |q_j(z)|\cdot(\|z\|+1)^{\delta^r-\deg(q_j)} \quad \textit{for any } z\in\mathbb{C}^n. \tag{2.5}$$

(ii) Moreover, if $0\in M$*, then the above conclusions remain valid with the constant* C *in (2.3) and (2.4) replaced by the smaller constant* $C' = C'(n,r,\delta,\lambda)$ *given by*

$$C' := \delta^r\big(2n\lambda + n^2 3^{n+1}\log n + rn^2 3^{n+2}\log\delta\big) \tag{2.6}$$

and with the expression $\|z\|+1$ *in (2.5) replaced by* $\|z\|$*.*

Remark 2.1. (i) The constant C in (2.3) grows at most simply exponentially in the parameter n as $n\to\infty$, i.e., $\log C$ is at most of polynomial growth in n (when the other parameters are kept constant). Actually, one easily checks that $\log C = o(n^{1+\epsilon})$ for any fixed $\epsilon > 0$. In a similar fashion, it is easy to see that C also grows at most simply exponentially in the parameter r, and C grows at most polynomially in δ. Most importantly, we note that C grows linearly in the parameter λ. It would be interesting to see whether Theorem 2.1 can be improved with substantially slower growth orders of C. We also remark that we have settled for a relatively simple expression for the constant C as given in (2.3) rather than a possibly much more complicated expression of marginally smaller value, as our main interest lies in its order of magnitude.
(ii) Let d denote the degree of the affine variety M in Theorem 2.1. Then it follows from Bézout's theorem that one always has $d\le\delta^r$. It will be clear from the proof of Theorem 2.1 that the constant C can be stated alternately (and at the expense of a more complicated expression) to indicate also its dependence on d and $\kappa :=$ pure $\dim_{\mathbb{C}} M$.
(iii) When $d=1$ (i.e., when M is an affine subspace of $\mathbb{C}^n$) or $\kappa = n-1$ (i.e., when M is an affine hypersurface of $\mathbb{C}^n$), it will be clear that one can simplify the proof of Theorem 2.1 substantially and the constant C in (2.3) can take a smaller value. Thus, in the proof of Theorem 2.1, we will always assume without loss of generality that $d\ge 2$ (and thus $\delta\ge 2$) and $\kappa\le n-2$.
(iv) Theorem 2.1(ii) holds for the important case when M is defined by a finite number of homogeneous polynomials.

2.2. *An application to bihomogeneous polynomials*

Next we consider bihomogeneous polynomials which are positive on an arithmetically defined variety in $\mathbb{C}^n$. For $m\ge 0$ and $n\ge 1$, we denote by $H_m(\mathbb{C}^n)$ the complex vector space of homogeneous holomorphic polynomials on $\mathbb{C}^n$ of degree m. As in [5], we denote by $BH_m(\mathbb{C}^n)$ the real vector space of real-valued bihomogeneous polynomials on $\mathbb{C}^n$ of degree m, i.e.,

$$BH_m(\mathbb{C}^n) = \Big\{ \sum_{|I|=|J|=m} a_{I,J} z^I \bar z^J \,\Big|\, a_{I,J} = \overline{a_{J,I}} \quad \text{for all } I,J\Big\}. \tag{2.7}$$

Here $z^I = z_1^{i_1} \cdots z_n^{i_n}$ and $|I| = i_1 + \cdots + i_n$ for $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$ and $I = (i_1, \cdots, i_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0$ denotes the set of non-negative integers. For an affine variety $M \neq \{0\}$ in $\mathbb{C}^n$ defined by a finite set of homogeneous polynomials, we let $P_m(\mathbb{C}^n, M)$ (resp. $\Sigma_m(\mathbb{C}^n, M)$) be the set of bihomogeneous polynomials of degree m which are positive on $M \setminus \{0\}$ (resp. whose restriction to M is equal to that of a sum of squares of (pointwise) norms of polynomials in $H_m(\mathbb{C}^n)$). More precisely,

$$\begin{aligned} P_m(\mathbb{C}^n, M) &:= \{f \in BH_m(\mathbb{C}^n) \,\big|\, f(z) > 0 \quad \text{for all } z \in M \setminus \{0\}\}, \text{ and} \\ \Sigma_m(\mathbb{C}^n, M) &:= \{f \in BH_m(\mathbb{C}^n) \,\big|\, f\big|_M = \sum_{i=1}^k |g_i|^2\big|_M \\ &\qquad \text{for some } g_i \in H_m(\mathbb{C}^n), 1 \le i \le k\}. \end{aligned} \tag{2.8}$$

We also denote

$$P_m(\mathbb{C}^n) := P_m(\mathbb{C}^n, \mathbb{C}^n) \quad \text{and} \quad \Sigma_m(\mathbb{C}^n) := \Sigma_m(\mathbb{C}^n, \mathbb{C}^n). \tag{2.9}$$

For any $f = \sum_{I,J \in \mathcal{I}(n,m)} a_{I,J} z^I \bar{z}^J \in P_m(\mathbb{C}^n, M)$, we let

$$\nu_M(f) := \frac{\inf\limits_{u \in M \cap S^{2n-1}} f(u)}{\max\limits_{I,J \in \mathcal{I}(n,m)} |a_{I,J}|} \in \mathbb{R}^+, \tag{2.10}$$

where $\mathcal{I}(n,m) := \{I \in \mathbb{N}_0^n \,\big|\, |I| = m\}$ and $S^{2n-1} := \{z \in \mathbb{C}^n \,\big|\, \|z\| = 1\}$. One easily checks that $\nu_M(f)$ is invariant under homothetic changes of f, and it satisfies the apriori inequality $\nu_M(f) \le \binom{n+m-1}{n-1}^2$. For positive integers n, r, δ and real number $\lambda > 0$, we denote by $\mathcal{M}^h_{\overline{\mathbb{Q}}}(n, r, \delta, \lambda)$ the subset of $\mathcal{M}_{\overline{\mathbb{Q}}}(n, r, \delta, \lambda)$ consisting of those affine varieties defined by homogeneous polynomials, i.e.,

$$\begin{aligned} \mathcal{M}^h_{\overline{\mathbb{Q}}}(n, r, \delta, \lambda) = \big\{M \subset \mathbb{C}^n \,\big|\, M = Z(f_1, \cdots, f_r) \quad \exists f_i \in \overline{\mathbb{Q}}[z_1, z_2, \cdots, z_n] \cap H_{\delta_i}(\mathbb{C}^n) \\ \text{with } \delta_i \le \delta,\ i = 1, \cdots, r,\ h^A(f_1, \cdots, f_r) \le \lambda, \\ \text{and } M \text{ is of pure dimension}\big\}. \end{aligned} \tag{2.11}$$

Our main application of Theorem 2.1 is the following

Theorem 2.2. *For positive integers n, m, r, δ and real number $\lambda > 0$, let $M \in \mathcal{M}^h_{\overline{\mathbb{Q}}}(n, r, \delta, \lambda)$ be an arithmetically defined affine variety, and let $f \in P_m(\mathbb{C}^n, M)$. Let $\mu_o = \mu_o(\nu_M(f), n, m, r, \delta, \lambda)$ be given by*

$$\begin{aligned} \mu_o := \binom{n + \delta^r - 1}{n-1}^2 \cdot e^{2C'} \cdot (\delta^r + 1)^{n-1} \cdot 2^{5\delta^r} n^{\delta^r} m^{2\delta^r} \binom{n+m-1}{n-1}^{4\delta^r} \cdot \frac{1}{\nu_M(f)^{2\delta^r}} \\ \cdot \Big[1 + 2\binom{n+m-1}{n-1}^2 \frac{1}{\nu_M(f)}\Big] + 2\binom{n+m-1}{n-1}^2 \frac{1}{\nu_M(f)}, \end{aligned} \tag{2.12}$$

where $C' = C'(n, r, \delta, \lambda) > 0$ is as in (2.6), and let $\ell_o = \ell_o(\nu_M(f), n, m, r, \delta, \lambda)$ be given by

$$\ell_o := \frac{n \max\{m, \delta^r\}(2\max\{m, \delta^r\} - 1) \cdot \mu_o}{\log 2} - n - m. \tag{2.13}$$

Then there exist a finite number of homogeneous polynomials $\{p_j\}_{1\le j\le j_o}$ *with* $j_o \le (\delta^r+1)^{n-1}$ *such that each* $p_j \in H_{d_j}(\mathbb{C}^n)$ *for some* $d_j \le \delta^r$*, each* p_j *vanishes on* M*, and for any integer* $\ell \ge \ell_o$*,*

$$\|z\|^{2\ell} f(z) + \sum_{j=1}^{j_o} |p_j(z)|^2 \|z\|^{2(m+\ell-d_j)} \in \Sigma_{m+\ell}(\mathbb{C}^n). \tag{2.14}$$

In particular, we have

$$\|z\|^{2\ell} f(z) \in \Sigma_{m+\ell}(\mathbb{C}^n, M). \tag{2.15}$$

Remark 2.2. Let $M,\ f$ be as in Theorem 2.2, and let $\mathbb{P}M \subset \mathbb{P}^{n-1}$ be the resulting projective algebraic variety upon projectivizing M. Then f corresponds to an indefinite Hermitian metric on the line bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(-m)$ whose restriction to $\mathbb{P}M$ is positive definite. We remark that Theorem 2.2 can be interpreted as an effective isometric embedding theorem for the indefinite Hermitian line bundle thus arisen. Such interpretation can easily be done and justified by following mutatis mutandis the statement and proof of [17], Corollary 4, which treated the special case when M is an affine hypersurface in $\mathbb{C}^n$. As such, we will skip the details and refer the reader to [17] for the discussion.

2.3. *Second form of effective Łojasiewicz inequality*

Combining Theorem 2.1 and the effective arithmetic Nullstellenstaz of Krick-Pardo-Sombra [11], one can obtain an effective Łojasiewicz inequality in terms of a given set of defining polynomials of the affine variety. We denote by $\overline{\mathbb{Z}}$ the integral closure of $\mathbb{Z}$. Our next main result is the following

Theorem 2.3. *For positive integers* n*,* r*,* δ *and real number* $\lambda > 0$*, let* $f_1, \cdots, f_r \in \overline{\mathbb{Z}}[z_1, \cdots, z_n]$ *be polynomials such that* $\max_{1\le i\le r} deg(f_i) \le \delta$*,* $h^A(f_1,\cdots,f_r) \le \lambda$*, and the affine variety* $M = Z(f_1,\cdots,f_r)$ *is of pure dimension. Let* $\tilde{C} = \tilde{C}(n,r,\delta,\lambda) > 0$*,* $\mu_1 = \mu_1(n,r,\delta) > 0$ *and* $\mu_2 = \mu_2(n,r,\delta) > 0$ *be given by*

$$\begin{aligned} \tilde{C} &:= (e^{2\lambda} n^{n\cdot 3^{n+1}} \delta^{rn\cdot 3^{n+2}})^{4n(n+1)(n+6)(\delta^r+1)^{n+3}}, \\ \mu_1 &:= 4(n+1)\delta^r(\delta^r+1)^{n+1}, \quad \text{and} \\ \mu_2 &:= 8(n+1)(\delta^r+1)^{n+2}. \end{aligned} \tag{2.16}$$

Then for any $z \in \mathbb{C}^n$*, one has*

$$dist(z, M)^{\mu_1} \le \tilde{C} \cdot \max_{1\le i\le r} |f_i(z)| \cdot (\|z\|+1)^{\mu_2}. \tag{2.17}$$

Remark 2.3. We remark that Theorem 2.3 can actually be applied to general affine varieties defined by polynomials in $\overline{\mathbb{Q}}[z_1,\cdots,z_n]$ in the following sense: Suppose $M = Z(g_1,\cdots,g_r)$, where the given defining polynomials g_i's are in $\overline{\mathbb{Q}}[z_1,\cdots,z_n]$. Upon multiplying each g_i by a suitable non-zero integer $c_i \in \mathbb{Z}$, one can then obtain a polynomial $f_i := c_i g_i \in \overline{\mathbb{Z}}[z_1,\cdots,z_n]$. Now one also has $M = Z(f_1,\cdots,f_r)$, and thus Theorem 2.3 can be applied to M via the new defining polynomials $f_1,\cdots,f_r$.

3. Some Background Results on Heights of Polynomials

In this section, we recall some standard definitions and results from algebraic number theory, which can be found in [8], p. 224-225, [11] and [12]. These results will be needed in Section 4 when we establish the effective Łojasiewicz inequality.

Consider a polynomial $f(z) = \sum_{I \in \mathcal{I}} a_I z^I \in \overline{\mathbb{Q}}[z_1, z_2, \cdots, z_n]$, where $\mathcal{I}$ is a finite index set, and $z^I = z_1^{i_1} \cdots z_n^{i_n}$ for $I = (i_1, \cdots, i_n) \in \mathcal{I}$. Without loss of generality, we may assume that all the a_I's lie in some number field K (for example, K can be the number field obtained by adjoining the a_I's to $\mathbb{Q}$). Let M_K be the (canonical) set of all absolute values on K whose restrictions to $\mathbb{Q}$ is either the archimedean absolute value $| \, |_\infty$ or the p-adic absolute value $| \, |_p$ for a prime p. Then the *(projective) height* $h(f)$ of f is given by

$$h(f) := \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} [K_v : \mathbb{Q}_v] \log |f|_v, \quad \text{where } |f|_v := \max_{I \in \mathcal{I}} |a_I|_v, \tag{3.1}$$

$[K : \mathbb{Q}]$ is the degree of K over $\mathbb{Q}$, and each $[K_v : \mathbb{Q}_v]$ is the degree of K_v over $\mathbb{Q}_v$ with K_v and $\mathbb{Q}_v$ denoting the completions of K and $\mathbb{Q}$ with respect to v respectively. In other words, $h(f)$ is the *absolute logarithmic height* of the point $[\cdots, a_I, \cdots]$ in $\mathbb{P}^{|\mathcal{I}|-1}(K)$ (cf. e.g. [8], p. 224-225). It follows from the product formula for M_K that $h(f)$ is invariant under homothetic change of f by a non-zero algebraic number, and that one always has $h(f) \geq 0$ for a nonzero f. Another closely related quantity is the *affine height* $h^A(f)$ of f given by

$$h^A(f) := \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} [K_v : \mathbb{Q}_v] \log \max\{1, |f|_v\}, \tag{3.2}$$

where $|f|_v$ is as in (3.1). In other words, $h^A(f)$ is the absolute logarithmic height of the point $[1, \cdots, a_I, \cdots]$ in $\mathbb{P}^{|\mathcal{I}|}(K)$. It is well-known that the values of both $h(f)$ and $h^A(f)$ remain unchanged if one replaces K by any other number field containing the a_I's. We also remark that the "height" of a non-zero algebraic number $\alpha \in \overline{\mathbb{Q}}$ is equal to $h^A(\alpha)$ when α is regarded as a monomial (notice that one always has $h(\alpha) = 0$). Finally the *affine height* of a finite number of polynomials $f_1, \cdots, f_r \in K[z_1, \cdots, z_n]$ is given by

$$h^A(f_1, \cdots, f_r) := \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} [K_v : \mathbb{Q}_v] \log \max\{1, |f_1|_v, \cdots, |f_r|_v\}. \tag{3.3}$$

Equivalently, $h^A(f_1, \cdots, f_r)$ is the absolute logarithmic height of the tuple formed by 1 and all the coefficients of $f_1, \cdots, f_r$. To facilitate ensuing exposition, we list the needed standard results on heights of polynomials as follows:

Lemma 3.1. *Let* $f = \sum_{I \in \mathcal{I}} a_I z^I$, $f_1, \cdots, f_r, g \in \overline{\mathbb{Q}}[z_1, \cdots, z_n]$, *where* $\mathcal{I}$ *denotes an index set. Then*

(i) $h(f) \leq h^A(f)$, *and* $\log \big(\max_{I \in \mathcal{I}} |a_I| \big) \leq h^A(f)$.

(ii) *If one of the coefficients of f is a root of unity, then* $h(f) = h^A(f)$. *Also, if* $f \in \mathbb{Z}[z_1, \cdots, z_n]$, *then* $h^A(f) = \log\big(\max_{I\in\mathcal{I}} |a_I|\big)$.

(iii) $h^A(f_1 + \cdots + f_r) \le h^A(f_1, \cdots, f_r) + \log r$.

(iv) *For a* <u>*monomial*</u> $\alpha \in \overline{\mathbb{Q}}[z_1, \cdots, z_n]$, *one has* $h^A(\alpha f) \le h^A(\alpha) + h^A(f)$.

(v) *For two* <u>*monomials*</u> $\alpha, \beta \in \overline{\mathbb{Q}}[z_1, \cdots, z_n]$ *with* α *divisible by* β, *one has*

$$h^A(\frac{\alpha}{\beta}) \le h^A(\alpha) + h^A(\beta).$$

(vi) $\max_{1\le i\le r} h^A(f_i) \le h^A(f_1, \cdots, f_r)$.

(vii) $h^A(f, f_1, \cdots, f_r) \le h^A(f) + h^A(f_1, \cdots, f_r) + \log 2$.

(viii) *Let* $p \in \overline{\mathbb{Q}}[w_1, \cdots, w_r]$. *Then*

$$\begin{aligned} & h^A(p(f_1, \cdots, f_r)) \\ & \le h^A(p) + \deg(p)\cdot\big(h^A(f_1, \cdots, f_r) + \log(r+1) + \log(n+1)\cdot \max_{1\le i\le r}\deg(f_i)\big). \end{aligned}$$

Proof. The first inequality of (i) is obvious. The second inequality of (i) follows from the standard fact that for any number field K and any absolute value v on $\mathbb{Q}$, the set of absolute values M_v on K extending v satisfies $\sum_{w\in M_v}[K_w : \mathbb{Q}_v] = [K : \mathbb{Q}]$ (see e.g. [12], p. 51). This fact, together with another standard fact that for a root of unity ξ, one has $|\xi|_v = 1$ for any $v \in M_K$ and any number field K containing ξ (see e.g. [12], p. 54), also lead to the first statement of (ii) readily. Moreover, these two standard facts, together with a third well-known fact that $|m|_p \le 1$ for any non-zero integer m and p-adic absolute value $|\ |_p$ on $\mathbb{Q}$, imply readily the second statement of (ii) (cf. e.g. [12], p. 52). (iii) follows from [11], Lemma 1.2(a). To prove (iv), we may assume without loss of generality that $\alpha \in \overline{\mathbb{Q}}$. For an appropriate number field K and $v \in M_K$, one obviously has

$$\max\{1, \max_{I\in\mathcal{I}} |\alpha a_I|_v\} = \max\{1, \max_{I\in\mathcal{I}} |\alpha|_v |a_I|_v\} \le \max\{1, |\alpha|_v\}\cdot\max\{1, \max_{I\in\mathcal{I}} |a_I|_v\}, \tag{3.4}$$

which, together with (3.2), implies (iv) readily. To prove (v), one may assume as in (iv) that $\alpha, \beta \in \overline{\mathbb{Q}}$. Then (v) follows from (iv) by letting $f = \dfrac{1}{\beta}$ and using the obvious fact that $h^A(\dfrac{1}{\beta}) = h^A(\beta)$. (vi) follows readily from (3.2) and (3.3). To prove (vii), we first remark that the following variant of (iii) holds:

$$h^A(f, g) \le h^A(f) + h^A(g) + \log 2 \tag{3.5}$$

(see e.g. [8], p.225-226, for a proof of (3.5)). Next it is easy to see that one can choose suitable monic monomials $p_i = z_1^{i_1}\cdots z_n^{i_n}$, $i = 1, \cdots, r$, so that any two monomial terms in the polynomials f, $p_1 f_1, \cdots, p_r f_r$ are not constant multiples of each other. Then one easily sees that

$$\begin{aligned} h^A(f, f_1, \cdots, f_r) &= h^A(f + p_1 f_1 + \cdots + p_r f_r) \\ &\le h^A(f) + h^A(p_1 f_1 + \cdots + p_r f_r) + \log 2 \quad \text{(by (3.5))} \\ &= h^A(f) + h^A(f_1, \cdots, f_r) + \log 2, \end{aligned} \tag{3.6}$$

which verifies (vii). Finally (viii) follows from [11], Lemma 1.2(c). □

4. First Form of Effective Łojasiewicz Inequality

4.1. *Lemma on orthogonalization*

In Section 4, we are going to prove Theorem 2.1. First we give a simple lemma.

Lemma 4.1. *Let $(V, \langle\ ,\ \rangle)$ be an n-dimensional inner product space over $\mathbb{C}$, and let $e_1, \cdots, e_n$ be an orthonormal basis of V. Suppose $u_1, \cdots, u_n$ is a basis of V such that each $u_j \in \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_n$. Then there exists an orthogonal basis $v_1, \cdots, v_n$ of V such that for each $1 \le j \le n$, one has*

(i) $v_j \in \mathbb{Z}e_1 + \cdots \mathbb{Z}e_n$,
(ii) $v_j \in \mathbb{C}u_1 + \cdots + \mathbb{C}u_j$, *and*
(iii) $\|v_j\| \le \Big(\max\limits_{1\le i\le n} \|u_i\|\Big)^{3^{n-1}}$.

Proof. Starting with the basis $u_1, \cdots, u_n$, we perform the (modified) Gram-Schmidt orthogonalization process as follows: Let $v_1 = u_1$, and for $2 \le j \le n$, define v_j recursively by letting

$$v_j := \Big(\prod_{i=1}^{j-1} \|v_i\|^2\Big) \cdot \Big(u_j - \sum_{i=1}^{j-1} \frac{\langle u_j, v_i\rangle}{\|v_i\|^2} v_i\Big) \tag{4.1}$$

$$= \Big(\prod_{i=1}^{j-1} \|v_i\|^2\Big) u_j - \sum_{i=1}^{j-1} \Big(\prod_{\substack{1 \le \ell \\ \ell \ne i}} \|v_\ell\|^2\Big) \langle u_j, v_i\rangle v_i. \tag{4.2}$$

First it is easy to see from (4.1) that $\{v_1, \cdots, v_n\}$ forms an orthogonal basis of V. Next, by using (4.2), one can easily prove by induction (on j) that each v_j satisfies Lemma 4.1(i) (and $\|v_j\|^2 \in \mathbb{N}$). Lemma 4.1(ii) also follows readily from (4.1). It is well-known and easy to check that for each $2 \le j \le n$,

$$\Big\|u_j - \sum_{i=1}^{j-1} \frac{\langle u_j, v_i\rangle}{\|v_i\|^2} v_i\Big\| \le \|u_j\|. \tag{4.3}$$

Together with (4.1), one knows that

$$\|v_j\| \le \|v_1\|^2 \cdot \|v_2\|^2 \cdots \|v_{j-1}\|^2 \cdot \|u_j\| \quad \text{for each } 2 \le j \le n. \tag{4.4}$$

Then using (4.4), one can easily prove by induction that

$$\|v_j\| \le \Big(\max_{1\le i\le j} \|u_i\|\Big)^{3^{j-1}} \quad \text{for each } j \ge 1, \tag{4.5}$$

which, together with the fact that each $\|u_j\| \ge 1$, lead to Lemma 4.1(iii) readily. □

4.2. *Distance estimates via projections*

Throughout the remainder of Section 4, we fix positive integers n, r, δ, d and a real number $\lambda > 0$. Also we fix a Euclidean coordinate system in $\mathbb{C}^n$ with the coordinate functions given by $z_1, \cdots, z_n$, and we denote the corresponding basis of coordinate unit vectors in $\mathbb{C}^n$ by $e_1, \cdots, e_n$ respectively. Thus a point $z = (z_1, \cdots, z_n)$ will be identified with the corresponding vector $z_1 e_1 + \cdots + z_n e_n$ in $\mathbb{C}^n$. Also we fix the associated Euclidean inner product $\langle\ ,\ \rangle$ and Euclidean norm $\|\ \|$ so that $\langle e_i, e_j\rangle = \delta_{ij}$ for $1 \le i, j \le n$, and $\|v\| = \sqrt{\langle v, v\rangle}$ for a vector $v \in \mathbb{C}^n$. Here δ_{ij} denotes the Kronecker delta symbol. Unless otherwise stated, any linear subspace of $\mathbb{C}^n$ will be endowed with the inner product and norm induced from $\langle\ ,\ \rangle$ and $\|\ \|$, which will be denoted by the same symbols. We also fix an arithmetically defined affine variety $M \in \mathcal{M}_{\overline{\mathbb{Q}}}(n, r, \delta, \lambda)$ as in Theorem 2.1 such that pure $\dim_{\mathbb{C}} M = \kappa$ with $0 \le \kappa \le n-2$, and $d := \deg(M) \ge 2$ (cf. Remark 2.1(iii)). Write $M = Z(f_1, \cdots, f_r)$ with each $f_i \in \overline{\mathbb{Q}}[z_1, z_2, \cdots, z_n]$ and $\deg(f_i) \le \delta$, and such that $h^A(f_1, \cdots, f_r) \le \lambda$ (cf. (2.1)).

Proposition 4.1. *There exists an orthogonal (but not necessarily orthonormal) basis $\{v_i\}_{1\le i\le n}$ of $\mathbb{C}^n$ such that*
(i) $v_i \in \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_n$ (and thus $\|v_i\|^2 \in \mathbb{N}$) for all $i = 1, \cdots, n$,
(ii) $\|v_i\| \le (\sqrt{n}\, d^2)^{3^{n-1}}$ for all $i = 1, \cdots, n$, and
(iii) $p_{\mathbb{C}^n, V_\kappa}\big|_M$ is a finite map of degree d. Here $V_\kappa := \mathbb{C}v_1 + \cdots + \mathbb{C}v_\kappa$, and $p_{\mathbb{C}^n, V_\kappa}$ denotes the unitary projection map from $\mathbb{C}^n$ onto V_κ.

Proof. By [11], Proposition 4.5 and Lemma 2.14, there exists a non-zero polynomial $G \in \overline{\mathbb{Q}}[b_{i,j} : 1 \le i \le \kappa,\ 1 \le j \le n]$ depending on M and satisfying the following two properties: (α) for each $1 \le i \le \kappa$, G is of degree $\le 2d^2$ in the group of variables $b_i := (b_{i,1}, \cdots, b_{i,n})$; and ($\beta$) for any $b_1, \cdots, b_\kappa \in \mathbb{C}^n$ such that $G(b_1, \cdots, b_\kappa) \ne 0$, the linear map $\phi_{b_1,\cdots,b_\kappa} : \mathbb{C}^n \to \mathbb{C}^\kappa$ given by

$$\phi_{b_1,\cdots,b_\kappa}(z) = (\langle z, b_1\rangle, \cdots, \langle z, b_\kappa\rangle) \tag{4.6}$$

is such that $\phi_{b_1,\cdots,b_\kappa}\big|_M : M \to \mathbb{C}^\kappa$ is a finite map of degree d. Since there are $2d^2 + 1$ integers in the interval $[-d^2, d^2]$, it is easy to see from (α) that one can choose $b_1, \cdots, b_\kappa \in \mathbb{Z}^n$ such that $G(b_1, \cdots, b_\kappa) \ne 0$, and

$$-d^2 \le b_{i,j} \le d^2 \quad \text{for each } 1 \le i \le \kappa \text{ and } 1 \le j \le n. \tag{4.7}$$

We fix such a choice of $b_1, \cdots, b_\kappa$, which are easily seen from (β) to be linearly independent. By adjoining suitable e_ℓ's to the b_i's, we get a basis $u_1, \cdots, u_n$ of $\mathbb{C}^n$ such that

$$u_1 = b_1, \cdots, u_\kappa = b_\kappa, \quad \text{and } u_i \in \{e_1, \cdots, e_n\} \quad \text{for each } i = \kappa+1, \cdots, n. \tag{4.8}$$

Together with (4.7), one easily sees that

$$\|u_i\| \le \sqrt{n}\, d^2 \quad \text{for all } i = 1, \cdots, n. \tag{4.9}$$

Then by Lemma 4.1, there exists an orthogonal basis $v_1, \cdots, v_n$ of $\mathbb{C}^n$ satisfying Lemma 4.1(i)-(iii). Then Proposition 4.1(i) follows from Lemma 4.1(i), and one obtains Proposition 4.1(ii) readily from Lemma 4.1(iii) and (4.9). Let $V_\kappa = \mathbb{C}v_1 + \cdots + \mathbb{C}v_\kappa$ and $p_{\mathbb{C}^n,V_\kappa} : \mathbb{C}^n \to V_\kappa$ be the unitary projection as in Proposition 4.1(iii). Then it follows from Lemma 4.1(ii) and (4.8) that $V_\kappa = \mathbb{C}b_1 + \cdots + \mathbb{C}b_\kappa$. Together with (4.6), one easily sees that for any $z \in \mathbb{C}^n$, one has

$$\phi^{-1}_{b_1,\cdots,b_\kappa}(\phi_{b_1,\cdots,b_\kappa}(z)) = p^{-1}_{\mathbb{C}^n,V_\kappa}(p_{\mathbb{C}^n,V_\kappa}(z)) = z + V_\kappa^\perp, \tag{4.10}$$

where $V_\kappa^\perp$ denotes the orthogonal complement of V_κ in $\mathbb{C}^n$. Finally Proposition 4.1(iii) follows from (4.10) and the fact that $\phi_{b_1,\cdots,b_\kappa}\big|_M$ is a finite map of degree d. □

By Proposition 4.1, there exists an orthogonal basis $\{v_i\}_{1\le i\le n}$ of $\mathbb{C}^n$ satisfying the properties listed in Proposition 4.1 (i)-(iii), and we will fix such a basis throughout the rest of Section 4. For each $\kappa \le i \le n$, we let $V_i := \mathbb{C}v_1 + \mathbb{C}v_2 + \cdots + \mathbb{C}v_i$, so that $V_n = \mathbb{C}^n$. For any $i, j \in \mathbb{N}$ with $2 \le i \le n$ and $0 \le j \le d$, we let

$$u_j^{(i)} := v_i + \frac{2j-d}{d} v_{i-1} \in V_i, \tag{4.11}$$

and we denote by $p_j^{(i)} : V_i \to V_{i-1}$ the unique linear projection map from V_i onto V_{i-1} with $\ker(p_j^{(i)}) = \mathbb{C}u_j^{(i)}$. Also we let $U_j^{(i)}$ be the conical open set in V_i given by

$$U_j^{(i)} := \big\{ v \in V_i \,\big|\, \|p_j^{(i)}(v)\| < \frac{1}{\sqrt{2}d \cdot (\sqrt{n}\, d^2)^{3^{n-1}}} \|v - p_j^{(i)}(v)\| \big\}. \tag{4.12}$$

Lemma 4.2. *$U_j^{(i)} \cap U_{j'}^{(i)} = \emptyset$ if $j \ne j'$.*

Proof. For each $2 \le i \le n$ and $0 \le j \le d$, we consider another open set of V_i given by

$$\begin{aligned}\hat{U}_j^{(i)} : = \big\{\, v \in V_i \,\big|\, v = \alpha(&u_j^{(i)} + \beta_1 v_1 + \cdots + \beta_{i-1}v_{i-1}) \\ &\text{for some } \alpha, \beta_1, \cdots, \beta_{i-1} \in \mathbb{C} \\ &\quad \text{with } \alpha \ne 0 \text{ and } |\beta_{i-1}| < \frac{1}{d} \,\big\}.\end{aligned} \tag{4.13}$$

From (4.11) and (4.12), one easily checks that $\hat{U}_j^{(i)} \cap \hat{U}_{j'}^{(i)} = \emptyset$ if $j \ne j'$. Also, one clearly has $\|u_j^{(i)}\| \le \sqrt{2}\max\{\|v_{i-1}\|, \|v_i\|\}$. Together with Proposition 4.1(ii), (4.12) and (4.13), one easily checks that $U_j^{(i)} \subset \hat{U}_j^{(i)}$, and Lemma 4.2 follows readily. □

For each $\kappa + 1 \le i \le n$, we denote by $\Pi_i : V_i \to V_\kappa$ the orthogonal projection map from V_i onto V_κ, and for a non-empty subset $Z \subset V_i$ and a point $w \in V_i$, we denote

$$\mathrm{dist}_{\Pi_i}(w, Z) := \inf_{y \in (\Pi_i|_Z)^{-1}(\Pi_i(w))} \|w - y\|. \tag{4.14}$$

The following lemma and its subsequent proposition follow from an adaptation of the arguments of Ji-Kollar-Shiffman [9], p. 815-816, to our effective setting here. For convenience of the reader, we give their self-contained proofs.

Lemma 4.3. *Let Z be a κ-dimensional affine variety in V_i ($i \geq \kappa+2$) and of degree $\leq d$ such that $\Pi_i\big|_Z : Z \to V_\kappa$ is a finite-to-one map (and thus $\Pi_i\big|_Z$ is of degree $\leq d$). Then the following statements hold.*
(i) For each $0 \leq j \leq d$, $p_j^{(i)}\big|_Z : Z \to V_{i-1}$ is a finite-to-one map of degree $\leq d$. Here $p_j^{(i)}$ is as in (4.12).
(ii) For $0 \leq j \leq d$, $\Pi_{i-1}\big|_{p_j^{(i)}(Z)} : p_j^{(i)}(Z) \to V_\kappa$ is a finite-to-one map of degree $\leq d$.
(iii) Moreover, for any $w \in V_i$,

$$dist_{\Pi_i}(w, Z) \leq n^{3^{n-1}} d^{3^n} \cdot \max_{0 \leq j \leq d} dist_{\Pi_{i-1}}(p_j^{(i)}(w), p_j^{(i)}(Z)). \tag{4.15}$$

Proof. Since $i \geq \kappa + 2$, it follows from (4.11) that $u_j^{(i)} \in \ker(\Pi_i)$ for $0 \leq j \leq d$. Thus, for any $w \in V_{i-1} \subset V_i$,

$$(p_j^{(i)}\big|_Z)^{-1}(w) = Z \cap (w + \mathbb{C}u_j^{(i)}) \subset (\Pi_i\big|_Z)^{-1}(\Pi_i(w)). \tag{4.16}$$

Together with the given condition that $\Pi_i\big|_Z : Z \to V_\kappa$ is a finite-to-one map of degree $\leq d$, it follows that $p_j^{(i)}\big|_Z$ is also a finite-to-one map of degree $\leq d$, which gives (i). To prove (ii) and (iii), we take an arbitrary vector $w \in V_i$. Since $\Pi_i\big|_Z$ is a finite-to-one map of degree $\leq d$, we may write $(\Pi_i\big|_Z)^{-1}(\Pi_i(w)) = \{y_1, \cdots, y_{d'}\}$ for some $d' \leq d$. For each $0 \leq j \leq d$, from the definition of $p_j^{(i)}$ (cf. (4.11)) and the fact that $i \geq \kappa + 2$, one easily sees that $\Pi_{i-1} \circ p_j^{(i)} = \Pi_i$ on V_i, and this implies readily that

$$\begin{aligned} p_j^{(i)}(Z) \cap \Pi_{i-1}^{-1}(\Pi_{i-1}(p_j^{(i)}(w))) &\subset p_j^{(i)}((\Pi_i\big|_Z)^{-1}(\Pi_i(w))) \\ &= \{p_j^{(i)}(y_1), \cdots, p_j^{(i)}(y_{d'})\}. \end{aligned} \tag{4.17}$$

Together with the fact that $p_j^{(i)}$ is a surjective map from V_i onto V_{i-1}, one obtains (ii) immediately. To prove (iii), for $0 \leq j \leq d$, we let $U_j^{(i)}$ be the open set in V_i associated to $p_j^{(i)}$ as given in (4.12). Then by the pigeonhole principle and Lemma 4.2, we must have $\{w - y_1, \cdots, w - y_{d'}\} \cap U_{j_o}^{(i)} = \emptyset$ for some $0 \leq j_o \leq d$. Thus, from (4.12), one has, for all $1 \leq \ell \leq d'$,

$$\begin{aligned} &\|p_{j_o}^{(i)}(w - y_\ell)\| \geq \epsilon \|w - y_\ell - p_{j_o}^{(i)}(w - y_\ell)\|, \quad \text{where } \epsilon := \frac{1}{\sqrt{2}d \cdot (\sqrt{n}\, d^2)^{3^{n-1}}} \\ \Longrightarrow \quad &\|w - y_\ell\| \leq (1 + \epsilon^{-1}) \|p_{j_o}^{(i)}(w) - p_{j_o}^{(i)}(y_\ell)\| \\ \Longrightarrow \quad &\text{dist}_{\Pi_i}(w, Z) \leq (1 + \epsilon^{-1}) \cdot \text{dist}_{\Pi_{i-1}}(p_{j_o}^{(i)}(w), p_{j_o}^{(i)}(Z)) \quad \text{(by (4.17))}. \end{aligned} \tag{4.18}$$

It is easy to check that $1 + \epsilon^{-1} = 1 + \sqrt{2}d \cdot (\sqrt{n}\, d^2)^{3^{n-1}} \leq n^{3^{n-1}} d^{3^n}$, which, together with (4.18), leads readily to (iii). $\square$

For any $m \in \mathbb{N}$, we denote

$$\mathcal{J}(m,d) := \{(j_1, \cdots, j_d) \in \mathbb{Z}^m \,\big|\, 0 \le j_1, \cdots, j_m \le d\}. \tag{4.19}$$

Observe that the cardinality of $\mathcal{J}(m,d)$ is $(d+1)^m$. For each $J = (j_1, \cdots, j_m) \in \mathcal{J}(m,d)$ with $m \le n-\kappa-1$, we let $p_J : \mathbb{C}^n \to V_{n-m}$ be the linear projection map arising as a composition of linear projection maps given by

$$p_J := p_{j_m}^{(n-m+1)} \circ p_{j_{m-1}}^{(n-m+2)} \circ \cdots \circ p_{j_1}^{(n)}, \tag{4.20}$$

where the $p_j^{(i)}$'s are as in (4.12).

Proposition 4.2. *(i) For any $z \in \mathbb{C}^n$, we have*

$$dist(z,M) \le (n^{3^{n-1}} d^{3^n})^{n-\kappa-1} \cdot \max_{J \in \mathcal{J}(n-\kappa-1,d)} dist_{\Pi_{\kappa+1}}(p_J(z), p_J(M)), \tag{4.21}$$

where $dist(z,M)$ denotes the Euclidean distance of z from M as in (2.2).
(ii) For each $J \in \mathcal{J}(n-\kappa-1,d)$, $p_J\big|_M : M \to V_{\kappa+1}$ is a finite-to-one map of degree $\le d$. Moreover, $\Pi_{\kappa+1}\big|_{p_J(M)} : p_J(M) \to V_\kappa$ is also a finite-to-one map of degree $\le d$.

Proof. First, one obviously has

$$\mathrm{dist}(z,M) \le \mathrm{dist}_{\Pi_n}(z,Z). \tag{4.22}$$

Recall from Proposition 4.1(iii) that $\Pi_n\big|_M : M \to V_\kappa$ is a finite-to-one map of degree $\le d$. Thus by Lemma 4.3(i), $p_j^{(n)}\big|_M : M \to V_{n-1}$ is a finite-to-one map of degree $\le d$. Also, by Lemma 4.3(iii), one has

$$\mathrm{dist}_{\Pi_n}(z,M) \le n^{3^{n-1}} d^{3^n} \cdot \max_{0 \le j_1 \le d} \mathrm{dist}_{\Pi_{n-1}}(p_{j_1}^{(n)}(z), p_{j_1}^{(n)}(M)). \tag{4.23}$$

Also, by Lemma 4.3(ii), $\Pi_{n-1}\big|_{p_{j_1}^{(n)}(M)} : p_{j_1}^{(n)}(M) \to V_{n-1}$ is a finite-to-one map of degree $\le d$. Then for each $0 \le j_1, j_2 \le d$, one can apply Lemma 4.3 (i) (to the Zariski closure of $p_{j_1}^{(n)}(M)$ in V_{n-1}) to deduce that $p_{j_2}^{(n-1)}\big|_{p_{j_1}^{(n)}(M)} : p_{j_1}^{(n)}(M) \to V_{n-2}$ is a finite-to-one map of degree $\le d$. Moreover, by Lemma 4.3(iii), one has

$$\begin{aligned}&\mathrm{dist}_{\Pi_{n-1}}(p_{j_1}^{(n)}(z), p_{j_1}^{(n)}(M)) \\ &\le n^{3^{n-1}} d^{3^n} \cdot \max_{0 \le j_2 \le d} \mathrm{dist}_{\Pi_{n-2}}(p_{j_2}^{(n-1)} \circ p_{j_1}^{(n)}(z),\, p_{j_2}^{(n-1)} \circ p_{j_1}^{(n)}(M)).\end{aligned} \tag{4.24}$$

By repeating the above arguments $n-\kappa-1$ times, one obtains Proposition 4.2(i), (ii) readily. □

4.3. *Proof of Theorem 2.1*

Notation as in subsection 4.2. We are going to derive our effective Łojasiewicz inequality for M. In light of Proposition 4.2, it remains to bound the quantity $\mathrm{dist}_{\Pi_{\kappa+1}}(p_J(z), p_J(M))$ for each $J \in \mathcal{J}(n-\kappa-1,d)$.

First we recall from subsection 4.2 that we have fixed a Euclidean coordinate system in $\mathbb{C}^n$ with coordinate functions $z_1, \cdots, z_n$ and an associated orthonormal basis of coordinate unit vectors $e_1, \cdots, e_n$. Also, we have constructed in Proposition 4.1 another orthogonal basis $\{v_1, \cdots, v_n\}$ of $\mathbb{C}^n$. From now on, we will denote by $w_1, \cdots, w_n$ the coordinate functions on $\mathbb{C}^n$ with respect to the basis vectors $v_1, \cdots, v_n$. With slight abuse of notation, for each $\kappa \leq i \leq n$, we will also regard $w_1, \cdots, w_i$ as coordinate functions on the linear subspace $V_i := \mathbb{C}v_1 + \cdots + \mathbb{C}v_i$ with respect to the basis vectors $v_1, \cdots, v_i$. For $\kappa + 1 \leq i \leq n$ and $0 \leq j \leq d$, one easily checks that with respect to the basis $\{v_1, \cdots, v_i\}$ of V_i and the basis $\{v_1, \cdots, v_{i-1}\}$ of V_{i-1}, the linear projection map $p_j^{(i)} : V_i \to V_{i-1}$ defined in subsection 4.2 (just prior to Lemma 4.2) is given by the $(i-1) \times i$ matrix

$$\begin{pmatrix} 1 & 0 & \cdots\cdots & & & 0 \\ 0 & 1 & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 & 0 \\ 0 & \cdots\cdots & & 0 & 1 & \frac{d-2j}{d} \end{pmatrix}. \tag{4.25}$$

Thus, for $J = (j_1, \cdots, j_{n-\kappa-1}) \in \mathcal{J}(n-\kappa-1, d)$, the $(\kappa+1) \times n$ matrix of $p_J : V_n(= \mathbb{C}^n) \to V_{\kappa+1}$ (as in (4.20)) with respect to the basis vectors v_i's is given by

$$A = \begin{pmatrix} 1 & 0 & \cdots\cdots & & & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & & & \vdots \\ \vdots & & \ddots & \ddots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots\cdots & & 0 & 1 & \frac{d-2j_{n-\kappa-1}}{d} & \prod_{\ell=n-\kappa-2}^{n-\kappa-1} \frac{d-2j_\ell}{d} & \cdots & \prod_{\ell=1}^{n-\kappa-1} \frac{d-2j_\ell}{d} \end{pmatrix}. \tag{4.26}$$

Next, for each $1 \leq i \leq n$, we write $v_i = v_{1i}e_1 + \cdots + v_{ni}e_n$. By Proposition 4.1(i), we have that $v_{ji} \in \mathbb{Z}$ for all i, j. Next we form the matrix $B = (v_{ji})_{1 \leq j, i \leq n}$, i.e., the i-th column of B is given by the components of v_i with respect to the basis $\{e_i\}$. It is easy to see that B^{-1} is the matrix corresponding to the change of basis on $\mathbb{C}^n$ from $\{e_i\}_{1 \leq i \leq n}$ to $\{v_i\}_{1 \leq i \leq n}$. Consider the $n \times n$ diagonal matrix

$$\begin{aligned} \Lambda : &= \operatorname{diag}\Big(\frac{1}{\|v_1\|^2}, \cdots, \frac{1}{\|v_n\|^2}\Big) \\ &= \frac{1}{\prod_{1=1}^{n} \|v_i\|^2} \operatorname{diag}\Big(\prod_{\substack{1 \leq i \leq n \\ i \neq 1}} \|v_i\|^2, \cdots, \prod_{\substack{1 \leq i \leq n \\ i \neq n}} \|v_i\|^2\Big). \end{aligned} \tag{4.27}$$

Since $\{v_i\}_{1 \leq i \leq n}$ forms an orthogonal basis of $\mathbb{C}^n$ with each $v_{ji} \in \mathbb{Z}$, one easily checks that $B^{-1} = \Lambda B^t$. Thus, with respect to the basis $\{e_i\}_{1 \leq i \leq n}$ on $\mathbb{C}^n$ and the basis $\{v_i\}_{1 \leq i \leq \kappa+1}$ on $V_{\kappa+1}$, p_J is given by the matrix

$$P^{(J)} := AB^{-1} = A\Lambda B^t. \tag{4.28}$$

For each $k, \ell \in \mathbb{N}$, we denote by $M_{k,\ell}(\mathbb{Z})$ (resp. $M_{k,\ell}(\mathbb{C})$) the set of $k \times \ell$ matrices with entries in $\mathbb{Z}$ (resp. $\mathbb{C}$). Also, we denote the (i,j)-th entry of $P^{(J)}$ by $P_{ij}^{(J)}$, etc.

Let

$$\eta := d^{n-\kappa-1} \cdot \prod_{i=1}^{n} \|v_i\|^2. \tag{4.29}$$

For a matrix $Q \in M_{k,\ell}(\mathbb{C})$, its matrix norm $\|Q\|$ is given by

$$\|Q\| := \sup_{\substack{u=(u_1,\cdots,u_\ell)^t \\ |u_1|^2+\cdots+|u_\ell|^2 \le 1}} \sqrt{\sum_{i=1}^{k} |(Qu)_i|^2} \tag{4.30}$$

Lemma 4.4. *(i) For each $J \in \mathcal{J}(n-\kappa-1, d)$, we have $\eta \cdot P^{(J)} \in M_{\kappa+1,n}(\mathbb{Z})$, and for all $1 \le i \le \kappa+1$ and $1 \le j \le n$,*

$$|\eta \cdot P^{(J)}_{ij}| \le n^{n\cdot 3^{n-1}} d^{4n\cdot 3^{n-1}}. \tag{4.31}$$

(ii) $\eta \in \mathbb{Z}$, and $1 \le \eta \le n^{n\cdot 3^{n-1}} d^{n+4n\cdot 3^{n-1}}$.
(iii) $\|P^{(J)}\| \le (n-\kappa)\cdot(\sqrt{n}\, d^2)^{3^{n-1}}$.

Proof. For each $J = (j_1, \cdots, j_{n-\kappa-1}) \in \mathcal{J}(n-\kappa-1, d)$, we have $0 \le j_\ell \le d$, and thus $|d-2j_\ell| \le d$ for each $1 \le \ell \le n-\kappa-1$ (cf. (4.19)). Then one easily checks from (4.26) that

$$d^{n-\kappa-1}A \in M_{\kappa+1,n}(\mathbb{Z}), \quad \text{and} \quad |d^{n-\kappa-1}A_{ij}| \le d^{n-\kappa-1} \quad \text{for all } i,j. \tag{4.32}$$

Using Proposition 4.1(i), one easily sees from definition that both B^t and $\big(\prod_{i=1}^n \|v_i\|^2\big)\cdot \Lambda$ are in $M_{n,n}(\mathbb{Z})$. Together with (4.28), (4.29) and (4.32), it follows readily that $\eta \cdot P^{(J)} \in M_{\kappa+1,n}(\mathbb{Z})$. By Proposition 4.1(ii), one has, for all $1 \le j,k \le n$,

$$|B^t_{jk}| = |v_{kj}| \le \max_{1\le i\le n} \|v_i\| \le (\sqrt{n}\, d^2)^{3^{n-1}} \quad \text{and} \tag{4.33}$$

$$\Big|\big(\prod_{i=1}^{n} \|v_i\|^2\big)\cdot \Lambda_{jk}\Big| \le \big(\max_{1\le i\le n} \|v_i\|\big)^{2n-2} \le (\sqrt{n}\, d^2)^{(2n-2)\cdot 3^{n-1}}. \tag{4.34}$$

From (4.28), one sees that $P^{(J)}_{ij} = \sum_{k=1}^n A_{ik}\Lambda_{kk}B^t_{kj}$. Together with (4.29), (4.32), (4.33) and (4.34), one has

$$|\eta \cdot P^{(J)}_{ij}| \le n\cdot d^{n-\kappa-1}\cdot(\sqrt{n}\, d^2)^{(2n-2)\cdot 3^{n-1}} \cdot (\sqrt{n}\, d^2)^{3^{n-1}} \le n^{n\cdot 3^{n-1}} d^{4n\cdot 3^{n-1}},$$

which gives (i). Next we remark that (ii) follows easily from (4.29), Proposition 4.1(i) and (ii) via a calculation similar to (i). To prove (iii), one first checks readily from the first line of (4.27) and Proposition 4.1(ii) that $\|\Lambda\| \le 1$. Also, for each $1 \le i \le n$, the i-th row of the matrix B^t is simply given by the components of v_i with respect to the standard basis $e_1, \cdots, e_n$. Together with the orthogonality of $\{v_i\}_{1\le i\le n}$, one can easily check that

$$\|B^t\| \le \max_{1\le i\le n} \|v_i\| \le (\sqrt{n}\, d^2)^{3^{n-1}}. \tag{4.35}$$

Observe from (4.26) that the $n-\kappa$ non-zero entries of the last row of the matrix A are all real numbers lying between 0 and 1. Using this, one can check readily from (4.26) that $\|A\| \leq n-\kappa$. Together with (4.28), one has

$$\begin{aligned}\|P^{(J)}\| = \|A\Lambda B^t\| \leq \|A\|\cdot\|\Lambda\|\cdot\|B^t\| &\leq (n-\kappa)\cdot 1\cdot(\sqrt{n}\,d^2)^{3^{n-1}}\\ &= (n-\kappa)\cdot(\sqrt{n}\,d^2)^{3^{n-1}},\end{aligned} \tag{4.36}$$

where the first inequality is a standard fact on matrix norms. Thus we have finished the proof of Lemma 4.4. □

Recall that we have fixed Euclidean coordinate functions $z_1,\cdots,z_n$ (with respect to the basis $\{e_i\}_{1\leq i\leq n}$) on $\mathbb{C}^n$ as well as Euclidean coordinate functions $w_1,\cdots,w_{\kappa+1}$ (with respect to the basis $\{v_i\}_{1\leq i\leq \kappa+1}$) on $V_{\kappa+1}$. Recall also from subsection 4.2 that $M = Z(f_1,\cdots,f_r) \in \mathcal{M}_{\overline{\mathbb{Q}}}(n,r,\delta,\lambda)$ with $\deg(M) = d \geq 2$ and pure $\dim_{\mathbb{C}} M = \kappa \leq n-2$, each $f_i \in \overline{\mathbb{Q}}[z_1,\cdots,z_n]$ with $\deg(f_i) \leq \delta$, $i = 1,\cdots,r$, and $h^A(f_1,\cdots,f_r) \leq \lambda$. For a fixed $J \in \mathcal{J}(n-\kappa-1,d)$, and let p_J and $P^{(J)}$ be as in (4.20) and (4.28) respectively. By Proposition 4.2(ii), $p_J\big|_M : M \to V_{\kappa+1}$ is a finite-to-one map of degree $\leq d$, and thus the Zariski closure $\overline{p_J(M)}$ of $p_J(M)$ in $V_{\kappa+1}$ is an affine hypersurface of degree $\leq d$. In particular, $\overline{p_J(M)}$ is the zero set of a single defining polynomial of degree $\leq d$ in $\mathbb{C}[w_1,\cdots,w_{\kappa+1}]$, which is unique up to a non-zero multiplicative constant.

Proposition 4.3. *Let*

$$C_1 = C_1(n,\delta,\lambda,r) := \delta^r(n\lambda + n^2 3^n \log n + rn^2 3^{n+1}\log\delta). \tag{4.37}$$

Then for each $J \in \mathcal{J}(n-\kappa-1,d)$, the hypersurface $\overline{p_J(M)}$ in $V_{\kappa+1}$ is defined by a non-zero polynomial $\phi_J \in \overline{\mathbb{Q}}[w_1,\cdots,w_{\kappa+1}]$ of degree $\leq d$ and satisfying

$$h^A(\phi_J) \leq C_1. \tag{4.38}$$

Proof. For each fixed $J \in \mathcal{J}(n-\kappa-1,d)$, it follows readily from [11], Proposition 2.5, that there exists a non-zero defining polynomial $\phi_J \in \overline{\mathbb{Q}}[w_1,\cdots,w_{\kappa+1}]$ for $\overline{p_J(M)}$ in $V_{\kappa+1}$ (necessarily of degree $\leq d$) such that

$$h^A(\phi_J) \leq h(M) + (\kappa+1)\big(h^A(P^{(J)}) + 8\log(n+\kappa+2)\big)d, \tag{4.39}$$

where $h(M)$ denotes the height of the variety M as defined in [11], p. 539, and the $(\kappa+1)\times n$ matrix $P^{(J)}$ is identified with an element in $\mathbb{Q}^{n(\kappa+1)}$ in the obvious way (see also related results in [1], [15]). By a special case of [11], Corollary 2.11, and as given in [11], p. 556, one has

$$\begin{aligned}h(M) &\leq \big(\kappa\lambda + (n+\kappa)\log(n+1)\big)\delta^{\min\{\kappa,r\}}\\ &\leq n\delta^r(\lambda + 2\log(n+1)) \quad (\text{since } \kappa \leq n-2).\end{aligned} \tag{4.40}$$

Next we have

$$\begin{aligned} h^A(P^{(J)}) &\le h^A(\eta \cdot P^{(J)}) + h^A(\frac{1}{\eta}) \quad \text{(by Lemma 3.1(iv))} \\ &= h^A(\eta \cdot P^{(J)}) + h^A(\eta) \quad (\text{since } h^A(\frac{1}{\eta}) = h^A(\eta) = h(1,\eta)) \\ &\le \log(n^{n\cdot 3^{n-1}} d^{4n\cdot 3^{n-1}}) + \log(n^{n\cdot 3^{n-1}} d^{n+4n\cdot 3^{n-1}}) \\ &\qquad \text{(by Lemma 3.1(ii) and Lemma 4.4(i)-(ii))} \\ &= \log(n^{2n\cdot 3^{n-1}} d^{n+8n\cdot 3^{n-1}}). \end{aligned} \tag{4.41}$$

Combining (4.39), (4.40), (4.41) and using the inequalities $\kappa \le n-2$, we have

$$\begin{aligned} & h^A(\phi_J) \\ \le\ & n\delta^r(\lambda + 2\log(n+1)) + (n-1)\big((\log(n^{2n\cdot 3^{n-1}} d^{n+8n\cdot 3^{n-1}}) + 8\log(2n)\big)d \\ \le\ & n\delta^r(\lambda + 2\log(n+1)) + (n-1)\big((\log(n^{2n\cdot 3^{n-1}} (\delta^r)^{n+8n\cdot 3^{n-1}}) + 8\log(2n)\big)\delta^r \\ & \qquad (\text{since } d \le \delta^r \text{ (cf. Remark 2.1(ii))}) \\ =\ & \delta^r\big(n\lambda + 2n\log(n+1) + (n-1)\cdot 2n\cdot 3^{n-1}\log n + (n-1)\cdot 8\log(2n) \\ & \qquad + (n-1)\cdot r(n + 8n\cdot 3^{n-1})\log\delta\big) \\ \le\ & \delta^r(n\lambda + n^2 3^n \log n + rn^2 3^{n+1}\log\delta) \\ =\ & C_1, \end{aligned} \tag{4.42}$$

where C_1 is as in (4.37), and the last inequality can easily be verified by considering separately the cases when $n = 2$ and when $n \ge 3$. This finishes the proof of Proposition 4.3. □

Now we are ready to give the proof of Theorem 2.1.

Proof of Theorem 2.1. Let $M \in \mathcal{M}_{\overline{\mathbb{Q}}}(n, r, \delta, \lambda)$ be as in Theorem 2.1 with $d := \deg(M) \ge 2$ and pure $\dim_{\mathbb{C}} M \le n-2$, so that $d \le \delta^r$ (cf. Remark 2.1(ii) and (iii)). We also recall the coordinate functions $z_1, \cdots, z_n$ with respect to the Euclidean orthonormal coordinate basis $\{e_i\}_{1\le i\le n}$ as well as the other coordinate system with coordinate functions $w_1, \cdots, w_n$ associated to the orthogonal basis $\{v_i\}_{1\le i\le n}$ on $\mathbb{C}^n$ constructed in Proposition 4.1. Recall also from (4.19) that the cardinality of $\mathcal{J}(n-\kappa-1, d)$ is $j_o := (d+1)^{n-\kappa-1} \le (\delta^r+1)^{n-1}$. For each $J \in \mathcal{J}(n-\kappa-1, d)$, let $\phi_J \in \overline{\mathbb{Q}}[w_1, \cdots, w_{\kappa+1}]$ be the non-zero polynomial defining the hypersurface $\overline{p_J(M)}$ in $V_{\kappa+1}$ as given by Proposition 4.3, where $p_J : \mathbb{C}^n \to V_{\kappa+1}$ is as in (4.20). Recall also that ϕ_J is of degree $d_J := \deg(\overline{p_J(M)}) \le d \le \delta^r$ and ϕ_J satisfies (4.38). For each $z \in \mathbb{C}^n$, one easily deduces from Proposition 4.2(i) that

$$\operatorname{dist}(z, M)^{\delta^r} \le C_2 \cdot \max_{J\in\mathcal{J}(n-\kappa-1,d)} \operatorname{dist}_{\Pi_{\kappa+1}}(p_J(z), p_J(M))^{d_J} \cdot \operatorname{dist}(z, M)^{\delta^r - d_J}, \tag{4.43}$$

where

$$C_2 := (n^{3^{n-1}} d^{3^n})^{(n-\kappa-1)\delta^r} \le (n^{3^{n-1}} \delta^{3^n r})^{(n-1)\delta^r}. \tag{4.44}$$

By Proposition 4.1(iii), one has $\deg(\Pi_n\big|_M) = d$, where $\deg(\Pi_n\big|_M)$ denotes the degree of the map $\Pi_n\big|_M$. As in the proof of Lemma 4.3, one easily sees from definitions that $\Pi_{\kappa+1} \circ p_J = \Pi_n$. It follows that one has

$$\deg(\Pi_{\kappa+1}\big|_{p_J(M)}) \cdot \deg(p_J\big|_M) = \deg(\Pi_n\big|_M) = d. \tag{4.45}$$

On the other hand, one clearly has $\deg(M) \geq \deg(p_J\big|_M) \cdot \deg(\overline{p_J(M)})$, i.e., $d \geq \deg(p_J\big|_M) \cdot d_J$. Together with (4.45), it follows that one has $\deg(\Pi_{\kappa+1}\big|_{p_J(M)}) \geq d_J$. Obviously we also have $\deg(\Pi_{\kappa+1}\big|_{p_J(M)}) = \deg(\Pi_{\kappa+1}\big|_{\overline{p_J(M)}}) \leq \deg(\overline{p_J(M)}) = d_J$, it follows that we have

$$\deg(\Pi_{\kappa+1}\big|_{\overline{p_J(M)}}) = d_J. \tag{4.46}$$

Upon rewriting ϕ_J in descending powers of $w_{\kappa+1}$ so that $\phi_J(w) = c_J w_{\kappa+1}^{d_J} + \cdots$, one easily sees from (4.46) that $c_J = \phi_J(0, \cdots, 0, 1) \neq 0$. Here, $(0, \cdots, 0, 1)$ is identified with $v_{\kappa+1}$. Now we factorize ϕ_J with respect to the last variable $w_{\kappa+1}$, so that we may write

$$\phi_J(w) = c_J \cdot \prod_{\alpha=1}^{d_J} \big(w_{\kappa+1} - w_{\kappa+1}(y_\alpha)\big), \tag{4.47}$$

where $\{y_\alpha\}_{1\leq\alpha\leq d_J}$ denote the d_J points of $\Pi_{\kappa+1}^{-1}\big(\Pi_{\kappa+1}(w)\big) \cap p_J(M)$ counting multiplicity, and $w_{\kappa+1}(y_\alpha)$ denotes the $w_{\kappa+1}$-coordinate of y_α. Then for $w \in V_{\kappa+1}$,

$$\begin{aligned}
\big(\mathrm{dist}_{\Pi_{\kappa+1}}(w, p_J(M))\big)^{d_J} &= \big(\min_{1\leq\alpha\leq d_J} \|w - y_\alpha\|\big)^{d_J} \\
&\leq \prod_{\alpha=1}^{d_J} \|w - y_\alpha\| \\
&= \Big(\prod_{\alpha=1}^{d_J} |w_{\kappa+1} - w_{\kappa+1}(y_\alpha)|\Big) \cdot \|v_{\kappa+1}\|^{d_J} \\
&= \frac{\|v_{\kappa+1}\|^{d_J}}{|c_J|} \cdot \big|\phi_J(w)\big|, \quad \text{(by (4.47))}.
\end{aligned} \tag{4.48}$$

From (4.38) and as in (4.41), one has

$$h^A(\frac{1}{|c_J|}) = h^A(|c_J|) \leq h^A(\phi_J) \leq C_1, \tag{4.49}$$

where C_1 is as in (4.37). Together with Lemma 3.1(i), Proposition 4.1(ii) and the inequality $d_J \leq d$, it follows that

$$\frac{\|v_{\kappa+1}\|^{d_J}}{|c_J|} \leq (\sqrt{n}\, d^2)^{3^{n-1} d_J} \cdot e^{h^A(\frac{1}{|c_J|})} \leq C_3, \quad \text{where } C_3 := (\sqrt{n}\, \delta^{2r})^{3^{n-1}\delta^r} \cdot e^{C_1}. \tag{4.50}$$

Next we consider the polynomial $\hat{q}_J \in \overline{\mathbb{Q}}[z_1, \cdots, z_n]$ given by

$$\hat{q}_J(z) := \phi_J(p_J(z)), \quad z \in \mathbb{C}^n. \tag{4.51}$$

From (4.47) and the fact that the map $p_J : \mathbb{C}^n \to V_{\kappa+1}$ is surjective, one sees that $\hat{q}_J$ is a nonzero polynomial. Moreover, since p_J is linear, it follows that $\hat{q}_J$ is also of degree d_J. Combining (4.48), (4.50) and (4.51), we have, for all $z \in \mathbb{C}^n$,

$$\big(\mathrm{dist}_{\Pi_{\kappa+1}}(p_J(z), p_J(M))\big)^{d_J} \le C_3|\hat{q}_J(z)|. \tag{4.52}$$

Together with (4.43), we have

$$\mathrm{dist}\,(z, M)^{\delta^r} \le C_2C_3 \cdot \max_{J\in\mathcal{J}(n-\kappa-1,d)} |\hat{q}_J(z)| \cdot \mathrm{dist}\,(z, M)^{\delta^r-d_J}. \tag{4.53}$$

On the other hand, we also have

$$\mathrm{dist}\,(z, M) \le \|z\| + \mathrm{dist}\,(0, M). \tag{4.54}$$

By (4.51), $\hat{q}_J(0) = \phi_J(p_J(0)) = \phi_J(0)$, and thus $\hat{q}_J(0)$ is the constant coefficient of ϕ_J. Hence as (4.49) and (4.50), we have $|\hat{q}_J(0)| \le e^{C_1}$. Together with (4.53) (with $z = 0$), we have

$$\begin{aligned} &\mathrm{dist}\,(0, M)^{\delta^r} \le C_2C_3\, e^{C_1} \cdot \max_{J\in\mathcal{J}(n-\kappa-1,d)} \mathrm{dist}\,(0, M)^{\delta^r-d_J} \\ \Longrightarrow\ &\mathrm{dist}\,(0, M) \le C_2C_3\, e^{C_1} \quad \text{(since each } d_J \ge 1). \end{aligned} \tag{4.55}$$

Combining (4.53), (4.54) and (4.55), we have

$$\begin{aligned} \mathrm{dist}\,(z, M)^{\delta^r} &\le C_2C_3 \cdot \max_{J\in\mathcal{J}(n-\kappa-1,d)} |\hat{q}_J(z)| \cdot (\|z\| + C_2C_3\, e^{C_1})^{\delta^r-d_J} \\ &\le C_2C_3 \cdot (C_2C_3\, e^{C_1})^{\delta^r-1} \cdot \max_{J\in\mathcal{J}(n-\kappa-1,d)} |\hat{q}_J(z)| \cdot (\|z\| + 1)^{\delta^r-d_J} \\ &\qquad \text{(since each } d_J \ge 1) \\ &= C_2^{\delta^r} C_3^{\delta^r}\, e^{(\delta^r-1)C_1} \cdot \max_{J\in\mathcal{J}(n-\kappa-1,d)} |\hat{q}_J(z)| \cdot (\|z\| + 1)^{\delta^r-d_J}. \end{aligned} \tag{4.56}$$

For each $J \in \mathcal{J}(n - \kappa - 1, d)$, we define the polynomial $q_J \in \overline{\mathbb{Q}}[z_1, \cdots, z_n]$ given by

$$q_J(z) := (\llcorner C_2^{\delta^r} C_3^{\delta^r}\, e^{(\delta^r-1)C_1} \lrcorner + 1) \cdot \hat{q}_J(z), \quad z \in \mathbb{C}^n. \tag{4.57}$$

Here $\llcorner a \lrcorner$ denotes the greatest integer less than or equal to a. Thus we have, for all $z \in \mathbb{C}^n$,

$$\mathrm{dist}\,(z, M)^{\delta^r} \le \max_{J\in\mathcal{J}(n-\kappa-1,d)} |q_J(z)| \cdot (\|z\| + 1)^{\delta^r-d_J}, \tag{4.58}$$

which gives (2.5). By (4.51), Lemma 3.1(viii) (with p and the map $(f_1, \cdots, f_r)$ there given here by ϕ_J and p_J respectively) and the fact that p_J is represented here by the matrix $P^{(J)}$ in (4.28), one has, for each $J \in \mathcal{J}(n - \kappa - 1, d)$,

$$\begin{aligned} &h^A(\hat{q}_J) \\ &\le h^A(\phi_J) + d_J\big(h^A(P^{(J)}) + \log(\kappa + 2) + \log(n + 1) \cdot 1\big) \\ &\le C_1 + d_J\big(\log(n^{2n\cdot 3^{n-1}} d^{n+8n\cdot 3^{n-1}}) + \log n + \log(n + 1)\big) \\ &\qquad \text{(by (4.38), (4.41) and the inequality } \kappa \le n - 2) \\ &\le C_1 + \delta^r \cdot \log(n^{3+2n\cdot 3^{n-1}} \delta^{r(n+8n\cdot 3^{n-1})}) \quad \text{(since } d_J \le d \le \delta^r). \end{aligned} \tag{4.59}$$

By Lemma 3.1(iv) and (4.57), we have

$$\begin{aligned}
&h^A(q_J)\\
&\le h^A(\llcorner C_2^{\delta^r} C_3^{\delta^r}\, e^{(\delta^r-1)C_1}\lrcorner + 1) + h^A(\hat{q}_J)\\
&\le \log(2C_2^{\delta^r} C_3^{\delta^r}\, e^{(\delta^r-1)C_1}) + C_1 + \delta^r \cdot \log(n^{3+2n\cdot 3^{n-1}} \delta^{r(n+8n\cdot 3^{n-1})})\\
&\qquad\qquad \text{(by Lemma 3.1(ii) and (4.59))}\\
&\le \delta^r (C_1 + \log C_2 + \log C_3 + \log 2 + \log(n^{3+2n\cdot 3^{n-1}} \delta^{r(n+8n\cdot 3^{n-1})})\\
&= \delta^r \big(2\, C_1 + \log(n^{3^{n-1}} \delta^{3^n r})^{(n-1)\delta^r} + \log(\sqrt{n}\, \delta^{2r})^{3^{n-1}\delta^r}\\
&\qquad\qquad + \log 2 + \log(n^{3+2n\cdot 3^{n-1}} \delta^{r(n+8n\cdot 3^{n-1})})\big) \quad \text{(by (4.44) and (4.50))}\\
&\le \delta^r \big(2\, \delta^r (n\lambda + n^2 3^n \log n + rn^2 3^{n+1} \log \delta) + (n-1)\delta^r (3^{n-1} \log n + 3^n r \log \delta)\\
&\qquad\qquad + 3^{n-1}\delta^r (\frac{1}{2} \log n + 2r \log \delta) + (3 + 2n \cdot 3^{n-1}) \log n\\
&\qquad\qquad + r(n + 8n \cdot 3^{n-1}) \log \delta + \log 2\big) \quad \text{(by (4.37))}\\
&\le \delta^{2r} \big(2n\lambda + (2n^2 3^n + (n-1)3^{n-1} + \frac{1}{2} \cdot 3^{n-1} + 3 + 2n \cdot 3^{n-1}) \log n\\
&\qquad\qquad + r(2n^2 3^{n+1} + (n-1)3^{n-1} + 2 \cdot 3^{n-1} + n + 8n \cdot 3^{n-1}) \log \delta + \log 2\big)\\
&\le \delta^{2r} \big(2n\lambda + n^2 3^{n+1} \log n + rn^2 3^{n+2} \log \delta\big)\\
&\qquad\qquad \text{(by straight-forward comparisons and using the inequality } n \ge 2)\\
&= C, && (4.60)
\end{aligned}$$

where C is as in (2.3), and thus we have verified (2.4). This finishes the proof of Theorem 2.1(i). As the proof of Theorem 2.1(ii) follows mutatis mutandis from that of Theorem 2.1(i), we will only indicate the necessary modifications. Under the additional condition that $0 \in M$, one easily sees that in place of (4.54) and (4.55), one has

$$\operatorname{dist}(z, M) \le \|z\| \quad \text{for all } z \in M. \tag{4.61}$$

Then one can use (4.61) and (4.53) to improve (4.56) to get

$$\begin{aligned}
\operatorname{dist}(z, M)^{\delta^r} &\le C_2 C_3 \cdot \max_{J \in \mathcal{J}(n-\kappa-1,d)} |\hat{q}_J(z)| \cdot \|z\|^{\delta^r - d_J}\\
&\le \max_{J \in \mathcal{J}(n-\kappa-1,d)} |q_J(z)| \cdot \|z\|^{\delta^r - d_J}, && (4.62)
\end{aligned}$$

where, in lieu of (4.57), each q_J is given in the present case by

$$q_J(z) := (\llcorner C_2 C_3 \lrcorner + 1) \cdot \hat{q}_J(z). \tag{4.63}$$

With the newly defined q_J and via a calculation similar to (4.60), one easily sees that

$$\begin{aligned}
h^A(q_J) &\le h^A(\llcorner C_2 C_3 \lrcorner + 1) + h^A(\hat{q}_J)\\
&\le \log C_2 + \log C_3 + \log 2 + C_1 + \delta^r \cdot \log(n^{3+2n\cdot 3^{n-1}} \delta^{r(n+8n\cdot 3^{n-1})})\\
&\le \delta^r \big(2n\lambda + n^2 3^{n+1} \log n + rn^2 3^{n+2} \log \delta\big)\\
&= C', && (4.64)
\end{aligned}$$

where C' is as in (2.6). We remark here that the third line of (4.64) follows readily from comparing the second line of (4.64) with the third line of (4.60). This finishes the proof of Theorem 2.1(ii). □

5. Application to Bihomogeneous Polynomials

Next we give the deduction of Theorem 2.2 from Theorem 2.1.

Proof of Theorem 2.2. As the deduction of Theorem 2.2 from Theorem 2.1 follows mutatis mutandis from the arguments in [17], Section 3.2, Proof of Theorem 3, which treated the special case when M is an affine hypersurface, we will only indicate here the necessary modifications and leave the details to the reader. Let $M \in \mathcal{M}^{h}_{\mathbb{Q}}(n, r, \delta, \lambda)$, f, μ_o and ℓ_o be as in Theorem 2.2. Clearly, $0 \in M$. Let $C' = C'(n, r, \delta, \lambda) > 0$, $\{q_j\}_{1 \le j \le j_o}$ with $j_o \le (\delta^r + 1)^{n-1}$ and each $d_j = \deg(q_j) \le \delta^r$ be as in Theorem 2.1(ii), so that each $h^A(q_j) \le C'$, and

$$\operatorname{dist}(z, M)^{\delta^r} \le \max_{1 \le j \le j_o} |q_j(z)| \cdot \|z\|^{\delta^r - d_j} \quad \text{for any } z \in \mathbb{C}^n. \tag{5.1}$$

For each $1 \le j \le j_o$, it is easy to see that q_j has at most $\binom{n+d_j-1}{n-1}$ non-zero monomial terms. Together with Lemma 3.1(i) and the triangle inequality, one sees that if $\|z\| \le 1$, then

$$|q_j(z)| \le \binom{n + d_j - 1}{n-1} \cdot e^{h^A(q_j)} \le \binom{n + \delta^r - 1}{n-1} \cdot e^{C'}. \tag{5.2}$$

Together with (5.1), we have

$$\operatorname{dist}(z, M)^{\delta^r} \le \binom{n + \delta^r - 1}{n-1} \cdot e^{C'} \cdot \max_{1 \le j \le j_o} \frac{|q_j(z)| \cdot \|z\|^{\delta^r - d_j}}{\sup_{u \in S^{2n-1}} |q_j(u)|} \quad \text{for any } z \in \mathbb{C}^n. \tag{5.3}$$

Let C_1, C_2 be as in [17], Proof of Theorem 3. We first remark that the polynomial in [17], (3.2.7) and (3.2.8), is replaced here by

$$F(z) := \begin{cases} f(z) + C_3 \sum_{j=1}^{j_o} \dfrac{\|z\|^{2(m-d_j)} |q_j(z)|^2}{\sup_{u \in S^{2n-1}} |q_j(u)|^2} & \text{if } m \ge \delta^r \\ \|z\|^{2(\delta^r - m)} f(z) + C_3 \sum_{j=1}^{j_o} \dfrac{\|z\|^{2(\delta^r - d_j)} |q_j(z)|^2}{\sup_{u \in S^{2n-1}} |q_j(u)|^2} & \text{if } m < \delta^r, \end{cases} \tag{5.4}$$

where

$$C_3 := \frac{\binom{n+\delta^r-1}{n-1}^2 \cdot e^{2C'} \cdot \left(\frac{1}{2} \inf_{u \in M \cap S^{2n-1}} f(u) + C_2\right)}{C_1^{2\delta^r}} > 0. \tag{5.5}$$

With [17], Proposition 3.1.2, replaced here by (5.3), one can easily check that the inequality in [17], (3.2.9), still holds (with the new definition of F), while [17], (3.2.10), is replaced by the following inequality:

$$\sup_{z \in S^{2n-1}} F(z) \le C_2 + C_3 j_o \le C_2 + C_3 \cdot (\delta^r + 1)^{n-1}, \tag{5.6}$$

from which one can check as in [17], (3.2.11), that

$$\epsilon(F) \geq \frac{1}{\mu_o}, \tag{5.7}$$

where $\epsilon(F)$ is as defined in [17], (1.1.4). Finally, by using (5.7) and letting each $p_j := \frac{\sqrt{C_3}}{\sup_{u\in S^{2n-1}}|q_j(u)|} q_j$, one can apply [17], Theorem 1, as in [17], (3.1.12), to obtain (2.14), from which (2.15) follows immediately. □

6. Effective Arithmetic Łojasiewicz Inequality - in Terms of Given Defining Polynomials

In this section, we are going to obtain a second form of the Łojasiewicz inequality for arithmetically defined affine varieties, where the distance function is bounded in terms of a given set of defining polynomials. This is obtained by combining Theorem 2.1 and the effective arithmetic Nullenstellensatz of Krick-Pardo-Sombra [11]. For a given number field K, we denote the ring of integers of K by $\mathcal{O}_K$. For our purpose of proving Theorem 2.3, we first recall a modified form of the latter result as follows:

Proposition 6.1. (Consequence of Krick-Pardo-Sombra's effective arithmetic Nullstellensatz [11], Theorem 3.6) *Let K be a number field, and let $q, f_1, \cdots, f_r \in \mathcal{O}_K[z_1, \cdots, z_n]$ be polynomials such that q vanishes on $Z(f_1, \cdots, f_r)$. Set*

$$\delta := \max\{\deg(q), \deg(f_i), \quad i = 1, \cdots, r\}, \quad \textit{and}$$
$$\lambda := h^A(q, f_1, \cdots, f_r).$$

Then there exist $s \in \mathbb{N}$, $a \in \mathcal{O}_K \setminus \{0\}$ and $g_1, \cdots, g_r \in \mathcal{O}_K[z_1, \cdots, z_n]$ such that
(i) $aq^s = g_1 f_1 + \cdots + g_r f_r$,
(ii) $s \leq 4(n+1)(\delta+1)^{n+1}$,
(iii) $\max_{1\leq i\leq r} \deg(g_i) \leq 4(n+1)(\delta+1)^{n+2}$,
(iv) $h^A(a) \leq 4(n+1)(n+2)(\delta+1)^{n+1}\big(\lambda + \log(r+1) + (\delta+1)(n+8)\log(n+2)\big)$, *and*
(v) $\max_{1\leq i\leq r} h^A(g_i) \leq 4(n+1)(n+4)(\delta+1)^{n+1}\big(\lambda+\log(r+1)+(\delta+1)(n+8)\log(n+2)\big)$.

Remark 6.1. For convenience of the reader, we give in the Appendix the deduction of Proposition 6.1 from the original form of Krick-Pardo-Sombra's effective arithmetic Nullstellensatz in [11]. See Section 7 for details.

Finally we are ready to give the proof of Theorem 2.3.

Proof of Theorem 2.3. Let $f_1, \cdots, f_r \in \overline{\mathbb{Z}}[z_1, \cdots, z_n]$, $M = Z(f_1, \cdots, f_r)$, δ, λ be as in Theorem 2.3, and let $d := \deg(M)$. Recall from Remark 2.1 that one has $d \leq \delta^r$, and we may assume that $n \geq 2$, $d \geq 2$ and thus also that $\delta \geq 2$. By Theorem 2.1, there exist polynomials $\{q_j\}_{1\leq j\leq j_o} \subset \overline{\mathbb{Q}}[z_1, \cdots, z_n]$ such that $j_o \leq (\delta^r+1)^{n-1}$, each q_j is of degree $d_j \leq d \leq \delta^r$ and vanishing on M, and (2.4) and (2.5) are satisfied with an explicit constant $C = C(n, r, \delta, \lambda)$ given as in (2.3). Next we fix a number field K such that $f_1, \cdots, f_r \in \mathcal{O}_K[z_1, \cdots, z_n]$ and each $q_j \in K[z_1, \cdots, z_n]$. For each $1 \le j \le j_o$, upon multiplying q_j by a suitable nonzero integer in $\mathcal{O}_K$ and cancelling out the common prime factors, one obtains a polynomial $\tilde{q}_j \in \mathcal{O}_K[z_1, \cdots, z_n]$ whose coefficients are relatively prime algebraic integers, and thus one has

$$h^A(\tilde{q}_j) = h(\tilde{q}_j) = h(q_j) \le h^A(q_j) \le C, \tag{6.1}$$

where the last inequality follows from (2.4). From the above construction and Lemma 3.1(i), one has $\tilde{q}_j = \dfrac{1}{c_j} \cdot q_j$ for some constant c_j such that $|c_j| \le e^{h^A(q_j)} \le e^C$. Together with (2.5), we have

$$\operatorname{dist}(z, M)^{\delta^r} \le e^C \cdot \max_{1 \le j \le j_o} |\tilde{q}_j(z)| \cdot (\|z\| + 1)^{\delta^r - d_j} \quad \text{for any } z \in \mathbb{C}^n. \tag{6.2}$$

Now we fix a $j \le j_o$. By Lemma 3.1(vii) and (6.1), we have

$$h^A(\tilde{q}_j, f_1, \cdots, f_r) \le h^A(\tilde{q}_j) + h^A(f_1, \cdots, f_r) + \log 2 \le C + \lambda + \log 2 =: \tilde{\lambda}. \tag{6.3}$$

By applying Proposition 6.1 to $\tilde{q}_j, f_1, \cdots, f_r$ (with δ, λ in Proposition 6.1 replaced by δ^r, $\tilde{\lambda}$ respectively), it follows that there exists $s_j \in \mathbb{N}$, $a_j \in \mathcal{O}_K \setminus \{0\}$ and $g_{j,1}, \cdots, g_{j,r} \in \mathcal{O}_K[z_1, \cdots, z_n]$ such that

$$a_j \tilde{q}_j^{s_j} = g_{j,1} f_1 + \cdots + g_{j,r} f_r, \tag{6.4}$$

$$s_j \le 4(n+1)(\delta^r + 1)^{n+1} := s, \tag{6.5}$$

$$\max_{1 \le i \le r} \deg(g_{j,i}) \le 4(n+1)(\delta^r + 1)^{n+2}, \quad \text{and} \tag{6.6}$$

$$\begin{aligned} \max_{1 \le i \le r} h^A(g_{j,i}) &\le 4(n+1)(n+4)(\delta^r + 1)^{n+1} \\ &\qquad \cdot \big(\tilde{\lambda} + \log(r+1) + (\delta+1)(n+8)\log(n+2)\big) \\ &= : \tilde{\lambda}'. \end{aligned} \tag{6.7}$$

Now we give an apriori pointwise upper bound for $\tilde{q}_j$. First we remark that $\tilde{q}_j$ has at most $\binom{n+d_j}{n}$ non-zero monomial terms. Together with Lemma 3.1(i) and the triangle inequality, one sees that for any $z \in \mathbb{C}^n$,

$$\begin{aligned} |\tilde{q}_j(z)| &\le \binom{n+d_j}{n} \cdot e^{h^A(\tilde{q}_j)} \cdot \max\{\|z\|, 1\}^{d_j} \\ &\le 2^{n+\delta^r} \cdot e^C \cdot (\|z\| + 1)^{d_j}, \end{aligned} \tag{6.8}$$

where the last inequality follows from (6.1) and the inequality $d_j \le \delta^r$. Similarly, we have, for each $1 \le i \le r$ and $z \in \mathbb{C}^n$,

$$\begin{aligned} |g_{j,i}(z)| &\le \binom{n + \deg(g_{j,i})}{n} \cdot e^{h^A(g_{j,i})} \cdot (\|z\| + 1)^{\deg(g_{j,i})} \\ &\le 2^{n+4(n+1)(\delta^r+1)^{n+2}} \cdot e^{\tilde{\lambda}'} \cdot (\|z\| + 1)^{4(n+1)(\delta^r+1)^{n+2}}, \end{aligned} \tag{6.9}$$

where the last inequality follows from (6.6) and (6.7). Next, we give another pointwise upper bound for $\tilde{q}_j$ using (6.4). Note that $|a_j| \geq 1$, since $0 \neq a_j \in \mathcal{O}_K$. Thus, we have, for $z \in \mathbb{C}^n$,

$$\begin{aligned} |\tilde{q}_j(z)|^{s_j} &\leq |a_j \tilde{q}_j(z)^{s_j}| \\ &= |g_{j,1}(z)f_1(z) + \cdots + g_{j,r}(z)f_r(z)| \quad \text{(by (6.4))} \\ &\leq r \cdot \max_{1\leq i\leq r} \big(|g_{j,i}(z)| \cdot |f_i(z)|\big) \\ &\leq r \cdot 2^{n+4(n+1)(\delta^r+1)^{n+2}} \cdot e^{\tilde{\lambda}'} \\ &\qquad \cdot (\|z\|+1)^{4(n+1)(\delta^r+1)^{n+2}} \cdot \max_{1\leq i\leq r} |f_i(z)|. \end{aligned} \tag{6.10}$$

Let μ_1 be as in (2.16), and let s be as in (6.5). Then one easily sees that $\mu_1 = \delta^r s$. Thus for any $z \in \mathbb{C}^n$, we have

$$\begin{aligned} &\operatorname{dist}(z, M)^{\mu_1} \\ &= \operatorname{dist}(z, M)^{\delta^r s} \\ &\leq e^{Cs} \cdot \max_{1\leq j\leq j_o} \big\{|\tilde{q}_j(z)|^{s-s_j} \cdot |\tilde{q}_j(z)|^{s_j} \cdot (\|z\|+1)^{(\delta^r-d_j)s}\big\} \quad \text{(by (6.2))} \\ &\leq e^{Cs} \cdot \max_{1\leq j\leq j_o} \big\{2^{(n+\delta^r)(s-s_j)} \cdot e^{C(s-s_j)} \cdot (\|z\|+1)^{d_j(s-s_j)} \cdot r \cdot 2^{n+4(n+1)(\delta^r+1)^{n+2}} \\ &\qquad\qquad \cdot e^{\tilde{\lambda}'} \cdot (\|z\|+1)^{4(n+1)(\delta^r+1)^{n+2}} \cdot \max_{1\leq i\leq r} |f_i(z)| \cdot (\|z\|+1)^{(\delta^r-d_j)s}\big\} \\ &\qquad \text{(by (6.8) and (6.10))} \\ &\leq \tilde{C}' \cdot \max_{1\leq i\leq r} |f_i(z)| \cdot (\|z\|+1)^{\mu}, \end{aligned} \tag{6.11}$$

where we may take

$$\tilde{C}' := r \cdot e^{2Cs+\tilde{\lambda}'} \cdot 2^{(n+\delta^r)s+n+4(n+1)(\delta^r+1)^{n+2}}, \quad \text{and} \tag{6.12}$$

$$\begin{aligned} \mu &:= 4(n+1)(\delta^r+1)^{n+2} + \delta^r s \\ &= 4(n+1)(\delta^r+1)^{n+2} + \delta^r \cdot 4(n+1)(\delta^r+1)^{n+1} \\ &\leq 8(n+1)(\delta^r+1)^{n+2} \\ &= \mu_2, \end{aligned} \tag{6.13}$$

where μ_2 is as in (2.16). By (6.12), we have

$$\begin{aligned} &\log \tilde{C}' \\ &= \log r + 2Cs + \tilde{\lambda}' + ((n+\delta^r)s + n + 4(n+1)(\delta^r+1)^{n+2}) \log 2 \\ &\leq \log r + 2C \cdot 4(n+1)(\delta^r+1)^{n+1} + 4(n+1)(n+4)(\delta^r+1)^{n+1}\big((C+\lambda+\log 2) \\ &\qquad + \log(r+1) + (\delta+1)(n+8)\log(n+2)\big) \\ &\qquad\quad + \big((n+\delta^r) \cdot 4(n+1)(\delta^r+1)^{n+1} + n + 4(n+1)(\delta^r+1)^{n+2}\big) \log 2 \\ &\qquad \text{(by (6.3), (6.5) and (6.7))} \\ &=: I_1 + I_2, \end{aligned} \tag{6.14}$$

where

$$
\begin{aligned}
I_1 : &= 4(n+1)(n+6)(\delta^r+1)^{n+1}C \\
&= 4(n+1)(n+6)(\delta^r+1)^{n+1} \cdot \delta^{2r}\big(2n\lambda + n^2 3^{n+1}\log n + rn^2 3^{n+2}\log\delta\big) \\
&\qquad (\text{by } (2.3)), \text{ and} \qquad (6.15) \\
I_2 : &= \log r + 4(n+1)(n+4)(\delta^r+1)^{n+1} \\
&\qquad \cdot\big(\lambda + \log 2 + \log(r+1) + (\delta+1)(n+8)\log(n+2)\big) \\
&\quad + \big((n+\delta^r)\cdot 4(n+1)(\delta^r+1)^{n+1} + n \\
&\qquad + 4(n+1)(\delta^r+1)^{n+2}\big)\log 2. \qquad (6.16)
\end{aligned}
$$

Using the inequalities $n \geq 2$, $\delta \geq 2$ and $r \geq 1$, it is easy to check that

$$I_2 \leq 4(n+1)(n+6)(\delta^r+1)^{n+1}\cdot(2\delta^r+1)\cdot(2n\lambda+n^2 3^{n+1}\log n+rn^2 3^{n+2}\log\delta). \quad (6.17)$$

Together with (6.15), one gets

$$I_1 + I_2 \leq 4(n+1)(n+6)(\delta^r+1)^{n+3}\cdot(2n\lambda + n^2 3^{n+1}\log n + rn^2 3^{n+2}\log\delta). \quad (6.18)$$

Combining (6.14) and (6.18), we get

$$\tilde{C}' \leq \big(e^{2n\lambda} n^{n^2 3^{n+1}} \delta^{rn^2 3^{n+2}}\big)^{4(n+1)(n+6)(\delta^r+1)^{n+3}} = \tilde{C}, \quad (6.19)$$

where $\tilde{C}$ is as in (2.16). Combining (6.11), (6.13) and (6.19), one obtains (2.17) readily. This finishes the proof of Theorem 2.3. □

7. Appendix: Deduction of Proposition 6.1 from Krick-Pardo-Sombra's Effective Arithmetic Nullstellensatz

For convenience of the reader, we give in this Appendix the deduction of Proposition 6.1 from Krick-Pardo-Sombra's effective arithmetic Nullstellensatz [11] using an effective adaptation of a well-known trick of Rabinowitsch (see e.g [4], p. 240). First we recall the following

Proposition 7.1. (Krick-Pardo-Sombra [11], Theorem 3.6) (Effective arithmetic Nullstellensatz) *Let K be a number field, and let $f_1, \cdots, f_r \in \mathcal{O}_K[z_1, \cdots, z_n]$ be polynomials without common zeros in $\mathbb{C}^n$. Set $\delta := \max_{1\leq i\leq r} \deg(f_i)$ and $\lambda := h^A(f_1, \cdots, f_r)$. Then there exist $a \in \mathcal{O}_K \setminus \{0\}$ and $\mu_1, \cdots, \mu_r \in \mathcal{O}_K[z_1, \cdots, z_n]$ such that*

(i) $a = \mu_1 f_1 + \cdots + \mu_r f_r$,
(ii) $\deg(\mu_i) \leq 4n\delta^n$ *for* $1 \leq i \leq r$, *and*
(iii) $h^A(a, \mu_1, \cdots, \mu_r) \leq 4n(n+1)\delta^n\big(\lambda + \log r + \delta(n+7)\log(n+1)\big)$.

Now we give the deduction of Proposition 6.1 from the Proposition 7.1.

Proof of Proposition 6.1. Let $q, f_1, \cdots, f_r \in \mathcal{O}_K[z_1, \cdots, z_n]$, λ, δ be as in Proposition 6.1. Following Rabinowitsch's trick as explained in [4], p. 240, we introduce an extra variable z_{n+1}. Then it is easy to see that the $r+1$ polynomials

$$f_1, \cdots, f_r,\ 1 - z_{n+1}q \in \mathcal{O}_K[z_1, \cdots, z_{n+1}], \tag{7.1}$$

and they have no common zeros in $\mathbb{C}^{n+1}$ (since q vanishes on $Z(f_1, \cdots, f_r)$). Moreover, one has

$$\begin{aligned} \deg(1 - z_{n+1}q) &\le \delta + 1, \quad \text{and} \\ h^A(f_1, \cdots, f_r, 1 - z_{n+1}q) &= h^A(q, f_1, \cdots, f_r) = \lambda, \end{aligned} \tag{7.2}$$

since the coefficients of $1 - z_{n+1}q$ are given by 1 and, up to the minus sign, the coefficients of q. By applying Proposition 7.1 to the polynomials in (7.1) (with the numbers n, r, δ in Proposition 7.1 replaced by $n+1$, $r+1$, $\delta+1$ respectively, while λ remains unchanged), one easily sees that there exist $a \in \mathcal{O}_K \setminus \{0\}$ and $\mu_1, \cdots, \mu_{r+1} \in \mathcal{O}_K[z_1, \cdots, z_{n+1}]$ such that

$$a = \mu_1 f_1 + \cdots + \mu_r f_r + \mu_{r+1}(1 - z_{n+1}q), \tag{7.3}$$

$$\deg(\mu_i) \le 4(n+1)(\delta+1)^{n+1}, \quad i = 1, \cdots, r+1, \quad \text{and} \tag{7.4}$$

$$\begin{aligned} h^A(a, \mu_1, \cdots, \mu_{r+1}) &\le 4(n+1)(n+2)(\delta+1)^{n+1} \\ &\quad \cdot \big(\lambda + \log(r+1) + (\delta+1)(n+8)\log(n+2)\big). \end{aligned} \tag{7.5}$$

Upon substituting $z_{n+1} = \frac{1}{q}$ in (7.3) and multiplying both sides of the resulting equality by q^s (to clear the denominators), where $s := \max\limits_{1 \le i \le r} \deg(\mu_i)$, one obtains

$$aq^s = g_1 f_1 + \cdots + g_r f_r, \quad \text{where } g_i := q^s \mu_i(z_1, \cdots, z_n, \frac{1}{q}) \tag{7.6}$$

is easy seen to be in $\mathcal{O}_K[z_1, \cdots, z_n]$, $i = 1, \cdots, r$, and this gives Proposition 6.1(i). Proposition 6.1(ii) follows directly from (7.4) and the definition of s. It is easy to see from (7.6) that for $1 \le i \le r$, one has

$$\begin{aligned} \deg(g_i) &\le s \cdot \deg(q) + \deg(\mu_i) \\ &\le 4(n+1)(\delta+1)^{n+1} \cdot \delta + 4(n+1)(\delta+1)^{n+1} \\ &\qquad \text{(by Proposition 6.1(ii), (7.4))} \\ &= 4(n+1)(\delta+1)^{n+2}, \end{aligned} \tag{7.7}$$

which gives Proposition 6.1(iii). Proposition 6.1(iv) follows from (7.5) and Lemma 3.1(vi). Finally we are going to prove Proposition 6.1(v), which amounts to bounding the $h^A(g_i)$'s. For each fixed i, we write $\mu_i = \sum c_{j_1 j_2 \cdots j_{n+1}} z_1^{j_1} \cdots z_{n+1}^{j_{n+1}}$, and define another polynomial given by

$$p_i(w_1, \cdots, w_{n+1}) := \sum c_{j_1 j_2 \cdots j_{n+1}} w_1^{j_1} \cdots w_n^{j_n} \cdot w_{n+1}^{s - j_{n+1}}. \tag{7.8}$$

From (7.4), (7.5), (7.6) and (7.8), we easily see that

$$g_i = p_i(z_1, \cdots, z_n, q), \tag{7.9}$$

$$\deg(p_i) \le s + \deg(\mu_i) \le 2s \le 8(n+1)(\delta+1)^{n+1}, \quad \text{and} \tag{7.10}$$

$$\begin{aligned} h^A(p_i) &= h^A(\mu_i) \\ &\le 4(n+1)(n+2)(\delta+1)^{n+1} \\ &\qquad \cdot\big(\lambda + \log(r+1) + (\delta+1)(n+8)\log(n+2)\big). \end{aligned} \tag{7.11}$$

Note also that $h^A(z_1, \cdots, z_n, q) = h^A(q) \le \lambda$. Then by (7.9) and Lemma 3.1(viii) (with the data r, p, $(f_1, \cdots, f_r)$ there given here by $n+1$, p_i, $(z_1, \cdots, z_n, q)$ respectively), we have

$$\begin{aligned} & h^A(g_i) \\ \le\ & h^A(p_i) + \deg(p_i) \cdot \big(\, h^A(z_1, \cdots, z_n, q) + \log(n+2) + \log(n+1) \cdot \deg\,(q)\big) \\ \le\ & 4(n+1)(n+2)(\delta+1)^{n+1}\big(\lambda + \log(r+1) + (\delta+1)(n+8)\log(n+2)\big) \\ & \qquad +8(n+1)(\delta+1)^{n+1} \cdot (\, \lambda + \log(n+2) + \log(n+1) \cdot \delta\,) \\ & \qquad\quad (\text{by (7.10) and (7.11)}) \\ \le\ & 4(n+1)(n+4)(\delta+1)^{n+1}\big(\lambda + \log(r+1) + (\delta+1)(n+8)\log(n+2)\big), \end{aligned}$$

which gives Proposition 6.1(v) upon varying i. □

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